# Gradualism and Irreversibility[1]

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**Abstract**: This paper considers a class of two-player dynamic games in which each player controls a one-dimensional variable which we interpret as a level of cooperation. In the base model, there is an irreversibility constraint stating that this variable can never be reduced, only increased. It otherwise satis...es the usual discounted repeated game assumptions. Under certain restrictions on the payo¤ function, which make the stage game resemble a continuous version of the Prisoners' Dilemma, we characterize e¢cient symmetric equilibria, and show that cooperation levels exhibit gradualism and converge, when payo¤s are smooth, to a level strictly below the one-shot ef-...cient level: the irreversibility induces a steady-state as well as a dynamic ine¢ciency. As players become very patient, however, payo¤s converge to (though never attain) the e¢cient level. We also show that a related model in which an irreversibility arises through players choosing an incremental variable, such as investment, can be transformed into the base model with similar results. Applications to a public goods sequential contribution model and a model of capacity reduction in a declining industry are discussed. The analysis is extended to incorporate partial reversibility, asymmetric equilibria, and sequential moves.

**Keywords**: Cooperation, repeated games, gradualism, irreversibility, public goods.

**JEL Classification Numbers**: C73, H41.

# 1. Introduction

We consider a model in which in every period, there is a Prisoner's Dilemma structure; agents have some mutual interest in cooperating, despite the fact that it is not in any agent's individual interest to cooperate. We suppose that this situation is repeated over time, and, crucially, subject to irreversibility, in the sense that an agent cannot reduce her level of cooperation once increased. In this setting, irreversibility has two opposing effects. First, it aids cooperation, through making deviations in the form of reduced cooperation impossible. Second, it limits the ability of agents to punish a deviator. We consider the complex interplay of these two forces.

The key role of irreversibility in affecting cooperation can be explained more precisely as follows. In the above model, suppose that every player has a (continuous) scalar action variable, which we interpret as a level of cooperation. We say that partial cooperation occurs in some time period if some player chooses a level of this action variable higher than the stage-game Nash equilibrium level, where the latter is the smallest feasible value of the action variable. Full cooperation is a level of this action variable that maximizes the joint payoff of the players[2]. In general, partial cooperation in any time-period can only be achieved if deviation by any agent can be punished by the other agents in some way.

Now the above model without reversibility is just a repeated Prisoner's Dilemma, and in that case, it is well-known that the most effective (and credible) punishments take the form of "sticks", i.e., threats to reduce cooperation back to the stage-game Nash equilibrium. With irreversibility, such punishments are no longer feasible; instead, deviators can only be punished by withdrawal of "carrots", that is, threats take the form of withdrawal of promised higher levels of cooperation in future. It follows immediately from this that irreversibility causes gradualism, i.e., any (subgame-perfect) sequence of actions involving partial cooperation cannot involve an immediate move to full cooperation[3].

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[2] The model is symmetric, i.e., players have identical per-period payoffs given a permutation of their actions. So, the full cooperation level is the same for each player.

[3] This observation is not entirely new; for example, Schelling (1960, p45) makes a similar point. Admati and Perry (1991) and Marx and Matthews (1998) present equilibria of a dynamic voluntary contribution game which exhibit gradualism. However, to the best of our knowledge, our paper provides the ...rst general characterization of gradualism in cooperation due to irreversibility.

Our first contribution is to refine and extend this basic insight. First, we show that any (subgame-perfect) equilibrium sequence of actions involving cooperation must have the level of cooperation rising in every period, but that full cooperation is never reached in finite time. So, as the level of cooperation in any period is bounded above by the full cooperation level, all equilibrium sequences will converge. We focus on the (symmetric) efficient equilibrium sequence i.e. the one that maximises the present value of payoffs of either player. A key question then is: to what value does this efficient equilibrium sequence converge? It turns out that if payoffs are smooth (differentiable) functions of actions, convergence will be to a level strictly below the full cooperation level, no matter how patient agents are. For the case where payoffs are linear up to some joint cooperation level, and constant or decreasing thereafter (the linear kinked case), the results are different — above some critical discount factor equilibrium cooperation can converge asymptotically to the fully efficient level. Below this critical discount factor, no cooperation at all is possible.

The reason for the asymptotic inefficiency in the smooth payoff case is that close to full cooperation, returns from additional mutual cooperation are second-order, whereas the benefits to deviation (not increasing cooperation when the equilibrium path calls for it) remain first-order. The future gains from sticking to an increasing mutually cooperative path will be insufficient to offset the temptation to deviate. It follows that it will be impossible to sustain equilibrium paths close to full cooperation.

Despite this result, inefficiency disappears in the limit as players become patient in the sense that the limit value of the sequence, and player payoffs, both converge to fully efficient levels as discounting goes to zero. However, the asymptotically efficient path of actions in our model is quite different that in the standard "folk theorem" for repeated games: that in the latter case, (without irreversibility) above some critical discount factor the efficient cooperation level can be attained exactly and immediately.

Later sections of the paper then extend the basic model in several directions. First, we recognize that our basic model is very stylized. In many economic applications, irreversibility arises more naturally when the level of "cooperation" is a stock variable which may benefit both players, and it is incremental investment in cooperation that is costly and non-negative, implying the stock variable is irreversible. Therefore, in Section 4, we

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present an "adjustment cost" model with these features, and show that it can be reformulated so that it is a special case of our base model. We then apply the adjustment cost model to study sequential public good contribution games (Admati and Perry (1991), Marx and Matthews (1998)) and capacity reduction in a declining industry (Ghemawat and Nalebu¤(1990)). These applications illustrate the extent to which our results are applicable to variety of disparate areas of economics.

A second key extension is to allow a small amount of irreversibility, so that any player can reduce his cooperation level by some (small) ...xed percentage. This has two countervailing e¤ects. The ...rst is to make deviation more pro...table; the deviator at t can lower his cooperation level below last period's, rather than just keeping it constant. The second e¤ect is to make punishment more severe; the worst possible perfect equilibrium punishment of the deviator is for the other player to reduce his cooperation over time, rather than just not increase it. A priori, it is not clear which e¤ect will dominate. Nevertheless, we are able to show that for a small amount of reversibility the second e¤ect dominates, and in the linear kinked case it dominates for any degree of reversibility. In our model, then, reversibility is desirable in that it allows more cooperative equilibria to be sustained.

The base model also assumes that (two) players move simultaneously, and that they both choose the same[4] path of actions (the symmetric path). In Section 6 we allow players to choose di¤erent action paths, and in this Section, we obtain a (partial) characterization of the Pareto-frontier of the set of equilibrium payo¤s, and how it changes with the discount factor. In Section 7, we allow payers to move sequentially. We show that the equilibrium payo¤s in this game are a subset of those in the simultaneous move game, but that as discounting goes to zero, the e¢cient symmetric payo¤ in the symmetric move game can be arbitrarily closely approximated by equilibrium payo¤s in the sequential game, so that asymptotically, the order of moves has little e¤ect on achievable payo¤s.

There is a small literature on games with the features we consider here. Admati and Perry (1991) and Marx and Matthews (1998) in particular have considered sequential public good contribution games in a formally similar context. Cooperation in such models

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[4]As the model is symmetric, i.e. players have identical per-period payo¤s given a permutation of their actions, this is a natural base case.

is the sum of an individual's contributions, and this is irreversible. Gale (1997) has considered a class of sequential move games which he dubs monotone games. For games with "positive spillovers", which include the class of games considered here, he characterizes long-run efficient outcomes when there is no discounting. In particular, his results imply that in a sequential-move version of our model without discounting, first-best outcomes are attainable.[5]

Of these papers, possibly the closest is Marx and Matthews (1998). The relationship between the two papers is as follows. First, the two papers consider quite different models, although there is some overlap. Marx and Matthews(1998) consider a number of different voluntary contribution games, where a number of players simultaneously make contributions to a public project over T periods, and where T may be finite or infinite. Each player gets a payoff that is linear in the sum of cumulative contributions, plus possibly a "bonus" when the project is completed. One case of their model (T infinite, two players, no bonus) can be reformulated as an "adjustment cost" variant of our model with linear kinked payoffs (as argued in detail in Section 4.1).

In this version of their model, Marx and Matthews (1998) construct a subgame-perfect equilibrium which is approximately efficient when discounting is negligible[6], whereas we are able to characterise efficient subgame-perfect equilibria for any fixed value of the discount factor. Specifically, our results show[7] that in their model, the equilibrium with completion which they construct is in fact efficient for any discount factor above a critical value, and conversely when the discount factor is below the critical value, there are no contributions made in the efficient equilibrium (see Section 4.1 for more details).

We see our model as being applicable to a wide variety of situations in addition to those already mentioned above. Nuclear disarmament between two countries is one example— here cooperation would be measured by the extent of disarmament. While it

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[5] The games considered in this literature allow for the possibility that a player's payoff may be increasing in his or her own cooperation level (on completion of the project in the public good model). The lack of this feature here allows us to obtain results without needing to impose linearity or no discounting.

[6] Corollary 3(ii), Marx and Matthews(1998). Note that their results are stated for n > 2 players also.

[7] We are also able to characterise equilibrium in the case of linear kinked payoffs (which includes the infinite-horizon contribution game without a bonus as a special case) when the two players contribute asymmetrically, whereas Marx and Matthews study only the symmetric equilibrium in this version of their model (although in their paper, they study other versions of their model where players behave asymmetrically).

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may be desirable to move immediately to total disarmament, this is not an equilibrium because either country would prefer to have the other destroy its stockpile while retaining its own. Disarmament must proceed gradually, and our results give conditions under which the limit of the process is complete or only partial disarmament.

Another example would be in trade negotiations. For example, GATT negotiations are known for their gradualism, although there has been little theoretical work on this (see Bagwell and Staiger, 1997). If concessions are irreversible, or if irreversibilities arise in investment such that shifting capital away from import competing technologies cannot easily be reversed, then a similar story to the one we give can be told to explain gradualism. A formal treatment of a related idea in the negotiation context is in Comte and Jehiel (1998) who consider the impact of outside options in a negotiation model where concessions by one party increase the payo¤ the other party gets in a dispute resolution phase.

A further fruitful application is to environmental problems. For example, environmental cooperation may take the form of installation of costly abatement technology. Once installed, this technology may be very expensive to replace with a "dirtier" technology, e.g., conversion of automobiles to unleaded petrol would be expensive to reverse. Consequently it will again be di¢cult to punish deviants by reversing the investment.[8] Similarly, destruction of capital which leads to over-exploitation of a common property resource (e.g., ...shing boats) will also ...t into the general framework of the paper if it is di¢cult to reverse.

## 2. The Model and Preliminary Results

There are two players[9] $i = 1, 2$: In each period, $t = 1, 2, \ldots,$ each player $i$ simultaneously chooses an action variable $c_i \in \Re_+$, measuring $i$'s level of cooperation[10]. The per-period payo¤ to player 1 is $\frac{1}{4}(c_1, c_2)$ with that of player 2 being $\frac{1}{4}(c_2, c_1)$: So, payo¤s of the two players are identical following a permutation of the pair of actions. Also, we assume that $\frac{1}{4}$ is continuous, strictly decreasing in $c_1$ and strictly increasing in $c_2$. Payo¤s over the

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[8] We are grateful to Anthony Heyes for suggesting this application.

[9] Our main results generalise straightforwardly to more than two players.

[10] The action spaces can also be bounded, i.e., $c_i \in [0, \bar{c}]$, as long as $\bar{c} \geq c^{\pi}$.

in...nite horizon are discounted by common discount factor $\pm$; $0 < \pm < 1$:

In this setting, we shall initially be restricting attention to symmetric equilibria. So, we can de...ne the ...rst-best e¢cient level(s) of cooperation as the value(s) of c, that maximise $w(c) := \frac{1}{4}(c; c)$: We assume the following weak property of $w(c)$ :

A1.   There exists a $c^{\pi} > 0$ such that $w(c)$ is strictly increasing in c for all $0 \cdot c < c^{\pi}$, and $w(c) \cdot w(c^{\pi})$ for all $c \, 2 <_{+}$.

This is satis...ed if $w(c)$ is concave with a ...nite maximum or even single-peaked: Note that $c^{\pi}$ is the smallest ...rst-best e¢cient level of cooperation: We assume that the choice of action is irreversible in every period, i.e.,

$$c_{i;t} \, _{\scriptscriptstyle \circ} \, c_{i;t_i \, 1}, \; i = 1; 2, \; t = 1; 2; :::; \tag{2.1}$$

where $c_{i;t}$ is i 's action in period t; and, without loss of generality, we set $c_{1;0} = c_{2;0} = 0$.

A game history at time t is de...ned in the usual way as $f(c_{1;\dot\iota}; c_{2;\dot\iota})g_{\dot\iota=1}^{t_i \, 1}$. Both players can observe game histories. A pure strategy for player $i = 1; 2$ is de...ned in the usual way as a sequence of mappings from game histories in periods $t = 1; 2:::$ to values of $c_{i;t}$ in $<_{+}$, and where every pair $(c_{i;t_i \, 1}; c_{i;t})$ satis...es (2.1). An outcome path of the game is a sequence of actions $f c_{1;t}; c_{2;t} g_{t=1}^{1}$ that is generated by a pair of pure strategies. We are interested in characterizing subgame perfect Nash equilibrium outcome paths. For the moment, we restrict our attention to symmetric equilibrium[11] outcome paths where $c_{1;t} = c_{2;t} = c_{t}$, $t = 1; 2; :::;$ and we denote such paths by the sequence $f c_{t} g_{t=1}^{1}$.

We now derive necessary and su¢cient conditions for some ...xed symmetric outcome path $f c_{t} g_{t=1}^{1}$ to be an equilibrium. Note that the worst punishment that j could impose on i for deviating at date t from such a path is for j to set $c_{j}$ as low as possible. So, if i deviates at t, the worst punishment is for j to set $c_{j;\dot\iota} = c_{j;t}$, all $\dot\iota > t$: Also whatever action is chosen by j, it is always a best response for i to set $c_{i}$ as low as possible. It follows that this punishment is credible, and, given the punishment, $i^{0}$s optimal deviation at t from the symmetric path $f c_{t} g_{t=1}^{1}$ is to set $c_{i;\dot\iota} = c_{t_i \, 1}$ for all $\dot\iota \, _{\scriptscriptstyle \circ} \, t$. Consequently, for a non-decreasing sequence $f c_{t} g_{t=1}^{1}$ to be an equilibrium outcome path it is necessary and

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[11] In the sequel, it is understood that "equilibrium" refers to subgame-perfect Nash equilibrium.

su¢cient that $\{c_t\}_{t=1}^{1}$ satis…es, for all $t \geq 1$; the inequalities

$$\frac{\mu(c_{t-1}; c_t)}{1-\delta} \leq \mu(c_t; c_t) + \delta\mu(c_{t+1}; c_{t+1}) + \cdots : \qquad (2.2)$$

So, as $c_t \geq c_{t-1}$ from the irreversibility constraint (2.1), the interpretation of (2.2) is that in the event of defection, both players stop increasing their levels of cooperation.

Let $C_{SE}$ be the set of non-decreasing paths $\{c_t\}_{t=1}^{1}$ that satisfy (2.2), and we refer to any path in $C_{SE}$ as a (symmetric) equilibrium path. We now note two basic properties of sequences in $C_{SE}$:

**Lemma 2.1.** If $\{c_t\}_{t=1}^{1}$ is an equilibrium path, then (i) $c_t < c^\ast$, for all $t \geq 1$; and (ii) if $c_t > c_{t-1}$ for some $t > 0$, then for all $\tau \geq 0$, there exists a $\tau^0 > \tau$ such that $c_{\tau^0} > c_\tau$ (i.e., the sequence never attains its limit):

**Proof.** (i) Suppose to the contrary that $c_t \geq c^\ast$ for some $t > 0$; with $c_{t-1} < c^\ast$. From the de…nition of $c^\ast$, and A1, we must have

$$\mu(c_t; c_t) \geq \mu(c_{t+1}; c_{t+1}); \quad \tau \geq 1$$

Consequently,

$$\mu(c_t; c_t) + \delta\mu(c_{t+1}; c_{t+1}) + \cdots < \frac{\mu(c_t; c_t)}{1-\delta}:$$

Then, by (2.2), we have

$$\frac{\mu(c_{t-1}; c_t)}{1-\delta} < \frac{\mu(c_t; c_t)}{1-\delta}:$$

But as $c_{t-1} < c_t$, and $\mu$ decreasing in its …rst argument, $\mu(c_{t-1}; c_t) > \mu(c_t; c_t)$, a contradiction.

(ii) If this is not the case, then $c_t > c_{t-1}$ for some $t > 0$, and there exists a $T \geq t$ with $c_\tau = e$ for all $\tau \geq T$ and $c_\tau < e$ for $\tau < T$. Player 1, by deviating at $T$, would receive

$$\frac{\mu(c_{T-1}; e)}{1-\delta} > \frac{\mu(e; e)}{1-\delta};$$

where the inequality follows from $\mu$ decreasing in its …rst argument: Thus the deviation is pro…table, contradicting the equilibrium assumption. ¤

Say that the path $\{b_t\}_{t=1}^{1} \in C_{SE}$ is e¢cient[12] (i.e., among symmetric equilibrium paths) if there does not exist another sequence $\{c_t^0\}_{t=1}^{1} \in C_{SE}$ such that

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t^0; c_t^0) \; > \; \sum_{t=1}^{\infty} \delta^{t-1} u(b_t; b_t):$$

We now have:

**Lemma 2.2.** An e¢cient sequence $\{b_t\}_{t=1}^{1}$ exists, and this sequence satis...es inequalities (2.2) with equality, i.e., for all $t \geq 1$;

$$\frac{u(b_{t-1}; b_t)}{1-\delta} = u(b_t; b_t) + \delta u(b_{t+1}; b_{t+1}) + \cdots : \qquad (2.3)$$

**Proof.** As all the inequalities in (2.2) are weak, existence follows from standard arguments. We refer to (2.2) holding at $t$ the $t$-constraint. To show that all the $t$-constraints hold with equality, suppose to the contrary that for some $t$,

$$\frac{u(b_{t-1}; b_t)}{1-\delta} < u(b_t; b_t) + \delta u(b_{t+1}; b_{t+1}) + \cdots :$$

Then, by continuity, we can increase $b_t$, holding $b_{t+1}; b_{t+2}; \ldots;$ ...xed; without violating the $t$-constraint. Moreover, the $t + 1$-constraint is relaxed by an increase in $b_t$, holding $b_{t+1}; b_{t+2}; \ldots$ ...xed, as $u$ is decreasing in its ...rst argument. Finally, we can hold $b_{t-1}; b_{t-2}; \ldots; b_1$ ...xed since the only e¤ect of an increase in $b_t$ is to relax the $\tau$-constraints, for $\tau < t$: $\quad\square$

It now follows quite straightforwardly from Lemmas 1 and 2 that the e¢cient path must satisfy a second-order di¤erence equation. First note that the e¢cient path must solve the sequence of equations (2.3). Let the sequence $\{c_t(c_1; \delta)\}_{t=1}^{1}$ solve the second-order di¤erence equation

$$u(c_t; c_{t+1}) = \frac{1}{\delta}[u(c_{t-1}; c_t) - u(c_t; c_t)] + u(c_t; c_t); \quad t > 1 \qquad (2.4)$$

with initial conditions $c_0 = 0; c_1 \geq 0$: It is easily checked[13] that any solution to this di¤erence equation is non-decreasing, so the sequence $\{c_t(c_1; \delta)\}_{t=1}^{1}$ has a limit $c_\infty(c_1; \delta)$ which is ...nite or $+1$ . Then we have:

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[12] We use the term '...rst-best' to refer to unconstrained e¢cient outcomes.
[13] This fact follows directly from the proof of Lemma 2.3 below.

Lemma 2.3. Any sequence $\{c_t\}_{t=1}^\infty$ solves (2.3) if and only if it solves (2.4) with initial conditions $c_0 = 0$, $c_1 \geq 0$, and $c_\infty := \lim_{t\to\infty} c_t < +\infty$.

Proof. Necessity. From the irreversibility constraint, $\{c_t\}_{t=1}^\infty$ is a non-decreasing sequence, so it converges to some finite limit $c_\infty$ or diverges to $+\infty$. Since (2.3) implies (2.2), $\{c_t\}_{t=1}^\infty$ is an equilibrium sequence and by Lemma 2.1, $\{c_t\}_{t=1}^\infty$ must converge to $c_\infty \leq c^*$. Now, (2.3) can be written

$$\frac{u(c_{t-1}, c_t)}{1-\beta} = S_t,$$

where we again write $S_t := u(c_t, c_t) + \beta u(c_{t+1}, c_{t+1}) + \dots$. Advancing by one period, we get

$$\frac{u(c_t, c_{t+1})}{1-\beta} = S_{t+1}.$$

Also,

$$S_t = u(c_t, c_t) + \beta S_{t+1}.$$

So,

$$\frac{u(c_{t-1}, c_t)}{1-\beta} = u(c_t, c_t) + \frac{\beta u(c_t, c_{t+1})}{1-\beta}. \qquad (2.5)$$

Rearrangement of (2.5) gives (2.4).

Sufficiency. As just shown above, (2.4) is equivalent to (2.5). By successive substitution using (2.5), we get

$$\frac{u(c_{t-1}, c_t)}{1-\beta} = u(c_t, c_t) + \dots + \beta^{n-1} u(c_{t+n-1}, c_{t+n-1}) + \frac{\beta^n u(c_{t+n-1}, c_{t+n})}{1-\beta} \qquad (2.6)$$

Now, as $\{c_t\}_{t=1}^\infty$ converges by assumption, we must have

$$\lim_{n\to\infty} \frac{\beta^n u(c_{t+n-1}, c_{t+n})}{1-\beta} = 0.$$

So, taking the limit in (2.6), we recover (2.3). $\square$

We now know that the efficient path solves the difference equation (2.4) with initial conditions $c_0 = 0$ and $c_1$ yet to be determined. The following lemma allows us to determine $c_1$ and hence the efficient path itself. This lemma shows that the efficient path is the upper envelope of all equilibrium paths (and hence it is unique). It then follows from Lemma 2.5 (ii) below that $c_1$ is simply the highest value consistent with convergence of the solution to the difference equation.

**Lemma 2.4.** The efficient path $\{b_t\}_{t=1}^{1}$ is the upper envelope of all equilibrium sequences, i.e., there does not exist a $\{c_t^0\}_{t=1}^{1} \in C_{SE}$ with $c_t^0 > b_t$, for some t:

**Proof.** See Appendix. ¤

As before, let the sequence $\{c_t(c_1; \pm)\}_{t=1}^{1}$ solve the difference equation (2.4), and consider the set of initial conditions $c_1$ such that $\{c_t(c_1; \pm)\}_{t=1}^{1}$ converges to a finite limit, i.e.,

$$C_1(\pm) = \{c_1 \mid c_1 (c_1; \pm) < +1\}:$$

Then we have our final result of this section:

**Lemma 2.5.** (i) If, for any $c_1 \geq 0$; $\{c_t(c_1; \pm)\}_{t=1}^{1}$ is a convergent sequence, then it is also an equilibrium sequence; (ii) The efficient path satisfies $\{b_t\}_{t=1}^{1} = \{c_t(b_1; \pm)\}_{t=1}^{1}$, where $b_1 = \max C_1(\pm)$, and $c_t(b_1; \pm) \geq c_t(c_1^0; \pm)$; all $c_1^0 \in C_1(\pm)$; all $t \geq 0$.

**Proof.** (i) In view of the fact that (2.3) guarantees the sequence is equilibrium, sufficiency implies (i) of Lemma 2.3.

(ii) From Lemma 2.2 and Lemma 2.3, the efficient path exists, solves (2.4) with initial conditions $c_0 = 0$; $c_1 \geq 0$ and must also converge. Consequently, $\{b_t\}_{t=1}^{1} = \{c_t(b_1; \pm)\}_{t=1}^{1}$ for some $b_1 \in C_1(\pm)$: Now suppose that there exists another $c_1^0 \in C_1(\pm)$ with $c_t(c_1^0; \pm) > c_t(b_1; \pm)$ at some $t > 0$. In this case, $\{c_t(c_1^0; \pm)\}_{t=1}^{1}$ is an equilibrium (by part (i)) with $c_t(c_1^0; \pm) > c_t(b_1; \pm)$ at some t, which contradicts Lemma 2.4. In particular this implies that $c_1^0 \in C_1(\pm)$ and $c_1^0 > b_1$ is not possible. ¤

# 3. Main Results

We know that the efficient path is the equilibrium path that is not crossed by any other, and which is the highest (at each point) of all convergent sequences that satisfy the difference equation (2.4). We now proceed to get an exact characterization of the limit $b_1$. To do this, we consider two particular cases.

The Differentiable Case.

¼ is twice continuously di¤erentiable, with $\frac{1}{4}_1 < 0$; $\frac{1}{4}_2 > 0$; $\frac{1}{4}_{11}$; $\frac{1}{4}_{22} < 0$; $\frac{1}{4}_{12} \cdot 0$:

## The Linear Kinked Case.

$$\frac{1}{4} = \begin{cases} \frac{1}{4}_1 c_1 + \frac{1}{4}_2 c_2 & \text{if } c_1 + c_2 \cdot 2c^\pi \\ 2\frac{1}{4}_2 c^\pi \text{¡} (\frac{1}{4}_2 \text{¡} \frac{1}{4}_1)c_1 & \text{if } c_1 + c_2 > 2c^\pi \end{cases}$$

where $\frac{1}{4}_1 < 0$; $\frac{1}{4}_2 > 0$ are constants[14] with $\frac{1}{4}_1 + \frac{1}{4}_2 > 0$.

Note that both these cases satisfy our assumption A1 above on the shape of w(c): In the di¤erentiable case, w(c) is strictly concave, as $w^{00} = \frac{1}{4}_{11} + \frac{1}{4}_{22} + 2\frac{1}{4}_{12} < 0$, with a unique maximum at $c^\pi$. In the linear kinked case, w(c) is linear and increasing in c until c reaches the e¢cient level $c^\pi$, and after that, higher cooperation yields negative bene…t.

Consider the di¤erentiable case …rst. De…ne the function

$$^\circ(c) := \frac{\text{¡} \frac{1}{4}_1(c; c)}{\frac{1}{4}_2(c; c)} > 0:$$

Note from the assumed properties of ¼; we have

$$^{\circ 0}(c) = \frac{\text{¡} 1}{\frac{1}{4}_2} [\frac{1}{4}_{11} + \frac{1}{4}_{12} + ^\circ(\frac{1}{4}_{22} + \frac{1}{4}_{12})] > 0;$$

and also that $c^\pi$ solves $^\circ(c^\pi) = 1$: Consequently, provided $^\circ(0) \cdot \pm$; there is a unique solution **b**($\pm$) to the equation

$$^\circ(\mathbf{b}) = \pm; \tag{3.1}$$

and moreover, **b**(:) is strictly increasing in $\pm$. If $^\circ(0) > \pm$; we set **b**($\pm$) = 0: Clearly **b**($\pm$) < $c^\pi$, $\pm < 1$, with $\lim_{\pm! 1} \mathbf{b}(\pm) = c^\pi$. We can now state our …rst main result:

**Proposition 3.1.** Assume the di¤erentiable case. Then the limit of the e¢cient symmetric path, $\mathbf{b}_1$; is equal to **b**($\pm$). Consequently, for all $\pm < 1$, the e¢cient path is uniformly bounded below the …rst-best e¢cient level of cooperation; i.e., $\mathbf{b}_t < \mathbf{b}(\pm) < c^\pi$ for all t.

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[14] An interpretation is that payo¤s depend positively on $(c_1 + c_2)$ up to $2c^\pi$ with a coe¢cient of $\frac{1}{4}_2$, but there is a marginal utility cost of $(\frac{1}{4}_2 \text{¡} \frac{1}{4}_1)$ to increasing one's own $c_i$: For $c_1 + c_2 > 2c^\pi$, there is no more bene…t from joint contributions, only the cost remains, so that joint payo¤s are declining in $(c_1 + c_2)$: For $c_1 + c_2 > 2c^\pi$; all that is needed for the results is that joint payo¤s are nonincreasing in $(c_1 + c_2)$ and also own payo¤s are declining in own $c_i$:

Proof. (a) By the Mean Value Theorem,

$$\text{¼}(c_{t_i 1}; c_t) \text{ ¡ } \text{¼}(c_{t_i 1}; c_{t_i 1}) = \text{¼}_2(c_{t_i 1}; \mu_t)\text{¢}c_t; \quad \mu_t \text{ 2 } [c_{t_i 1}; c_t]$$

$$\text{¼}(c_{t_i 2}; c_{t_i 1}) \text{ ¡ } \text{¼}(c_{t_i 1}; c_{t_i 1}) = \text{ ¡ } \text{¼}_1(\mu_{t_i 1}; c_{t_i 1})\text{¢}c_{t_i 1}; \quad \mu_{t_i 1} \text{ 2 } [c_{t_i 2}; c_{t_i 1}];$$

where $\text{¢}c_t := c_t \text{ ¡ } c_{t_i 1}$: So, substituting in (2.4) and rearranging , we get

$$\text{¢}c_t = \text{ ¡ } \frac{\text{¼}_1(\mu_{t_i 1}; c_{t_i 1})}{\pm\text{¼}_2(c_{t_i 1}; \mu_t)}\text{¢}c_{t_i 1} \qquad (3.2)$$
$$\; \acute{} \; a(c_{t_i 1}; c_t)\text{¢}c_{t_i 1}:$$

(b) Suppose that $\mathbf{b}_1 > \mathbf{b}(\pm)$. There must, by $\text{¼}(\text{¢}; \text{¢})$ being twice continuously di¤er-entiable and $a(\mathbf{b}_1; \mathbf{b}_1) = {}^\circ(\mathbf{b}_1)=\pm > 1$, exist a T such that for $t > T$, $a(c_{t_i 1}; c_t) > 1$. But then from (3.2), for all $t > T$; $\text{¢}c_t > \text{¢}c_{t_i 1}$ whenever $\text{¢}c_{t_i 1} > 0$ and by Lemma 2.1 (ii), $\text{¢}c_{t_i 1} > 0$ for some $t \text{ ¡ } 1 > T$; so $c_t$ cannot converge, contrary to hypothesis. We conclude $\mathbf{b}_1 \cdot \mathbf{b}(\pm)$

(c) Suppose that $0 < \mathbf{b}_1 < \mathbf{b}(\pm)$: We show that this is impossible. Find a neighborhood around $\mathbf{b}_1$; $(\mathbf{b}_1 \text{ ¡ } "; \mathbf{b}_1 + ")$, such that $a(c; c^0) < k < 1$ for all $c, c^0 \text{ 2 } (\mathbf{b}_1 \text{ ¡ } "; \mathbf{b}_1 + ")$: De…ne $\tilde{A} := (1 \text{ ¡ } k)"$, and consider T such that $c_T(\mathbf{b}_1; \pm) > \mathbf{b}_1 \text{ ¡ } \tilde{A}$ (this must exist by de…nition of $\mathbf{b}_1$). Now, since $c_T(\mathbf{b}_1; \pm) < c_{T+1}(\mathbf{b}_1; \pm) < \mathbf{b}_1$; by $c_T(c_1; \pm)$ being continuous in $c_1$; we can …nd $c_1^0 > \mathbf{b}_1$ such that $c_T(c_1^0)$ and $c_{T+1}(c_1^0) \text{ 2 } (\mathbf{b}_1 \text{ ¡ } \tilde{A}; \mathbf{b}_1)$, and moreover, since $0 < c_{T+1}(\mathbf{b}_1; \pm) \text{ ¡ } c_T(\mathbf{b}_1; \pm) < \tilde{A}$, $c_1^0$ can also be chosen so that $0 < c_{T+1}(c_1^0; \pm) \text{ ¡ } c_T(c_1^0; \pm) < \tilde{A}$. Hence for all $t > T$; $\text{¢}c_t < k\text{¢}c_{t_i 1}$ by (3.2), and consequently $fc_t(c_1^0; \pm)g_{t=1}^1$ must converge to some $c_1(c_1^0; \pm) < \mathbf{b}_1 + \frac{\tilde{A}}{1_i k} (= \mathbf{b}_1 + ")$: Since $fc_t(c_1^0; \pm)g_{t=1}^1$ is a convergent path it is also an equilibrium path (Lemma 2.5(i)) and $c_1^0 > \mathbf{b}_1$; which contradicts the envelope property of the e¢cient equilibrium (Lemma 2.4). Finally, a minor modi…cation to this argument establishes that $\mathbf{b}_1 = 0$ is impossible whenever $\mathbf{b}(\pm) > 0$: ¤

Next, consider the linear kinked case. Here, we have the following striking result.

Proposition 3.2. Assume the linear kinked case. If there is su¢ciently little discounting $(\pm > \text{ ¡ } \text{¼}_1=\text{¼}_2)$, then the limit of the e¢cient symmetric sequence, $\mathbf{b}_1$; equals $c^\alpha$, i.e., …rst-best e¢cient cooperation can be asymptotically obtained. Otherwise, no cooperation can ever be obtained, i.e., $\mathbf{b}_t = 0$, all t:

Proof. From Lemma 2.1, we can restrict attention to those paths with $c_t < c^{\pi}$, all $t$, as no other path can be an equilibrium one. Writing out (2.4) for this case, using the de…nition of ¼ for the kinked linear case, we get:

$$\text{¼}_1 c_t + \text{¼}_2 c_{t+1} = \frac{1}{\pm}[\text{¼}_1 c_{t_i 1} + \text{¼}_2 c_t \text{ ¡ } \text{¼}_1 c_t \text{ ¡ } \text{¼}_2 c_t] + \text{¼}_1 c_t + \text{¼}_2 c_t;$$

which rearranges to

$$\text{¢} c_t = a \text{¢} c_{t_i 1};\tag{3.3}$$

where $a := \text{ ¡ } \frac{\text{¼}_1}{\pm \text{¼}_2}$ ; $\text{¢} c_t := c_t \text{ ¡ } c_{t_i 1}$. Thus, $\text{¢} c_t = a^{t_i 1} \text{¢} c_1$ where $\text{¢} c_1 = c_1 \text{ ¡ } c_0 = c_1$, and $c_1$ can be chosen freely. So, we have

$$c_t = \sum_{\text{¿}=1}^{\text{X}} \text{¢} c_{\text{¿}} = (1 + a + \text{:::} a^{t_i 1}) c_1.\tag{3.4}$$

First suppose that $a \text{ , } 1$: If $c_1 > 0$, then from (3.4), $c_t \text{ ! } \text{ 1 }$ as $t \text{ ! } \text{ 1 }$; contradicting the assumption that $c_t < c^{\pi}$, all $t$: So, we must have $c_1 = 0$, in which case $c_t = 0$, all $t$. Thus if $a \text{ , } 1$ () $\pm \cdot$ (¡ ¼$_1$=¼$_2$); no cooperation is possible as claimed. Now suppose that $a < 1$: Then the series in (3.4) converges, so we get

$$c_1 = \frac{1}{1 \text{ ¡ } a} c_1 = \frac{1}{1 + \frac{\text{¼}_1}{\pm \text{¼}_2}} c_1.$$

So by appropriate choice of $c_1$, we can choose a path that converges to $c^{\pi}$; and this must be the e¢cient path by virtue of Lemma 2.4. ¤

Note that in both cases, we have shown that as $\pm \text{ ! } \text{ 1 }$, the limiting level of cooperation on the e¢cient equilibrium path, $c_1$, tends to the …rst-best e¢cient level, $c^{\pi}$. It turns out that this fact implies that payo¤s also converge to their e¢cient levels as $\pm \text{ ! } \text{ 1 }$; i.e., there is no limiting ine¢ciency in this model.

Corollary 3.3. In either the di¤erentiable or linear kinked cases, as $\pm \text{ ! } \text{ 1 }$, the normalized discounted payo¤ from the e¢cient path, $\hat{\text{¦}} = (1 \text{ ¡ } \pm) \sum_{t=1}^{\text{P}} \pm^{t_i 1} \text{¼}(\mathbf{b}_t; \mathbf{b}_t)$; converges to the …rst-best payo¤ $\text{¼}(c^{\pi}; c^{\pi})$:

Proof. Consider, for some …xed $\pm$; rewriting the equilibrium condition (2.2) as, for each $t$;

$$\text{¼}(c_{t_i 1}; c_t) \cdot (1 \text{ ¡ } \pm) \sum_{\text{¿}=t}^{\text{X}} \pm^{\text{¿} i t} \text{¼}(c_{\text{¿}}; c_{\text{¿}});\tag{3.5}$$

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Now, if $\{c_t\}_{t=1}^1$ is an equilibrium sequence at $\pm$; then $\{c_t\}_{t=1}^1$ is also an equilibrium at any $\pm^0 > \pm$ since, as $\frac{1}{4}(c_t; c_t)$ is a non-decreasing sequence, the R.H.S. of (3.5) is non-decreasing in $\pm$; and the L.H.S. is constant.

Now for the di¤erentiable case, de…ne $\mathbf{b}(\pm)$ as in (3.1), and in the linear kinked case, de…ne

$$\mathbf{b}(\pm) = \begin{cases} c^\pi & \text{if } \pm > \text{¡ } \frac{1}{4}_1 = \frac{1}{4}_2 \\ 0 & \text{otherwise} \end{cases}$$

So, for any $" > 0$; …nd a $\bar{\pm}$ such that $\frac{1}{4}(\mathbf{b}(\bar{\pm}); \mathbf{b}(\bar{\pm})) > \frac{1}{4}(c^\pi; c^\pi)$ ¡ $"$ (where in the di¤erentiable case, we use the continuity of $\frac{1}{4}(\text{¢}; \text{¢})$; and, as already remarked, $\lim_{\pm! 1} \mathbf{b}(\pm) = c^\pi$). From Propositions 3.1 and 3.2, at $\bar{\pm}$; $\mathbf{b}_t$ ! $\mathbf{b}(\bar{\pm})$; so holding $\{\mathbf{b}_t\}_{t=1}^1$ …xed, $\lim_{\pm! 1}(1$ ¡ $\pm) \mathbf{P}_{t=1}^1 \pm^{t\text{¡}1} \frac{1}{4}(\mathbf{b}_t; \mathbf{b}_t)$ ! $\frac{1}{4}(\mathbf{b}(\bar{\pm}); \mathbf{b}(\bar{\pm}))$; and hence there exists a $\pm^0 > \bar{\pm}$ such that for $\pm$ satisfying $\pm^0 < \pm < 1$; $(1$ ¡ $\pm) \mathbf{P}_{t=1}^1 \pm^{t\text{¡}1} \frac{1}{4}(\mathbf{b}_t; \mathbf{b}_t) > \frac{1}{4}(c^\pi; c^\pi)$ ¡ $"$: Since $\{\mathbf{b}_t\}_{t=1}^1$ is an equilibrium sequence for such $\pm$; the e¢cient path at such $\pm$ must also give a payo¤ greater than $\frac{1}{4}(c^\pi; c^\pi)$ ¡ $"$: As $"$ is arbitrary, this completes the proof. ¤

An alternative way of viewing this result is to note that if we shrink the period length, holding payo¤s per unit of time constant, then ine¢ciency disappears as period length goes to zero.[15]

# 4. A Model with Adjustment Costs

The model studied above is very stylized. In many economic applications, irreversibility arises more naturally when there is a stock variable which bene…ts both players, and a ‡ow or incremental variable which is costly to increase, and is nonnegative. This non-negativity constraint implies that the value of the stock variable can never fall i.e. the stock variable is irreversible. Here, we present a model with these features, and show that it can be reformulated so that it is a special case of our base model.

Player $i^0$s payo¤ at time t is

$$u(c_{i;t}; c_{jt}) \text{ ¡ } ^\circledR(c_{i;t} \text{ ¡ } c_{i;t\text{¡}1}); \tag{4.1}$$

---

[15] If $\frac{1}{4}$ is discontinuous but otherwise satis…es our assumptions then asymptotic e¢ciency can fail. Consider an example in which player i bene…ts only from j's $c_j$; with an upwards jump in payo¤ at completion ($c_j = c^\pi$), and su¤ers continuous (increasing) costs from $c_i$: Lemma 2.1 still applies, so $c_{i;t} < c^\pi$; all t; and the payo¤ jump is never realised no matter how patient the players.

with u increasing in both arguments, and with $\circledR > 0$ being the cost of adjustment: Here, $c_{i;t}$ is to be interpreted as $i$'s cumulative investment in, or the stock level of, the cooperative activity. We assume that the investment flow is nonnegative, which implies that the stock level of cooperation is irreversible, i.e., $c_{i;t} \geq c_{i;t_i 1}$, $i = 1; 2$:

We now proceed as follows. The present value payoff for $i$ in this model is

$$
\begin{aligned}
\P_i \quad &= \quad u(c_{i;1}; c_{j;1}) \; _i \; \circledR(c_{i;1} \; _i \; c_{i;0}) + \pm[u(c_{i;2}; c_{j;2}) \; _i \; \circledR(c_{i;2} \; _i \; c_{i;1})] + ::: \\
&= \quad \sum_{t=1} \pm^{t_i \; 1}[u(c_{i;t}; c_{j\,t}) \; _i \; \circledR(1 \; _i \; \pm)c_{i;t}] + \circledR c_{i;0}:
\end{aligned}
$$

As initial levels of cooperation $c_{1;0}; c_{2;0}$ are fixed, we can think of this model as a special case of the model of the previous section (i.e. without adjustment costs) where per-period payoffs are

$$
\frac{1}{4}(c; c^0) = u(c; c^0) \; _i \; \circledR(1 \; _i \; \pm)c: \tag{4.2}
$$

Of course, we require that $\frac{1}{4}$ defined in (4.2) satisfies the conditions imposed in Section 2, and also satisfies the relevant conditions of either the differentiable or linear kinked case. If this is the case, then Propositions 3.1 and 3.2 apply directly.

We now study two important economic applications using this extension of our basic model. These are not the only topics that can be studied in this way, but they are chosen to illustrate the power and flexibility of our approach.

## 4.1. Dynamic Voluntary Contribution Games

There is now a small literature (Admati and Perry (1991), Fershtman and Nitzan (1991), Marx and Matthews (1998)), on dynamic games where players can simultaneously or sequentially make contributions towards the cost of a public project. The paper in this literature (Marx and Matthews (1998)) that is closest to our work is one where contributions are made simultaneously, and where the benefits from the project are proportional to the amount contributed (up to a maximum, at which point the project is completed). We will show that a special case of Marx and Matthews' model can be written as an adjustment cost game as above, and that Proposition 3.2 above can be applied to extend some of their results.

Marx and Matthews (1998) consider a model in which $N$ individuals simultaneously make nonnegative private contributions, in each of a ﬁnite or inﬁnite number of periods, to a public project. We assume that $N = 2$, and let $c_{i,t}$ be the cumulative contribution of a numeraire private good by $i$ towards the public project. Individuals obtain a ﬂow of utility $u = (1-\delta)v(c)$ from the aggregate cumulative contribution $c_{1,t} + c_{2,t}$, where $v(c)$ is piecewise linear:

$$v(c_1, c_2) = \begin{cases} \gamma(c_1 + c_2) & \text{if } c_1 + c_2 < 2c^* = C^* \\ \gamma C^* + b & \text{if } c_1 + c_2 \geq C^* \end{cases}$$

where we follow as closely as possible the notation of Marx and Matthews. Thus agents get beneﬁt $\gamma$ from each unit of cumulative contribution, and an additional beneﬁt $b \geq 0$ when the project is "completed", i.e., when the sum of cumulative contributions reaches $C^*$. Also, the cost to $i$ of an increment $c_{i,t} - c_{i,t-1}$ in the cumulative contribution is simply $c_{i,t} - c_{i,t-1}$. We consider the case where $b = 0$ and the time horizon is inﬁnite (the $b = 0$ case unravels otherwise). Also it is assumed that $0.5 < \gamma < 1$, so that it is socially eﬃcient to complete the project (immediately, in fact), but not privately eﬃcient to contribute anything.

Then, from (4.2), per period payoﬀs in the equivalent repeated game are

$$\begin{aligned} \pi(c_1, c_2) &= (1-\delta)v(c_1, c_2) - (1-\delta)c_1 \\ &= \begin{cases} (1-\delta)[(\gamma-1)c_1 + \gamma c_2] & \text{if } c_1 + c_2 < 2c^* = C^* \\ (1-\delta)\gamma C^* - (1-\delta)c_1 & \text{if } c_1 + c_2 \geq C^* \end{cases} \end{aligned}$$

So, $\pi_1 = (1-\delta)(\gamma-1) < 0$, $\pi_2 = (1-\delta)\gamma > 0$. Thus, all the conditions of the linear kinked case are satisﬁed, and so Proposition 3.2 applies directly to this version of the Marx-Matthews model.

First, we can deﬁne the critical value of $\delta$ in Proposition 3.2 as

$$\hat{\delta} = \frac{-\pi_1}{\pi_2} = \frac{(1-\gamma)}{\gamma}.$$

Two results then follow directly from our Proposition 3.2 and its proof:

1. If $\delta > \hat{\delta}$, there is a class of equilibria, indexed by the initial condition $c_1$, where each player's cumulative contribution $c_t$ converges to $c^*$, or indeed to any value less than or equal to $c^*$. Along the equilibrium path, incremental contributions fall at rate $\frac{(1-\gamma)}{\delta\gamma}$.

The e¢cient symmetric equilibrium has initial contribution $c_1 = c^{\pi}(1 - \frac{(1-\delta)}{\pm_\circ})$; and each player's cumulative contribution $c_t$ converges to $c^{\pi}$:

2. If $\pm \cdot \hat{\pm}$, then no contributions are made in any equilibrium.

Result 1 sharpens Proposition 3 and Corollary 3(ii) of Marx and Matthews, who show that for $\pm > \hat{\pm}$; there is an equilibrium with $c_t \to c^{\pi}$, and that for $\pm \approx 1$, this equilibrium is approximately e¢cient. In the special case of $n = 2$ and $b = 0$; we not only con…rm their results, but also show that the equilibrium they construct is the e¢cient equilibrium for any $\pm > \hat{\pm}$: Also, Result 2 is a complete converse result to their Proposition 3.

## 4.2. Capacity Reduction in a Declining Industry

There is now a literature on the equilibrium evolution of capacity in an industry where demand is declining over time (See Ghemawat and Nalebu¤ (1990) and the references therein). For tractability, this literature assumes that product demand declines asymptotically to zero; a backward induction argument can then be used to establish the equilibrium pattern of capacity reduction by …rms. Our framework allows us to deal with the more general case where demand does not decline to zero.

The model is a modi…cation of that of Ghemawat and Nalebu¤ (1990). There is a duopoly where each …rm $i = 1; 2$; has initial capacity at time zero of $k_0$. In any period, the output of …rm $i$ must be no greater than capacity, i.e., $x_{i;t} \cdot k_{i;t}$. Demands and costs are as follows. At time $t$, each …rm faces the linear inverse demand schedule $p_t = a_t - x_{1t} - x_{2t}$. There is no short-run cost of production, but there is a per-period cost of maintaining capacity $\tilde{A} > 0$, and a cost $\frac{3}{4} > 0$ of scrapping capacity, with the ‡ow cost of scrapping less than maintenance, i.e., $\frac{3}{4}(1 - \pm) < \tilde{A}$. It is assumed that capacity, once withdrawn, cannot be reintroduced (for example, the capital stock may consist of specialized capital goods which are no longer manufactured).

Within a period, the production decision is delegated to myopic managers who engage in Cournot competition, so output conditional on capacity is

$$x_{i;t} = \min\{k_{i;t}; a_t/3\}; \tag{4.3}$$

where $a_t/3$ is unconstrained Cournot output at time $t$: We assume that at the beginning

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of period 1, $a_t$ falls permanently from $a_0$ to $a_1$, i.e., the size of the market declines once and for all.[16] We suppose that initial capital stocks have been set so as to force managers to produce at joint pro…t-maximizing outputs, taking into account the cost of capital, and adjustment costs, at the initial level of demand, i.e.,

$$k_0 = \frac{(a_0 - \tilde{A} + \gamma(1 - \delta))}{4}. \tag{4.4}$$

A story consistent with this is that in the (distant) past, this industry has already been hit by a negative demand shock, and has adjusted to the old long-run equilibrium.[17] Note that cutting capacity can act as a way of committing to a lower level of output than the Cournot solution. The question is, if demand falls, can the …rms cut their capacities su¢ciently so as to reach the joint pro…t maximising level?

It is convenient to assume that the decline in the market is not too large, i.e.,

$$\frac{3a_0}{4} < a_1. \tag{4.5}$$

In this case, managers will always be constrained by capacity.[18] So, if (4.5) holds, pro…t in period $t$ can be written

$$\pi_{i;t} = a_1 k_{i;t} - k_{i;t} - k_{i;t}k_{j;t} - \tilde{A}k_{i;t} - \gamma(k_{i;t-1} - k_{i;t})$$
$$\equiv \pi(k_{i;t}; k_{j;t}) - \gamma(k_{i;t-1} - k_{i;t}).$$

So, the fully e¢cient capital stock at the new level of demand, $k^*$; maximizes $\sum_{t=1}^{\infty} \delta^{t-1}(\pi_{1;t} + \pi_{2;t})$, i:e:;

$$k^* = \frac{a_1 - \tilde{A} + \gamma(1 - \delta)}{4};$$

and adjustment should be immediate. Note that $k^* < k_0$.

Now de…ne the level of cooperation of …rm $i$ to be the amount of capital scrapped, $c_{i;t} := k_0 - k_{i;t}$, so $c_{i;0} = 0$, $c^* = k_0 - k^*$. So, from (4.2) we can write pro…t as a function of cooperation levels:

$$\pi(c_{i;t}; c_{j;t}) := \pi(k_0 - c_{i;t}; k_0 - c_{j;t}) - \gamma(1 - \delta)c_{i;t}. \tag{4.6}$$

---

[16] This is in contrast to Ghemawat and Nalebu¤ who make the assumption of a constantly declining market, an assumption which implies a backwards unravelling result and a unique equilibrium. By contrast here there will be many equilibria.

[17] Although, as we shall see, this statement is only approximately correct if $\delta$ is near 1:

[18] To see this, note that (4.5) implies $k_{it} < k_0 = \frac{(a_0 - \tilde{A} + \gamma(1 - \delta))}{4} < \frac{a_1}{3}$ as $\tilde{A} > \gamma(1 - \delta)$ by assumption.

As $\pi(c_{i,t}, c_{j,t})$ is non-linear, the relevant case is the differentiable case. To apply Proposition 3.1, we need to verify the assumptions of the differentiable case. By direct calculation, we have:

$$\pi_1 = -a_1 + \tilde{A} + 2k_{i,t} + k_{j,t} - \tfrac{3}{4}(1 - \delta);$$
$$\pi_2 = k_{i,t};$$
$$\pi_{11} = -2; \quad \pi_{22} = 0; \quad \pi_{12} = -1.$$

So, all the differentiable case conditions are satisfied if $\pi_1 < 0$, which in turn is satisfied if (4.5) holds and capacity (net of scrapping) costs are small[19].

Our results for the differentiable case then apply directly. In particular, on the efficient symmetric path $c_{i,t}$ rises asymptotically to $\mathbf{b}$, where $\mathbf{b}$ is defined in (3.1) above. We can express this in terms of the capital stock: $k_{it}$ declines asymptotically to $\hat{k}$, where $\hat{k}$ solves

$$\frac{\pi_1(\hat{k}, \hat{k})}{\pi_2(\hat{k}, \hat{k})} = \delta.$$

Or, using (4.6), we get:

$$\frac{a_1 - \tilde{A} - 2\hat{k} - \hat{k} + \tfrac{3}{4}(1 - \delta)}{\hat{k}} = \delta.$$

Solving, we get

$$\hat{k} = \frac{a_1 - \tilde{A} + \tfrac{3}{4}(1 - \delta)}{3 + \delta} > k^{\ast}.$$

So, for $\delta < 1$, the duopolists cannot credibly reduce capacity to the new joint profit-maximizing level $k^{\ast}$, even asymptotically. All they can manage is to force down capital stocks to $\hat{k}$, so there will be excess capacity and output in the industry (relative to joint profit maximization), even in the long-run. As $\delta \to 1$, the amount of excess capacity goes to zero.

## 5. Reversible Cooperation

So far, we have assumed that cooperation is completely irreversible. This is clearly a strong assumption. In this section, we examine to what extent our results are robust to

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[19] To see this note that $\pi_1 < 0$ if $k_{it} < \frac{(a_1 - \tilde{A} + \tfrac{3}{4}(1 - \delta))}{3}$. But if capacity (net of scrapping) costs are small $(\tilde{A} \simeq \tfrac{3}{4}(1 - \delta))$, $k_{it} \leq k_0 = \frac{(a_0 - \tilde{A} + \tfrac{3}{4}(1 - \delta))}{4} \simeq \frac{a_0}{4} < \frac{a_1}{3} \simeq \frac{a_1 - \tilde{A} + \tfrac{3}{4}(1 - \delta)}{3}$ as required.

a relaxation of this assumption. Suppose that we modify the irreversibility constraint to

$$c_{i,t} \geq \lambda c_{i,t-1}, \quad 0 \le \lambda \le 1,$$

where the degree of irreversibility is parameterized by $\lambda$; complete irreversibility is $\lambda = 1$, and a standard repeated game is $\lambda = 0$. The first—and important—point is that the effect of lowering $\lambda$ from 1 on the efficient symmetric path is not clear without further analysis, because of two effects that work in opposite directions.

The first effect of a smaller $\lambda$ is to make deviation more profitable; the deviator at $t$ can lower his cooperation level at $t$ to $\lambda c_{t-1} < c_{t-1}$, rather than keep it at $c_{t-1}$. The second effect is to make punishment more severe; the worst possible perfect equilibrium punishment of the deviator is for the other player to reduce his cooperation as fast as possible over time, rather than just not increase it. A priori, it is not clear which effect will dominate. Nevertheless, we are able to show that for a small amount of reversibility the second effect dominates, and in the linear case it dominates for any degree of reversibility.

Specifically, we show that lowering $\lambda$ slightly from $\lambda = 1$ relaxes the incentive constraints; that is, any path that is an equilibrium when $\lambda = 1$ is also an equilibrium path when $\lambda$ is slightly lower than one, and moreover because the incentive constraints become slack, an improved path can be found, so that payoffs increase.

Consider a deviation by $i$ from some symmetric path $\{c_t\}_{t=1}^{\infty}$ at $t$. The worst subgame-perfect punishment that $j$ can impose on $i$ is to reduce cooperation by the maximum amount in every period following $t$, i.e., to set $c_{j,t+1} = \lambda c_t$; $c_{j,t+2} = \lambda^2 c_t$, etc. Consequently, the most profitable deviation $i$ can make is to lower his cooperation by the maximum feasible amount at $t$, i.e., set $c_{i,t} = \lambda c_{t-1}$. So, the maximal payoff to deviation at $t$ is

$$\Phi(\lambda; c_{t-1}; c_t) := u(\lambda c_{t-1}; c_t) + \delta u(\lambda^2 c_{t-1}; \lambda c_t) + \delta^2 u(\lambda^3 c_{t-1}; \lambda^2 c_t) + \cdots$$

Then, $\{c_t\}_{t=1}^{\infty}$ is an equilibrium path if and only if it satisfies for all $t \geq 1$:

$$\Phi(\lambda; c_{t-1}; c_t) \le u(c_t; c_t) + \delta u(c_{t+1}; c_{t+1}) + \delta^2 u(c_{t+2}; \lambda c_{t+2}) + \cdots \tag{5.1}$$

An efficient (symmetric) equilibrium path is defined now as the path that maximizes the utility of either agent subject to the sequence of constraints (5.1).

In order to characterize e¢cient payo¤s, the relevant results extending Lemmas 1-4 are collected below:

Lemma 5.1. With reversibility, there exists an e¢cient symmetric equilibrium sequence $f\mathbf{b}_t g_{t=1}^1$ such that (i) $\mathbf{b}_{t_i 1} \cdot \mathbf{b}_t \cdot c^{\pi}$ for all $t \, _{\circ} 1$; (ii) if $\mathbf{b}_t < c^{\pi}$; then (5.1) holds with equality, (iii) $f\mathbf{b}_t g$ is the upper envelope of all equilibrium sequences which never exceed $c^{\pi}$:

Proof. See Appendix. ¤

If $c^{\pi}$ is the unique maximizer of $\frac{1}{4}(c; c)$; then the sequence $f\mathbf{b}_t g_{t=1}^1$ characterized in the lemma is the unique e¢cient symmetric equilibrium outcome path; otherwise there may be multiple e¢cient paths di¤ering only in the interchange of e¢cient levels of c; but they do not di¤er before such levels are attained. In what follows, the 'e¢cient equilibrium path' is understood to refer to the one which does not exceed $c^{\pi}$:

Using Lemma 5.1, we now turn to discuss the impact of a small amount of irreversibility, and we begin with the di¤erentiable case. Let $f\mathbf{b}_t(\frac{1}{2}) g_{t=1}^1$ be the e¢cient equilibrium path in the $\frac{1}{2}_i$ reversible game, let $\mathbf{b}_1(\frac{1}{2})$ be its limit (which exists by Lemma 5.1), and let

$$\uparrow_{\mathbf{i}}(\tfrac{1}{2}) := (1 \,_{\mathbf{i}}\, \pm) \sum_{t=1}^{\mathbf{X}} \pm^{t_i 1} \tfrac{1}{4}(\mathbf{b}_t(\tfrac{1}{2}); \mathbf{b}_t(\tfrac{1}{2}))$$

be the payo¤ from this e¢cient path, all for some ...xed discount factor $\pm < 1$. Then we have the following:

Proposition 5.2. In the di¤erentiable case, provided $\mathbf{b}_1(1) > 0$; there exists $\bar{\bar{\mathcal{V}}}; 1 > \bar{\bar{\mathcal{V}}} > 0$; such that if $1 > \frac{1}{2} > \bar{\bar{\mathcal{V}}}$, then (i) if $f\mathbf{b}_t(\frac{1}{2}) g_{t=1}^1$ is the e¢cient equilibrium path in the irreversible case, it is also an equilibrium path in the $\frac{1}{2}_i$ reversible case; (ii) $\mathbf{b}_1(\frac{1}{2}) > \mathbf{b}_1(1)$ $( ´ \mathbf{b})$; (iii) $\uparrow_{\mathbf{i}}(\frac{1}{2}) > \uparrow_{\mathbf{i}}(1)$:

Proof. See Appendix. ¤

The reasoning behind this result is that a small amount of irreversibility relaxes the incentive constraints in every time period, allowing every components of the e¢cient path to be raised slightly as $\frac{1}{2}$ decreases slightly from 1. This in turn implies that the limit

value of the e¢cient path is higher, as well as the present discounted payo¤ from the e¢cient path.

We now turn to the linear kinked case. We shall ...rst characterize the sequence $fc_tg_{t=1}^1$ described in Lemma 5.1. From (ii) of the lemma, if $c_t < c^{\pi}$ and $c_{t+1} < c^{\pi}$ then (5.1) holds with equality at both dates, and substituting out the continuation equilibrium payo¤s after $t + 1$ yields

$$\overset{\infty}{\underset{j=1}{X}} \pm^{j\,i\,1}(\text{¼}_1\text{½}^j c_{t_i\,1} + \text{¼}_2\text{½}^{j\,i\,1}c_t) = \text{¼}_1c_t + \text{¼}_2c_t + \pm\left(\overset{\infty}{\underset{j=1}{X}} \pm^{j\,i\,1}(\text{¼}_1\text{½}^j c_t + \text{¼}_2\text{½}^{j\,i\,1}c_{t+1})\right)$$

or

$$\frac{\text{¼}_1\text{½}c_{t_i\,1} + \text{¼}_2c_t}{1\,_i\,\text{½}\pm} = \text{¼}_1c_t + \text{¼}_2c_t + \frac{\text{¼}_1\text{½}c_t + \text{¼}_2c_{t+1}}{1\,_i\,\text{½}\pm};$$

which can be simpli...ed to

$$c_{t+1\ i}\ \text{½}c_t = \ _i\ \frac{\text{¼}_1}{\pm\text{¼}_2}(c_{t\ i}\ \text{½}c_{t_i\,1}):$$

Given that $c_1\,_i\,\text{½}c_0 = c_1;$ this can be solved for

$$c_t = (\text{½}^{t_i\,1} + \text{½}^{t_i\,2}a + \text{½}^{t_i\,3}a^2 ::: + \text{½}a^{t_i\,2} + a^{t_i\,1})c_1; \tag{5.2}$$

where $a = \ _i\ \frac{\text{¼}_1}{\pm\text{¼}_2}$ as before; and note that for $\text{½} = 1$ (irreversibility), (5.2) reduces to (3.4). (If $\text{½} \mathbf{6} a$ then the solution can be written $c_t = \frac{(\text{½}^t_i\,a^t)}{(\text{½}_i\,a)}c_1:$)

We can now prove:

**Proposition 5.3.** In the linear kinked case, (i) if $a(= \ _i\ \frac{\text{¼}_1}{\pm\text{¼}_2}) < 1$ (so a non-trivial equilibrium exists with irreversibility) then payo¤s in e¢cient symmetric equilibrium are a strictly decreasing function of ½ whenever they are below the ...rst-best level (which they are at $\text{½} = 1$). Moreover if $\text{½} < 1$ the project is completed in ...nite time (i.e., $c_t = c^{\pi}$ for some $t < 1$): (ii) If $a > 1;$ then $c_t = 0$ for all t; for all $\text{½} 2 (0; 1]$ in any symmetric equilibrium. (iii) If $a = 1;$ then the project is completed asymptotically for $\text{½} 2 (0; 1):$

**Proof.** See Appendix. ¤

Recall that if $\text{½} = 1;$ no non-trivial equilibrium exists if $a \ _, \ 1;$ while if $\text{½} = 0$ (repeated game) it can be checked that the ...rst best is attainable (immediately) if $a \cdot 1;$ otherwise

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there is no non-trivial equilibrium. The path used in the proof of part (i), which satis...es (5.2) up to its maximum value, is not the e¢cient path unless this maximum occurs at $t = 1$; since each incentive constraint up to $t^{\pi}$ is slack, violating Lemma 5.1(ii). So the e¢cient path also satis...es (5.2) so long as $c_t < c^{\pi}$; but $c_1$ is higher than in the construction of the proof (otherwise Lemma 5.1(iii) is violated).

# 6. Asymmetric Cooperation

So far, we have only considered symmetric paths, i.e., where $c_{1;t} = c_{2;t} = c_t$: A natural question is whether the agents could achieve higher (expected) equilibrium payo¤s by playing asymmetrically. A further related question concerns the characteristics of e¢cient equilibria in a model where agents are constrained to move sequentially; as we shall see, this is a closely related issue and will be considered below.

We shall consider these questions for the linear kinked case only. Let $fc_{1;t}; c_{2;t}g_{t=1}^1$ be an arbitrary (possibly asymmetric) path. Then, by a similar argument to that given in Section 2, such a path is an equilibrium path if and only if for $i; j = 1; 2; i \mathbf{6} j;$ $t = 1; 2; :::;$

$$\frac{\frac{1}{4}_1 c_{i;t_i 1} + \frac{1}{4}_2 c_{j;t}}{1_i \pm} \cdot \frac{1}{4}_1 c_{i;t} + \frac{1}{4}_2 c_{j;t} + \pm \left(\frac{1}{4}_1 c_{i;t+1} + \frac{1}{4}_2 c_{j;t+1}\right) + ::: \qquad : \qquad (6.1)$$

Let $C_E$ be the set of equilibrium paths (i.e. sequences that satisfy (2.1) and (6.1)). Also, let $|_i(fc_{1;t}; c_{2;t}g_{t=1}^1)$ be the normalized (multiplied through by $(1_i \pm)$) present discounted values of payo¤ to $i$ associated with a path, and let $|_E$ be the image of $C_E$ in the space of normalized present discounted values of payo¤s„ i.e.,

$$|_E = f(|_1; |_2) j |_i = |_i(fc_{1;t}; c_{2;t}g_{t=1}^1); \ fc_{1;t}; c_{2;t}g_{t=1}^1 \ 2 \ C_E, \ i = 1; 2g$$

Our focus in on the shape of the e¢cient frontier of $|_E$: As far as symmetric equilibria go, we know from Proposition 3.2 if $\pm \cdot \hat{\pm} = _i \frac{1}{4}_1=\frac{1}{4}_2$; no cooperation is possible, whereas if $\pm > \hat{\pm}$, completion equilibria exist. From the symmetry assumption on payo¤s, $|_E$ is symmetric about the $45^o$ line. One issue concerns the possibility that $|_E$ may be a non-convex set, in which case it may be optimal for the players to randomize between two pure-strategy equilibria rather than play the e¢cient symmetric equilibrium. The

following result, which characterizes $\pi_E$ when $\delta > \hat{\delta}$, establishes that this is not the case, and moreover shows that the efficient frontier of $\pi_E$ is linear with slope -1 near the 45°line, so in terms of joint payoffs, a degree of asymmetry does not matter. This part of the frontier consists of payoffs from sequences which satisfy the incentive constraints with equality (this is no longer true for efficient paths with sufficiently asymmetric payoffs).

**Proposition 6.1.** Assume that $\delta > \hat{\delta} = -\mu_1=\mu_2$: Then, $\pi_E$ is convex. Moreover, the efficient frontier of $\pi_E$ has the following form. There exist points $A = (\pi^0; \pi^{00})$, $B = (\pi^{00}; \pi^0)$; on the efficient frontier of $\pi_E$ with $\pi^0 > \pi^{00} > 0$ such that between A and B, $\pi_1$ and $\pi_2$ sum to a constant $S$ (i.e., the frontier of $\pi_E$ is linear between A and B with slope -1): For any point on the frontier below A or above B, the sum of utilities is strictly less than $S$:

**Proof.** See Appendix. ∎

The Proposition is illustrated in Figure 1 below,

Figure 1 in here

which shows the general shape of the frontier (although we have no results about the shape of the frontier to the left of B or below A, except that it must be described by a concave function). We can also say something about how the frontier shifts as $\delta$ changes:

**Proposition 6.2.** The segment of the efficient frontier between A and B is increasing in $\delta$ in the sense that both $\pi^0=\pi^{00}$ and $S$ are increasing in $\delta$; and converges to the ﬁrst-best frontier as $\delta ! 1$ (i.e., $\pi^{00}=\pi^0 ! 0$ and $S ! 2(\mu_1 + \mu_2)c^{\pi}$): As $\delta ! \hat{\delta} = -\mu_1=\mu_2$ from above, $A ! B$ and $S ! 0$:

**Proof.** See Appendix. ∎

Proposition 4 is illustrated in Figure 2 below, where the solid line represents the frontier at a lower $\delta$ and the dotted line the frontier at a higher value of $\delta$.

Figure 2 in here

Note that as $\pm \rightarrow 1$, the efficient frontier becomes linear everywhere with slope equal to minus one -1, i.e., it converges to the first-best efficient frontier. So, Proposition 6.2 generalizes Corollary 3.3 to the case of asymmetric paths, at least in the linear kinked case.

# 7. Sequential Moves

So far, we have assumed that players can move simultaneously. However, it may be that players can only move sequentially, e.g., Admati-Perry (1991), Gale (1997). In certain public good contribution games, the assumption made can affect the conclusions substantially. In the Admati-Perry model, where players move sequentially, a no contribution result holds when no player individually would want to complete the project, even though it might be jointly optimal to do so, but this result may disappear if the players can move simultaneously (see Marx and Matthews (1997) for a full discussion of this issue). By contrast, we shall find that in our model, equilibria in the two cases are closely related; indeed, the efficient symmetric equilibrium can "approximately" be implemented in the sequential move game.

Suppose w.l.o.g. that player 1 can move at even periods and player 2 at odd periods. Then, this move structure imposes the constraint that

$$c_{1;t} = c_{1;t_i 1}, \; t = 1; 3; 5::::\tag{7.1}$$
$$c_{2;t} = c_{2;t_i 1}, \; t = 2; 4; 6::::$$

Let the set of all paths that satisfy (7.1) be $C^{seq}$: To be an equilibrium in the sequential game, any path $fc_{1;t}; c_{2;t}g$ must satisfy the following incentive constraints. When player 1 moves at $t = 2; 4; ::::$; he prefers to raise his level of cooperation from $c_{t_i 2}$ to $c_t$ only if

$$\frac{\frac{1}{4}(c_{1;t_i 2}; c_{2;t_i 1})}{1 \; {}_i \; \pm} \cdot \; \frac{1}{4}(c_{1;t}; c_{2;t_i 1}) + \pm\frac{1}{4}(c_{1;t}; c_{2;t+1}) + ::::; \; t = 2; 4; 6:::\tag{7.2}$$

Similarly, when player 2 moves at $t = 3; 5::::$; he prefers to raise his level of cooperation from $c_{2;t_i 2}$ to $c_{2;t}$ only if

$$\frac{\frac{1}{4}(c_{2;t_i 2}; c_{1;t_i 1})}{1 \; {}_i \; \pm} \cdot \; \frac{1}{4}(c_{2;t}; c_{1;t_i 1}) + \pm\frac{1}{4}(c_{2;t}; c_{1;t+1}) + ::::; \; t = 3; 5; 7:::\tag{7.3}$$

When player 2 moves at period 1, (7.3) is modi…ed by the fact that 2 can revert to $c_0 = 0$, rather than $c_{i-1}$, but otherwise the incentive constraint is the same, i.e.,

$$w \frac{\pi(0;0)}{1-\delta} \cdot \pi(c_{2;1};0) + \delta\pi(c_{2;1};c_{1;2}) + \ldots : \qquad (7.4)$$

Let the set of paths in $C^{seq}$ that satisfy (7.2),(7.3) and (7.4) be $C_E^{seq} \subseteq C^{seq}$:

However, note that a path is in $C_E^{seq}$ if and only if it is an (asymmetric) equilibrium path satisfying (7.1) in the simultaneous move game studied above. This is because in the simultaneous move game, the incentive constraints in the periods where agents do not have to move are automatically satis…ed, as no agent likes to choose a higher $c_{i;t}$ than necessary (from $\pi$ decreasing in its …rst argument). So, $C_E^{seq}$ is simply that subset of $C_E$ also in $C^{seq}$, i.e.,

$$C_E^{seq} = C_E \setminus C^{seq}:$$

So, the set of feasible present-value payo¤s $\Pi_E^{seq}$ is the image of $C_E^{seq}$ in $\Re^2$ under the payo¤ function , and consequently

$$\Pi_E^{seq} \subseteq \Pi_E:$$

To say more than this, we shall go to the linear kinked case, in which case we have the following. De…ne $A := (\Pi^0; \Pi^{00})$ as in Proposition 6.1 above, and let $\hat{\Pi}$ be the present value payo¤ from the e¢cient symmetric path in the simultaneous move game, so that $S := (\hat{\Pi}; \hat{\Pi})$ is the equal utility point on the Pareto-frontier for that game.

**Proposition 7.1.** $\Pi_E^{seq}$ is convex. Also, A is in $\Pi_E^{seq}$; and for any …xed $\varepsilon > 0$, there is a $\delta(\varepsilon) < 1$; and a point $B = (\hat{\Pi}_1^{seq}; \hat{\Pi}_2^{seq}) \in \Pi_E^{seq}$ such that $\hat{\Pi}_i^{seq} > \hat{\Pi}_i - \varepsilon$; $i = 1;2$ for $\delta \geq \delta(\varepsilon)$: Consequently, as $\delta \to 1$, the Pareto frontier of $\Pi_E^{seq}$ is asymptotically linear between S and A.

**Proof.** See Appendix. ¤

This Proposition is illustrated in Figure 3 below. It shows that in the sequential move game, for low discounting, we can approximate "half" the linear part of the Pareto-frontier of the simultaneous move game, so sequential moves need not be a barrier to e¢ciency.

Figure 3 in here

# 8. Conclusions

This paper has studied a simple dynamic game where the level of cooperation chosen by each player in any period is irreversible. We have shown that irreversibility causes gradualism, i.e., any (subgame-perfect) sequence of actions involving partial cooperation cannot involve an immediate move to full cooperation, and we have re...ned and extended this basic insight in various ways. First, we showed that if payo¤s are di¤erentiable in actions, then (for a ...xed discount factor), the level of cooperation asymptotes to a limit strictly below full cooperation, and this limit value is easily characterized. For the case where payo¤s are linear up to some joint cooperation level, and constant or decreasing thereafter, the results are di¤erent — above some critical discount factor equilibrium cooperation can converge asymptotically to the fully e¢ cient level. Below this critical discount factor, no cooperation is possible.

Later sections of the paper then extend the basic model in several directions. First, we studied an "adjustment cost" model which is applicable to a variety of economic situations, and showed that it can be reformulated so that it is a special case of our base model. We then applied the adjustment cost model to study sequential public good contribution games and capacity reduction in a declining industry.

Other extensions were to allow for irreversibility, asymmetry, and sequential moves. However, in all these variants of the base case, we have continued to assume that the underlying model is symmetric, i.e., both players have the same payo¤s, given a permutation of their action variables. This is somewhat restrictive; in many situations where irreversibility arises naturally, e.g. Coasian bargaining without enforceable contracts but where actions are irreversible, payo¤s will be asymmetric. Another limitation of the model is that players only have a scalar action variable; in many applications, players have several action variables, as in, for example, capacity reduction games, where ...rms control both capacity and output. Extending the model in these directions is a project for the future.

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# A. Appendix

**Proof of Lemma 4.** Suppose to the contrary there exists a $\{c_t^0\}_{t=1}^1$ in $C_{SE}$ with $c_t^0 > b_t$ for some t. De…ne for all $t \geq 0$; $\epsilon_t = \max\{b_t; c_t^0\}$: It is clear from Assumption A1 and Lemma 2.1 (i) that

$$\frac{1}{4}(\epsilon_t; \epsilon_t) \geq \frac{1}{4}(b_t; b_t), \text{ all } t; \qquad (A.1)$$

with at least one strict inequality, so that $\{\epsilon_t\}_{t=1}^1$ gives both agents a higher payo¤ than $\{b_t\}_{t=1}^1$. So, if we can show that $\{\epsilon_t\}_{t=1}^1$ is an equilibrium sequence, this will contradict the assumed e¢ciency of $\{b_t\}_{t=1}^1$ and the result is then proved.

Say the sequences $\{b_t\}_{t=1}^1$; $\{c_t^0\}_{t=1}^1$ have a crossing point at $\tau$ if $c_{\tau-1}^0 \leq b_{\tau-1}$; $c_\tau^0 \geq b_\tau$ with at least one strict inequality, or $c_{\tau-1}^0 \geq b_{\tau-1}$; $c_\tau^0 \leq b_\tau$ with at least one strict inequality. Also, de…ne $S_t = \frac{1}{4}(c_t; c_t) + \pm\frac{1}{4}(c_{t+1}; c_{t+1}) + \cdots$; so that $S_t \geq S_t; S_\tau^0$ by (A.1).

There are then two possibilities at any time $\tau$: The …rst is that there is no crossing point at $\tau$. Then, either $(\epsilon_{\tau-1}; \epsilon_\tau) = (b_{\tau-1}; b_\tau)$ or $(\epsilon_{\tau-1}; \epsilon_\tau) = (c_{\tau-1}^0; c_\tau^0)$. Without loss of generality, assume the former. As $\{b_t\}_{t=1}^1$ is an equilibrium sequence, we have $\frac{1}{4}(b_{\tau-1}; b_\tau) = (1 - \pm) \cdot S_\tau$; so that $(\epsilon_{\tau-1}; \epsilon_\tau) = (b_{\tau-1}; b_\tau)$ and $S_\tau \geq S_\tau$ together imply $\frac{1}{4}(\epsilon_{\tau-1}; \epsilon_\tau) = (1 - \pm) \cdot S_\tau$; i.e., the $\tau-1$ constraint is satis…ed for $\{\epsilon_t\}_{t=1}^1$.

Now assume that $\{b_t\}_{t=1}^1$ and $\{c_t^0\}_{t=1}^1$ have a crossing point at $\tau$; and assume w.l.o.g. that

$$c_{\tau-1}^0 \leq b_{\tau-1}; c_\tau^0 \geq b_\tau: \qquad (A.2)$$

Then as $\{c_t^0\}_{t=1}^1$ is an equilibrium sequence, $\frac{1}{4}(c_{\tau-1}^0; c_\tau^0) = (1 - \pm) \cdot S_\tau^0$. Also, $S_\tau \geq S_\tau^0$ and from (A.2), $\epsilon_\tau = c_\tau^0$. Consequently,

$$\frac{\frac{1}{4}(c_{\tau-1}^0; \epsilon_\tau)}{1 - \pm} \cdot S_\tau: \qquad (A.3)$$

Finally, again from (A.2), $c_{\tau-1}^0 \leq b_{\tau-1} = \epsilon_{\tau-1}$: Using this fact, plus $\frac{1}{4}$ decreasing in its …rst argument, we have $\frac{1}{4}(\epsilon_{\tau-1}; \epsilon_\tau) \cdot \frac{1}{4}(c_{\tau-1}^0; \epsilon_\tau)$; so from (A.3) the $\tau-1$ constraint holds for $\{\epsilon_t\}_{t=1}^1$. Consequently all $\tau-1$ constraints hold for the sequence $\{\epsilon_t\}_{t=1}^1$, so it is an equilibrium sequence, as required. ¤

**Proof of Lemma 5.1.** (i) Take an e¢cient path $\{\epsilon_t\}_{t=1}^1$—such a sequence exists by a similar argument to that of Lemma 2—and de…ne $\tau \geq 1$ to be the …rst period such that $\epsilon_\tau > c^\pi$ (if such a period does not exist, then (i) holds immediately): De…ne a new sequence with $b_t := \epsilon_t$; for $t < \tau$; and $b_t := c^\pi$ for $t \geq \tau$: $\{b_t\}_{t=1}^1$ clearly yields as much utility as $\{\epsilon_t\}_{t=1}^1$ at every point, and it will be shown that it also satis…es (5.1) for all t: First, (5.1) holds at $\tau$ since $\Phi(\frac{1}{2}; \{\epsilon_{\tau-1}; \epsilon_\tau\}) > \Phi_\tau(\frac{1}{2}; \{b_{\tau-1}; b_\tau\})$ as $b_\tau < \epsilon_\tau$ while $b_{\tau-1} = \epsilon_{\tau-1}$ (and using $\frac{1}{4}$ increasing in its second argument); moreover the RHS of (5.1) is no smaller. Likewise, for $t' > \tau$; we have $\Phi(\frac{1}{2}; \{c_{t'-1}; c_{t'}\}) < \Phi(\frac{1}{2}; \{c_{\tau-1}; c_\tau\})$ since $b_{t'} = b_\tau$; and $b_{t'-1} > b_{\tau-1}$; while

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continuation path payoffs (RHS of (5.1)) are the same at $\tau$ and $t^0$: So (5.1) holds at $t^0$; it clearly holds at $t < \tau$ as the LHS is unchanged relative to the $\{a_t\}_{t=1}^1$ sequence while the RHS is no smaller. The proof of $\mathbf{b}_{t_i 1} \cdot \mathbf{b}_t$ is straightforward but tedious and is omitted. (ii) The argument is similar to the proof of Lemma 2.2. (iii) Assume the contrary, so there is an equilibrium sequence $\{c_t^0\}_{t=1}^1$ yielding a higher payoff than $\{b_t\}_{t=1}^1$; and both sequences lie below or equal to $c^\pi$: Hence the construction of Lemma 2.4 can be followed to create a new sequence $\{e_t\}_{t=1}^1$ which yields a higher overall payoff. That it satisfies (5.1) at each t follows from similar arguments. $\square$

Proof of Proposition 5.2. (a) Let $\mathbf{b}_t(1) = \mathbf{b}_t$ to ease notation. To prove part (i), it is sufficient to show that we can find $\hat{b}$ such that

$$\Phi(\tfrac12; \mathbf{b}_{t_i 1}; \mathbf{b}_t) < \Phi(1; \mathbf{b}_{t_i 1}; \mathbf{b}_t g); \quad t = 1; 2; \ldots; \quad 1 > \tfrac12 > \hat{b}: \tag{A.4}$$

For then, for $1 > \tfrac12 > \hat{b}$, $\{b_t g\}_{t=1}^1$ satisfies the incentive constraints (5.1).

(b) Fix t; then

$$\Phi_t(\tfrac12) \; _i \; \Phi_t(1) = \; _i \; \Phi_t^0(1)'' + \frac{1}{2}\Phi_t^{00}(1)''^2 + O(''^3); \tag{A.5}$$

where $'' := 1 _i \tfrac12$; and to ease notation, we set $\Phi_t(\tfrac12) := \Phi(\tfrac12; \{b_{t_i 1}; b_t g)$. Routine calculation gives:

$$\Phi_t^0(1) = A_t(1 + 2\pm + 3\pm^2 + 4\pm^3 + \ldots) \tag{A.6}$$

$$\Phi_t^{00}(1) = A_t(2\pm + 6\pm^2 + 12\pm^3 + \ldots) + B_t \tag{A.7}$$

where $A_t = \tfrac14_1 \mathbf{b}_{t_i 1} + \pm\tfrac14_2 \mathbf{b}_t$, and $B_t$ is the sum of terms involving $\tfrac14_{11}; \tfrac14_{22}; \tfrac14_{12}$, and where it is understood that all derivatives of $\tfrac14$ are evaluated at $(\mathbf{b}_{t_i 1}; \mathbf{b}_t)$. Also the series $1 + 2\pm + 3\pm^2 + 4\pm^3 + \ldots$ and $2\pm + 6\pm^2 + 12\pm^3 + \ldots$ both converge (to $s_1; s_2 > 0$ respectively). Useful properties of $A_t; B_t$; proved in (c) below, are: $A_t > 0$; $B_t < 0$, $\lim_{t! 1} A_t = 0$, $\lim_{t! 1} B_t < 0$:

Consequently, we can write

$$_i \; \Phi_t^0(1)'' + \frac{1}{2}\Phi_t^{00}(1)''^2 = (\tfrac14_1\mathbf{b}_{t_i 1} + \pm\tfrac14_2\mathbf{b}_t)(_i \; s_1'' + 0.5s_2''^2) + 0.5''^2 B_t: \tag{A.8}$$

Clearly there exists $''_t$ such that for $''$ satisfying $0 < '' < ''_t$, the RHS of (A.8) is negative. It follows from (A.5) that for $'' < ''_t$, $\Phi_t(\tfrac12) < \Phi_t(1)$.

(c) (Properties of $A_t; B_t$): First we show that $A_t > 0$: We have $\mathbf{b}_t \geq \mathbf{b}_{t_i 1}$, so (as $\tfrac14_2 > 0$) we only need show that

$$\tfrac14_1(\mathbf{b}_{t_i 1}; \mathbf{b}_t) + \pm\tfrac14_2(\mathbf{b}_{t_i 1}; \mathbf{b}_t) > 0: \tag{A.9}$$

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Now, we know from Section 3 that provided the maximum attainable level of cooperation $\mathbf{b} > 0$, then $\mathbf{b}_t < \mathbf{b}$ all $t$; and thus $\varphi'(\mathbf{b}_t) < \gamma_1(\mathbf{b}_t;\mathbf{b}_t)=\gamma_2(\mathbf{b}_t;\mathbf{b}_t) < \pm$, which implies

$$\gamma_1(\mathbf{b}_t;\mathbf{b}_t) + \pm\gamma_2(\mathbf{b}_t;\mathbf{b}_t) > 0: \tag{A.10}$$

Also, from the assumptions on $\gamma$ that $\gamma_{11} < 0$; $\gamma_{12} \cdot 0$; we have

$$\gamma_1(\mathbf{b}_{t-1};\mathbf{b}_t) \ge \gamma_1(\mathbf{b}_t;\mathbf{b}_t); \quad \gamma_2(\mathbf{b}_{t-1};\mathbf{b}_t) \ge \gamma_2(\mathbf{b}_t;\mathbf{b}_t): \tag{A.11}$$

Consequently, (A.9) follows from (A.10) and (A.11). Also note

$$\begin{aligned}
\lim_{t \to 1} A_t &= \gamma_1(\mathbf{b}_{t-1};\mathbf{b}_t)\mathbf{b}_{t-1} + \pm\gamma_2(\mathbf{b}_{t-1};\mathbf{b}_t)\mathbf{b}_t \\
&= [\gamma_1(\mathbf{b};\mathbf{b}) + \pm\gamma_2(\mathbf{b};\mathbf{b})]\mathbf{b} \\
&= 0
\end{aligned}$$

where the term in the square brackets is zero by de...nition of $\mathbf{b}$: The properties of $B_t$ follow from the fact that $B_t$ is the sum of terms involving $\gamma_{11}; \gamma_{22}; \gamma_{12}$ with coe¢cients bounded (in $t$) above zero.

(d) We now show that the sequence $\{\lambda_t\}_{t=1}^1 := \{1 - \tau_t\}_{t=1}^1$ can be chosen to be bounded below 1; this would imply (A.4) with $\bar{\lambda} := \sup \lambda_t < 1$. If such a sequence does not exist, then there must be a subsequence which w.l.o.g. we take to be $\{\lambda_t\}_{t=1}^1$ itself, converging to 1; i.e., $\lambda_t \to 1$ and

$$\Phi(\lambda_t;\mathbf{b}_{t-1};\mathbf{b}_t) \ge \Phi(1;\mathbf{b}_{t-1};\mathbf{b}_t); \qquad \text{all } t: \tag{A.12}$$

But now as $t \to 1$; $\mathbf{b}_t \to \mathbf{b}$, so from (A.5), we have

$$\begin{aligned}
\Phi(\tfrac{1}{2};\mathbf{b};\mathbf{b}) - \Phi(1;\mathbf{b};\mathbf{b}) &\to \lim_{t \to 1} f - \Phi'_t(1)'' + \frac{1}{2}\Phi'_t(1)''^2 g \\
&= \lim_{t \to 1} 0.5''^2 B_t = 0.5''^2\overline{B} < 0:
\end{aligned}$$

So, for some ...xed $\mu > 0$, there exists $\lambda_\mu < 1$ such that

$$\Phi(\tfrac{1}{2};\mathbf{b};\mathbf{b}) < \Phi(1;\mathbf{b};\mathbf{b}) - 3\mu; \quad 1 > \lambda > \lambda_\mu: \tag{A.13}$$

Also, as $t \to 1$; $\mathbf{b}_t \to \mathbf{b}$; and $\Phi_t(\lambda)$ is continuous in $\lambda$ and $\mathbf{b}_{t-1};\mathbf{b}_t$, there exists a $T_\mu$ such that for all $t \ge T_\mu$ :

$$\begin{aligned}
\Phi(\lambda;\mathbf{b}_{t-1};\mathbf{b}_t) &< \Phi(\lambda;\mathbf{b};\mathbf{b}) + \mu; \quad 1 > \lambda > \lambda_\mu; \\
\Phi(1;\mathbf{b};\mathbf{b}) &< \Phi(1;\mathbf{b}_{t-1};\mathbf{b}_t) + \mu: 
\end{aligned} \tag{A.14}$$

Combining (A.13) and (A.14), we get

$$\Phi(\lambda;\mathbf{b}_{t-1};\mathbf{b}_t) < \Phi(1;\mathbf{b}_{t-1};\mathbf{b}_t) - \mu; \quad 1 > \lambda > \lambda_\mu; \ t \ge T_\mu: \tag{A.15}$$

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But (A.12) and (A.15) are in contradiction.

(e) To prove part (ii) of the Proposition, let

$$
\epsilon_t = \begin{cases} \mathbf{b}_t & t < T_\mu \\ \mathbf{b}_t + \acute{}\ & t \geq T_\mu \end{cases}^{\frac{1}{2}}
$$

Also, choose $\acute{} < c^\pi - \mathbf{b}$ small enough so that (by continuity)

$$\mathbb{C}\,(\tfrac{1}{2};\epsilon_{t-1};\epsilon_t) < \mathbb{C}\,(\tfrac{1}{2};\mathbf{b}_{t-1};\mathbf{b}_t) + \mu{=}2;\ 1 > \tfrac{1}{2} > \tfrac{1}{2}_\mu;\ t \geq T_\mu: \tag{A.16}$$

We show that $f\epsilon_t g_{t=1}^1$ is an equilibrium symmetric path in the $\tfrac{1}{2}-$ reversible game, if $1 > \tfrac{1}{2} > \max f \sup \tfrac{1}{2}_t; \tfrac{1}{2}_\mu g$. To see this, note ...rst that $\epsilon_t < c^\pi$; so for any t the continuation payo¤ from $f\epsilon_t g_{t=1}^1$ is strictly greater than that from $f\mathbf{b}_t g_{t=1}^1$: Hence, it su¢ces to show that the deviation payo¤ in the $\tfrac{1}{2}-$ reversible game from $f\epsilon_t g_{t=1}^1$ is no higher than the deviation payo¤ from $f\mathbf{b}_t g_{t=1}^1$ in the irreversible case. But from (A.15) and (A.16), we have

$$\mathbb{C}\,(\tfrac{1}{2};\epsilon_{t-1};\epsilon_t) < \mathbb{C}\,(1;\mathbf{b}_{t-1};\mathbf{b}_t) - \mu{=}2;\ 1 > \tfrac{1}{2} > \tfrac{1}{2}_\mu;\ t \geq T_\mu$$

as required; provided $\tfrac{1}{2} > \mathbf{b}\ \acute{}\ \sup \tfrac{1}{2}_t;$ (A.4) ensures (from (a)-(d) above) that (5.1) holds for $t < T_\mu$: Thus setting $\bar{\tfrac{1}{2}} = \max f \sup \tfrac{1}{2}_t; \tfrac{1}{2}_\mu g$ implies that (5.1) holds for all $1 > \tfrac{1}{2} > \bar{\tfrac{1}{2}};\ t \geq 1$: Then from Lemma 5.1 (iii), $\mathbf{b}_1\,(\tfrac{1}{2}) \geq \mathbf{b}_1\,(1) + ":$

(f) To prove part (iii), it follows immediately from the construction of $f\epsilon_t g_{t=1}^1$ that

$$\dagger := (1 - \pm) \sum_{t=1}^{X} \pm^{t-1} \tfrac{1}{4}(\epsilon_t;\epsilon_t) > \hat{\,}(1)$$

and as $f\epsilon_t g_{t=1}^1$ is an equilibrium (but not necessarily the e¢cient) path in the $\tfrac{1}{2}-$ reversible game, $\hat{\,}(\tfrac{1}{2}) \geq \dagger$ and so the result is proved. ¤

Proof of Proposition 5.3. Let $\tfrac{1}{2} = 1;$ and suppose $fc_t g_{t=1}^1$ is an e¢cient path; assuming $a < 1;$ this path is increasing by earlier arguments. The derivative of $\mathbb{C}_t(\tfrac{1}{2}; fc_t g_{t=1}^1)\ \acute{}\ (\tfrac{1}{4}_1 \tfrac{1}{2} c_{t-1} + \tfrac{1}{4}_2 c_t){=}(1 - \tfrac{1}{2} \pm)$ with respect to $\tfrac{1}{2}$ has the sign of $c_t - a c_{t-1};$ which is positive for all $t \geq 1$ as $a < 1$ and $c_t > c_{t-1} \geq 0:$ Hence for any $\mathbf{b} \in [0;1),$ $fc_t g_{t=1}^1$ remains an equilibrium path as the deviation payo¤ $\mathbb{C}_t(\mathbf{b}; fc_t g_{t=1}^1)$ is smaller than at $\tfrac{1}{2} = 1,$ while the continuation payo¤ is unchanged. By Lemma 5.1(i) and (iii), there exists a non-decreasing e¢cient path for $\mathbf{b} < 1;$ say $f\mathbf{b}_t g_{t=1}^1;$ which lies no lower than $fc_t g_{t=1}^1$ and no higher than $c^\pi$ at each point. Next, the above argument can be repeated for any $\tfrac{1}{2}^0 < \mathbf{b} < 1;$ so that at $\tfrac{1}{2}^0;$ $f\mathbf{b}_t g_{t=1}^1$ is an equilibrium path. Moreover, the incentive constraint at each t is strictly looser, so that by Lemma 5.1(ii) if the ...rst-best is not attainable at $\tfrac{1}{2},$ i.e., if $\mathbf{b}_t < c^\pi$ for some t, $\mathbf{b}_t$ is not part of an e¢cient equilibrium path for $\tfrac{1}{2}^0:$ The conclusion is then that at $\tfrac{1}{2}^0;$ $f\mathbf{b}_t g_{t=1}^1$ is equilibrium but not e¢cient, i.e., there is an equilibrium path yielding a higher payo¤ than $f\mathbf{b}_t g_{t=1}^1:$ To prove that $c^\pi$ is attained in ...nite time, consider the path generated by

(5.2) for some choice of $c_1$: Note that $(\frac{1}{2}^{t_i 1} + \frac{1}{2}^{t_i 2}a + \ldots + \frac{1}{2}a^{t_i 2} + a^{t_i 1})$ attains a maximum at some $t^{\tt x} \geq 1$; and declines to zero. Choose $c_1 = e_1$ so that $e_{t^{\tt x}} = c^{\tt x}$: If (5.2) is followed for all $t$; the same argument as in Lemma 2.4 establishes that the incentive constraint holds for all $t$ as $\lim_{t! 1} e_t = 0$ ($< 1$): (It does not matter if this path violates $e_t \geq \frac{1}{2}e_{t_i 1}$ beyond $t^{\tt x}$:) Now change the path by setting $e_t = c^{\tt x}$ for $t > t^{\tt x}$: Continuation payo¤s are increased at each date. Deviation payo¤s are the same at each date up to $t^{\tt x}$; and since the incentive constraint is thus satis…ed at $t^{\tt x}$ it must also be satis…ed at all $t > t^{\tt x}$: Thus this path satis…es all incentive constraints and $c^{\tt x}$ is attained in …nite time. By Lemma 5.1(iii) there is an e¢cient path that attains $c^{\tt x}$ by $t^{\tt x}$ or earlier. (ii) If $a \geq 1$; then consider the incentive condition for a stationary path at $c$:

$$\frac{\frac{1}{4}_1 \frac{1}{2}c + \frac{1}{4}_2 c}{1_i \frac{1}{2}\pm} \cdot \frac{\frac{1}{4}_1 c + \frac{1}{4}_2 c}{1_i \pm}: \tag{A.17}$$

Rearranging, this is equivalent to $a \cdot 1$: Hence if $a > 1$; if $c^{\tt x}$ is attained, the incentive constraint is violated at $c^{\tt x}$ (likewise if a higher e¢cient level is attained, should one exist); if $c_t < c^{\tt x}$ for all $t$, then the path must satisfy (5.2) for all $t$; implying $c_t ! 1$ if $c_1 > 0$; ; a contradiction; hence $c_1 = 0$; so $c_t = 0$ all $t$. If $a = 1$; (A.17) holds with equality; if $c^{\tt x}$ is attained at $t$; the incentive constraint at $t$ is stricter than (A.17), and so is violated; hence $c_t < c^{\tt x}$ all $t$; in which case (5.2) applies, and setting $c_1 = (1_i \frac{1}{2})c^{\tt x}$ implies that $\lim_{t! 1} c_t = c^{\tt x}$; and because the limit is …nite, all incentive constraints are satis…ed (as argued earlier).¤

**Proof of Proposition 6.1.** First, we show that $\rceil_E$ is a convex set. First; the constraints in (6.1) are linear. Consequently, if $fc^0_{1;t}; c^0_{2;t}g^1_{t=1}$ and $fc^{00}_{1;t}; c^{00}_{2;t}g^1_{t=1}$ satisfy (6.1), a convex combination of the two must also satisfy (6.1) and so $C_E$ is a convex set. Also, adapting Lemma 2.1, any sequence in $C_E$ must have $c_{1;t} + c_{2;t} < 2c^{\tt x}$, all $i$; $t$, so payo¤s are linear in any path in $C_E$: It follows immediately that $\rceil_E$ is a convex set also.

Let $C_{EE} \mu C_E$ be the set of all paths $fc_{1;t}; c_{2;t}g^1_{t=1}$ which satisfy the incentive constraints (6.1) with equality at each $t \geq 1$; and $\rceil_{EE} \mu \rceil_E$ the corresponding set of payo¤s. Straightforward manipulation implies that these paths can be written as a system of two linked …rst-order di¤erence equations in di¤erences $¢c_{i;t} = c_{i;t i} c_{i;t_i 1}$;

$$¢c_{1;t} = a¢c_{2;t_i 1} \tag{A.18}$$

$$¢c_{2;t} = a¢c_{1;t_i 1} \tag{A.19}$$

where $a = \frac{i \frac{1}{4}_1}{\frac{1}{4}_2 \pm}$ as before. As $\pm > \hat{\pm}$; it follows that $a < 1$: Also, note that the initial conditions

$$¢c_{i;1} = c_{i;1 i} c_{i;0} = c_{i;1}, \quad i = 1; 2$$

can be set freely. Routine manipulation of the system (A.18), (A.19) gives the solutions

$$c_{i;t} = \begin{cases} \frac{1}{1_i a^2}[c_{i;1}(1_i a^{t+1}) + ac_{j;1}(1_i a^{t_i 1})]; & t \text{ odd} \\ \frac{1}{1_i a^2}[c_{i;1}(1_i a^t) + ac_{j;1}(1_i a^t)]; & t \text{ even} \end{cases} ; \quad i; j = 1; 2; j \neq i: \tag{A.20}$$

Taking limits in (A.20), we get two equations that give, as $a < 1$, the limit values of $c_{1;t}, c_{2;t}$ as functions of the initial values:

$$\lim_{t \to \infty} c_{1;t} = c_{1;\infty} = \frac{1}{1 - a^2}[c_{1;1} + ac_{2;1}];$$

$$\lim_{t \to \infty} c_{2;t} = c_{2;\infty} = \frac{1}{1 - a^2}[c_{2;1} + ac_{1;1}].$$

Inverting and solving, we get

$$c_{1;1} = c_{1;\infty} - ac_{2;\infty}; \quad c_{2;1} = c_{2;\infty} - ac_{1;\infty}. \tag{A.21}$$

Note that we can think of $c_{1;1}$ and $c_{2;1}$ as being determined by $c_{1;\infty}$ and $c_{2;\infty}$ where the latter can be freely chosen subject to the constraint that $c_{1;\infty} + c_{2;\infty} \leq 2c^{\pi}$ and that $c_{i;1} \geq 0$, $i = 1, 2$. The latter requires

$$\frac{c_{2;\infty}}{a} \geq c_{1;\infty} \geq ac_{2;\infty}. \tag{A.22}$$

$C_{EE}$ is characterized by sequences satisfying (A.20) and (A.22) since convergent sequences satisfying (A.18) and (A.19) also satisfy (6.1) with equality as in Lemma 2.4.

Substituting (A.20) back in the payoffs gives, after some rearrangement, for $i, j = 1, 2$, $j \neq i$,

$$\pi_i = (1 - \pm)\sum_{t=1}^{\infty} \pm^{t-1}(\mu_1 c_{i;t} + \mu_2 c_{j;t})$$

$$= \frac{1}{1 - a^2}[\mu_1(c_{i;1} + ac_{j;1}) + \mu_2(c_{j;1} + ac_{i;1})]$$

$$+ \frac{(1 - \pm)}{(1 - a^2)(1 - a^2\pm^2)}\mu_1\left[a(ac_{i;1} + c_{j;1}) + \pm a^2(c_{i;1} + ac_{j;1})\right]$$

$$+ \frac{(1 - \pm)}{(1 - a^2)(1 - a^2\pm^2)}\mu_2\left[a(ac_{j;1} + c_{i;1}) + \pm a^2(c_{j;1} + ac_{i;1})\right].$$

Now, from (A.21), we have

$$c_{i;1} + ac_{j;1} = (1 - a^2)c_{i;\infty}. \tag{A.23}$$

So, we get, after some manipulation,

$$\pi_i = \left[1 - \frac{(1 - \pm)(a + a^2\pm)}{(1 - a^2\pm^2)}\right](\mu_1 c_{i;\infty} + \mu_2 c_{j;\infty}); \quad i = 1, 2$$

and so

$$\pi_1 + \pi_2 = \acute{A}(\pm)(\mu_1 + \mu_2)(c_{1;\infty} + c_{2;\infty}); \tag{A.24}$$

where $\acute{A}(\pm) := \left[1 - \frac{(1-\pm)(a+a^2\pm)}{(1-a^2\pm^2)}\right]$.

So as long as $c_{1;1} + c_{2;1} = 2c^{\pi}$, $\pi_1 + \pi_2 = A(\delta)(\gamma_1 + \gamma_2)2c^{\pi}$, no matter how the sum $c_{1;1} + c_{2;1}$ is distributed. This says that the frontier is linear between two endpoints defined by the restrictions (A.22). Let A be one endpoint, defined by the condition that $c_{1;1} = ac_{2;1}$, and B the other endpoint, defined by $c_{2;1} = ac_{1;1}$ (B is symmetric to A) Combining this with $c_{1;1} + c_{2;1} = 2c^{\pi}$ implies that A is generated by the path with endpoints

$$c_{1;1} = \frac{2ac^{\pi}}{1+a}; \ c_{2;1} = \frac{2c^{\pi}}{1+a};$$

and therefore with payoffs $(\pi'; \pi'')$ where

$$\pi' = \frac{2c^{\pi}}{1+a}\left[1 - \frac{(1-\delta)(a+a^2\delta)}{(1-a^2\delta^2)}\right][\gamma_1 a + \gamma_2];$$

$$\pi'' = \frac{2c^{\pi}}{1+a}\left[1 - \frac{(1-\delta)(a+a^2\delta)}{(1-a^2\delta^2)}\right][\gamma_1 + a\gamma_2]:$$

So,

$$\frac{\pi'}{\pi''} = \frac{\gamma_1 a(\delta) + \gamma_2}{\gamma_1 + a(\delta)\gamma_2}: \tag{A.25}$$

Now, it is easily checked that $\pi'; \pi'' > 0$ and that the RHS of (A.25) is strictly greater than 1, so $\pi' > \pi'' > 0$ as claimed.

To complete the proof, we need to show that points A and B lie on the frontier of $\pi_E$; the convexity of $\pi_E$ then implies that the whole of line segment AB lies on this frontier. First, note that the point S where the line segment AB crosses the $45^{\circ}$ line is generated by the symmetric path

$$c_t^{\pi} = 0.5c_{1;t} + 0.5c_{2;t};$$

where $\{c_{1;t}; c_{2;t}\}_{t=1}^1$ is the path supporting A; so every incentive constraint holds with equality for $\{c_t^{\pi}\}_{t=1}^1$. But then $\{c_t^{\pi}\}_{t=1}^1$ is the symmetric efficient path characterized in Sections 2 and 3. So, S must be on the frontier since otherwise there is an asymmetric path which Pareto-dominates S; and by symmetry another path with the player indices switched which also Pareto dominates S; a convex combination of these two paths is a symmetric path which Pareto dominates S; a contradiction of the definition of S:

Suppose finally that points A; B are not on the frontier of $\pi_E$. Then, there must be points C; D where C (resp. D) Pareto-dominates A (resp. B) which are on the frontier of $\pi_E$: But if S; C; D are all on the frontier of $\pi_E$, it must be non-convex, contrary to the result already established. $\blacksquare$

**Proof of Proposition 6.2.** From the proof of Proposition 6.1, we have

$$\frac{\pi'}{\pi''} = \frac{\gamma_1 a(\delta) + \gamma_2}{\gamma_1 + a(\delta)\gamma_2}: \tag{A.26}$$

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As a is decreasing in $\pm$, and the right-hand side of (A.26) is decreasing in a, $\pi' = \pi''$ is increasing in $\pm$. Moreover, as $\pm \to 1$, $\pi' = \pi'' \to 0$; and as $\pm \to \hat{\pm}_+$, $\pi' = \pi'' \to 1$, as required. Likewise from (A.24) in the proof of Proposition 6.1, on the line segment AB,

$$\S = \pi_1 + \pi_2 = \acute{A}(\pm)(\mu_1 + \mu_2)2c^{\alpha}$$

where $\acute{A}(\pm) := \left[1 - \frac{(1-\pm)(a+a^2\pm)}{(1-a^2\pm^2)}\right]$. Rearrangement gives $\acute{A}(\pm) = \left[1 - \frac{\hat{\pm}(1-\pm)}{\pm(1-\hat{\pm}^2)}\right]$. It is then clear that $\acute{A}(\hat{\pm}) = 0$; $\acute{A}(1) = 1$; and $\acute{A}'(\pm) > 0$; $\pm \in (\hat{\pm}, 1)$; and so $\S$ has the desired properties on the line segment AB: $\square$

**Proof of Proposition 7.1.** To prove convexity of $\pi_E^{seq}$, note that since $C_E$; $C^{seq}$ are both convex, so $C_E^{seq} = C_E \setminus C^{seq}$ is also convex. Consequently, $\pi_E^{seq}$ is also convex, by linearity of payoffs.

To prove A in $\pi_E^{seq}$, we proceed as follows. Point A is generated by a path described in (A.20) with $c_{1;1} = 0$. All we have to do is show that this path is in $C^{seq}$ as this path is already in $C_E$ by construction. Now setting $c_{1;1} = 0$ in (A.20), we see that the path generating A satisfies:

$$c_{1;t}^A = \begin{cases} \frac{1}{2}\frac{1}{1-a^2}[ac_{2;1}(1-a^{t-1})]; & t \text{ odd} \\ \frac{1}{1-a^2}[ac_{2;1}(1-a^t)]; & t \text{ even} \end{cases}$$

$$c_{2;t}^A = \begin{cases} \frac{1}{2}\frac{1}{1-a^2}[c_{2;1}(1-a^{t+1})] & t \text{ odd} \\ \frac{1}{1-a^2}[c_{2;1}(1-a^t)]; & t \text{ even} \end{cases}$$

So, by inspection, $\{c_{1;t}^A; c_{2;t}^A\}_{t=1}^{1}$ has the property that player 1 only changes her level of cooperation in even periods, and player 2 in odd periods.

Next, let $\{\mathbf{b}_t\}_{t=1}^{1}$ be the (unique) symmetric efficient path in the simultaneous move game. Now define the asymmetric path $\{\mathbf{b}_{1;t}; \mathbf{b}_{2;t}\}_{t=1}^{1}$ in $C^{seq}$ as follows:

$$\mathbf{b}_{1;t} = \mathbf{b}_{1;t+1} = \mathbf{b}_t; \quad t = 0; 2; 4; 6:::;$$

$$\mathbf{b}_{2;t} = \mathbf{b}_{2;t+1} = \mathbf{b}_t; \quad t = 1; 3; 5:::$$

This is simply the path where an agent whose turn it is to move at t chooses $\mathbf{b}_t$. Next, we show that $\{\mathbf{b}_{1;t}; \mathbf{b}_{2;t}\}_{t=1}^{1}$ is incentive-compatible, i.e., in $C_E^{seq}$ in the sequential move game. Define as before $\phi_t := \mathbf{b}_t - \mathbf{b}_{t-1}$; and recall $\phi_t = a\phi_{t-1}$ on the efficient path. For the player who moves at $t \geq 2$; and writing $\phi$ for $\phi_{t-1}$; the constraints (7.2) and (7.3) can be written as:

$$\frac{\mu_1 c_{t-2} + \mu_2(c_{t-1} + \phi)}{1 - \pm} \cdot \mu_1(c_{t-2} + \phi + a\phi) + \mu_2(c_{t-1} + \phi) \qquad (A.27)$$

$$+ \pm(\mu_1(c_{t-2} + \phi + a\phi) + \mu_2(c_{t-1} + \phi + a\phi + a^2\phi))$$

$$+ \pm^2(\mu_1(c_{t-2} + \phi + ::: + a^3\phi) + \mu_2(c_{t-1} + \phi + a\phi + a^2\phi)) + :::$$

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or

$$\frac{\text{¼}_2 \text{¢}}{1 \; \text{¡} \; \pm} \cdot \frac{(1 + a)\text{¼}_1\text{¢} + (1 \; \text{¡} \; \pm^2 a^2 + \pm a + \pm a^2)\text{¼}_2\text{¢}}{(1 \; \text{¡} \; \pm)(1 \; \text{¡} \; \pm^2 a^2)};$$

which holds with equality as $a = \text{¡}\,\text{¼}_1 = (\pm\text{¼}_2)$. Thus $f\mathbf{b}_{1;t}; \mathbf{b}_{2;t}g_{t=1}^{1}$ satis...es equilibrium conditions from $t = 2$ onwards; at $t = 1$ the constraint would hold with equality if player 2's inherited $c$ was $\text{¡}\,\text{¢}_1 = a$; since it is higher, the constraint will be slack (as $\text{¼}_1 < 0$):

The payo¤s from the path $f\mathbf{b}_{1;t}; \mathbf{b}_{2;t}g$ are;

$$\Uparrow_1^{\text{seq}} = (1 \; \text{¡} \; \pm)f[\text{¼}_2\mathbf{b}_1] + \pm[\text{¼}_1\mathbf{b}_2 + \text{¼}_2\mathbf{b}_1] + \pm^2[\text{¼}_1\mathbf{b}_2 + \text{¼}_2\mathbf{b}_3] + \cdots$$

$$\Uparrow_2^{\text{seq}} = (1 \; \text{¡} \; \pm)f[\text{¼}_1\mathbf{b}_1] + \pm[\text{¼}_1\mathbf{b}_1 + \text{¼}_2\mathbf{b}_2] + \pm^2[\text{¼}_1\mathbf{b}_3 + \text{¼}_2\mathbf{b}_2] + \cdots$$

Now since the payo¤s from the e¢cient symmetric path in the simultaneous move game are

$$\Uparrow = (1 \; \text{¡} \; \pm)f[\text{¼}_1\mathbf{b}_1 + \text{¼}_2\mathbf{b}_1] + \pm[\text{¼}_1\mathbf{b}_2 + \text{¼}_2\mathbf{b}_2] + \pm^2[\text{¼}_1\mathbf{b}_3 + \text{¼}_2\mathbf{b}_3] + \cdots;$$

we get

$$
\begin{aligned}
\Uparrow \; \text{¡} \; \Uparrow_1^{\text{seq}} &= (1 \; \text{¡} \; \pm)f\text{¼}_2\mathbf{b}_1 + \pm\text{¼}_1(\mathbf{b}_2 \; \text{¡} \; \mathbf{b}_1) + \pm^2\text{¼}_2(\mathbf{b}_3 \; \text{¡} \; \mathbf{b}_2) + \pm^3\text{¼}_1(\mathbf{b}_4 \; \text{¡} \; \mathbf{b}_3) + \cdots g \\
&= (1 \; \text{¡} \; \pm)\mathbf{b}_1 f\text{¼}_2\mathbf{b}_1 + \pm\text{¼}_1 a\mathbf{b}_1 + \pm^2\text{¼}_2 a^2\mathbf{b}_1 + \pm^3\text{¼}_1 a^3\mathbf{b}_1\cdots g \\
&= (1 \; \text{¡} \; \pm)\mathbf{b}_1 \; \text{¼}_2(1 + \pm^2 a^2 + \pm^4 a^4 + \cdots) + \pm a\text{¼}_1(1 + \pm^2 a^2 + \pm^4 a^4 + \cdots)^{\text{¤}} \\
&= \frac{(1 \; \text{¡} \; \pm)\mathbf{b}_1}{1 \; \text{¡} \; \pm^2 a^2}[\text{¼}_2 + \pm a\text{¼}_1] \\
&< (1 \; \text{¡} \; \pm)\frac{\mathbf{b}_1\text{¼}_2}{1 \; \text{¡} \; (\text{¼}_1 = \text{¼}_2)^2}
\end{aligned}
$$

So, rearranging, $\Uparrow \; \text{¡} \; (1 \; \text{¡} \; \pm)\mu < \Uparrow_1^{\text{seq}}$, $\mu > 0$. Consequently, for any $" > 0$, $\Uparrow \; \text{¡} \; " < \Uparrow_1^{\text{seq}}$ for all $\pm \; \text{¸} \; \pm(") = 1 \; \text{¡} \; "=\mu$; as required. (A similar argument applies for $i = 2$). $\text{¤}$