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Abstract: We introduce the framework of noncooperative pregames and demonstrate that for all games with sufficiently many players, there exists approximate (ε) Nash equilibria in pure strategies. Moreover, an equilibrium can be selected with the property that most players choose the same strategies as all other players with similar attributes. More precisely, there is an integer K, depending on ε but not on the number of players so that any sufficiently large society can be partitioned into fewer than K groups, or cultures, consisting of similar players, and all players in the same group play the same pure strategy. In ongoing research we are extending the model to cover a broader class of situations, including incomplete information.

Although this research was initated some time ago this is the first complete "published" version. We would be grateful for any comments that might help us improve the paper.

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1 Introduction: Learning from similar individuals.

A society or culture is a group of individuals who have commonalities of language, social and behavioral norms, and customs. Social learning consists, at least in part, in learning the norms and behavior patterns of the society into which one is born and in those other societies which one may join – our professional associations, our workplace, and our community, for example. Social learning may also include learning a set of skills from others that will enable us to "fit into the society." The society in question may be broad as "Western civilization" or Canada, or as small as the Econometric Society. If most people observe "similar" people and learn by mimicing other individuals, then a stable society depends on the existence of an equilibrium where most individuals who are similar choose the same strategies. If most individuals learn from and mimic similar individuals, then the existence of such equilibria is important; indeed, it is fundamental to the social sciences.

To ask whether equilibria where most players choose the same strategies as similar players exist, we must first have an appropriate model. One of the main contributions of the current paper is the introduction of a noncooperative counterpart to the pregame framework of cooperative game theory.[1] In cooperative game theory this framework has lead to a number of results, especially results showing that large games with small effective groups resemble, or *are,* competitive economies. It appears that our framework of noncooperative games may be equally useful. In this paper, we demonstrate, for all games with sufficiently many players, existence of ε-equilibria in pure strategies and, with a more restrictive space of player types, existence of ε-equilibria in pure strategies where almost all players (that is, all except at most some bounded and finite number) choose the same strategy as all sufficiently similar players.

As in the cooperative pregame framework, we take as given a set of attributes of players; here, these attributes index payoff functions. We require two anonymity assumptions. The first is that payoffs do not depend on the identities of other players, only their attributes and, of course, the strategies they choose. The second is that in when there are many (but still finite numbers) of players, then the actions of a small subset of players do not

[1] See, for example, Wooders (1983,1994) and Wooders and Zame (1984).

significantly affect payoffs of members of the complementary player set. We require also some continuity conditions and demonstrate two main results:

Theorem 1: Existence. Given $\varepsilon > 0$ there exists an integer $\eta(\varepsilon)$ with the property that every game with at least $\eta(\varepsilon)$ players has an ε-equilibrium in pure strategies.

Theorem 2: Social conformity. When the set of player attributes can be parameterized by the interval $\ = [0, 1]$ there is an integer L such that the ε-equilibrium can be chosen so that for some partition of into fewer than L intervals, $\{\ ^{\ell}\}_{\ell=1}^{L}$, all players with attributes represented by points in the same interval $\ ^{\ell}$ play the same strategy.

To compare our noncooperative pregame framework to the cooperative pregame framework, it is important to note a major and significant difference. In the cooperative framework, the payoff to a coalition is fixed and independent of the society in which that coalition is embedded. Although this is possible within the current noncooperative framework, it is not built into the model and thus may or may not hold. Noncooperative games derived from a (noncooperative) pregame are parameterized by the numbers of players of each type in the player set and may vary considerably depending on the attributes of the players actually represented in the society. For example, there may be little relationship between derived games where all players have the same attribute, male for example, and games where some players have a different attribute, female, for example. Moreover, even in the case where all players are identical, there is no necessary relationship between a game with n players and another with $n + 1$ players. That said, however, it should be noted that asymptotically, only the distribution of players' attributes matters, that is, the games become anonymous. Just to be sure this is clear, in games with many players, the percentages of males and females is still relevant, but whether a male is called i or j is irrelevant and a few males or females, more or less, is of no great consequence.

One interesting similarity between the two frameworks is that, in the cooperative pregame framework, the condition of *small group effectiveness* plays an important role, cf. Wooders (1994).[2] This condition dictates that

[2] A strong form of this condition, a sort of *strict* small group effectiveness was originally introduced in Wooders (1979) and earlier versions of that paper. For our pruposes here, this form of small group effectiveness is not useful. Neither is the 'boundedness of marginal contributions' of Wooders and Zame (1984).

all or almost all gains to collective activities can be realized by groups of players bounded in size. An *equivalent* condition is that *small groups are negligible*: in large cooperative games derived from pregames, small groups are effective if and only if small groups cannot have significant effects on aggregate per capita payoff (Wooders 1993). The main substantive condition of this paper can be interpreted as the negligibility of small groups of players; that is, the effects of the actions of any small set of players on the complementary set of players become negligible in games with many players. For cooperative pregames with side payments, the condition of small group negligibility implies that large games are market games, as defined by Shapley and Shubik (1969). The full implications of the condition for noncooperative pregames have not been fully explored but we expect there to be many.

Our first result is actually a "purification result," showing that, for all sufficiently large games, every mixed strategy equilibrium generates a pure strategy ε-equilibrium. Our result differs from purification results in the literature in that prior papers all have a continuum of players (cf. Schmeidler 1973, Mas-Colell 1984, Khan 1989, Pascoa 1993,1998, Khan et al. 1997, Araujo and Pascoa 2000). With a continuum of players, small group negligibility is *built into* the framework and thus does not appear as a separate assumption. Of course for a number of these results, it is easy to see that one could consider a sequence of large finite games with player distribution converging to the distribution of player types in the given continuum, and from the results for the continuum, establish existence of ε-equilibrium in pure strategies for all sufficiently large games in the sequence. Our results differ in that we are not restricted to one limiting distribution of player types; our results hold for all sufficiently large games derived from a noncooperative pregame. In particular, our results allow for the possibility that there are player types who appear in arbitrarily small percentages in large finite games. An important part of our work is defining the model of noncooperative pregames and establishing that a set of conditions, especially small group negligibility, that allow us to obtain our purification result.

2 The Model. Noncooperative Pregames.

We first introduce the concept of a society, then strategies and the set of 'weight functions' derived from the possible strategies chosen by the members of the society. We conclude by introducing the game corresponding to a society.

2.1 Societies.

We assume a compact metric space of player types . Let N be a finite set and let α be a mapping from N to , called an *attribute function*. The pair (N, α) is a *society*.

Let Z_+ denote the set of non-negative integers. The *profile of a society* (N, α) is a function $\rho(N, \alpha) : \quad \to Z_+$ given by

$$\rho(N, \alpha)(\omega) = \left|\alpha^{-1}(\omega) \cap N\right|$$

Thus, the profile of a player set tells us the number of players with each attribute in the set. Let *support(*$\rho(N, \alpha)$) denote the support of the function $\rho(N, \alpha)$, that is,

$$\textit{support}\,(\rho(N, \alpha)) = \{\omega \in \quad : \rho(N, \alpha)(\omega) \neq 0\}.$$

Let $P(\quad)$ denote the set of all functions from to Z_+^T with finite support. Note that for each possible society (N, α) the profile of N is in $P(\quad)$. Note also that the sum of profiles (defined pointwise) is also a profile.

Before introducing the game corresponding to any society, we require some preliminary concepts.

Let $S = \{s_1, ..., s_K\}$ be a finite set of pure strategies. Let $\Delta(S)$ denote the set of mixed strategies. In each game $\Gamma(N, \alpha)$, each player will have the strategy choice set $\Delta(S)$. The support of a mixed strategy σ_i is denoted by *support*(σ_i), where "support" is defined as above. A mixed strategy is called *pure* if it puts unit weight on a single pure strategy.

A *strategy vector* is given by $\sigma = (\sigma_1, ..., \sigma_{|N|}) \in \times_{i \in N} \Delta(S)$ where σ_i denotes the strategy of player i and σ_{ik} denotes the probability with which player i plays pure strategy s_k. We denote the set of all strategy vectors by Σ. A strategy vector σ is called *degenerate* if for each i, for some k, $\sigma_{ik} = 1$; that is, each player's strategy is a pure strategy.

Given an attribute function α (or a profile $\rho(N, \alpha)$) we define a *weight function* $w(\cdot, \cdot; \alpha)$ (or $w(\cdot, \cdot; \rho(N, \alpha))$) as a mapping from $\quad \times S$ into $\mathcal{R}_+$ satisfying the conditions that

$$\sum_k w(\omega, s_k; \alpha) = \rho(N, \alpha)(\omega).$$

Thus, given an attribute function α, a weight function is an assignment of a non-negative real number to each attribute-strategy pair (ω, s) so that the sum, over strategies, of the weights attached to the pairs (ω, s) equals the number of players with that attribute. It follows that

$$\sum_k \sum_{\omega \in support(\rho(N,\alpha))} w(\omega, s_k; \alpha) = |N| \, .$$

It is convenient to also define weight functions relative to strategy vectors. Given an attribute function α and a strategy vector σ, define a *weight function* $w(\cdot,\cdot;\alpha,\sigma)$ *relative to* α *and* σ by:

$$w(\omega, s_k; \alpha, \sigma) = \sum_{i \in N : \alpha(i) = \omega} \sigma_i(s_k)$$

for each $s_k \in S$ and for each $\omega \in$. We interpret $w(\omega, s; \alpha, \sigma)$, as showing, for each $\omega \in$, the 'weight' given to pure strategy s_k in the strategy vector σ by players assigned type ω by α. That is, given the society (N, α) and the strategy vector σ, $\frac{w(\omega, s_k; \alpha, \sigma)}{\rho(N,\alpha)(\omega)}$ is the expected proportion of times strategy s_k will be played by players of type ω. It is immediate that a weight function relative to an attribute function α and a strategy vector σ is a weight function, as defined above. In particular

$$\sum_k w(\omega, s_k; \alpha, \sigma) = p(N, \alpha)(\omega)$$

for each $\omega \in$. Note that there may be many strategy vectors that generate a given weight function relative to an attribute function. Let W_α denote the set of all possible weight functions for the society (N, α).

For a strategy $p \in \Delta(S)$ and attribute $\omega_0 \in$ we denote by $\chi(\cdot,\cdot;\omega_0,p)$ the *individual weight function.*

$$\chi(\omega, s_k; \omega_0, p) = p_k \text{ if } \omega = \omega_0 \text{ and}$$

$$\chi(\omega, s_k; \omega_0, m) = 0 \text{ otherwise.}$$

for each $s_k \in S$, where p_k denotes the probability pure strategy $s_k \in S$ is played, given the mixed strategy p.

We highlight a subset G_α of the set of weight functions W_α, those with integer values; thus, a member $g(\cdot,\cdot;\alpha)$ of G_α is a weight function mapping

$\times\ S$ into the non-negative integers $\mathcal{Z}_+$,rather than the nonnegative real numbers. If $g(\cdot,\cdot;\alpha) \in G_\alpha$ then there exists strategy vectors σ such that σ_i is degenerate for all players $i \in N$ and $g(\cdot,\cdot;\alpha,\sigma)(\omega) = g(\cdot,\cdot;\alpha)(\omega)$ for all $\omega \in$. Moreover, every degenerate strategy vector σ generates an integer-valued weight function. Given the profile $\rho(N,\alpha)$ and a degenerate strategy vector σ, the interpretation is that, for each attribute ω and strategy $s_k \in S$, $g(\omega, s_k, \alpha)$ denotes the number of players i in N with attribute ω whose strategy σ_i places weight 1 on some pure strategy s_k.

Given an attribute function α and a strategy vector $\sigma \in \Sigma$ with corresponding weight function $w(\cdot,\cdot;\alpha,\sigma) \in W_\alpha$, let $w_{-i}(\cdot,\cdot;\alpha,\sigma)$ denote the weight function in which player i's contribution is not included. That is,

$$w_{-i}(\omega, s; \alpha, \sigma) = w(\omega, s; \alpha, \sigma) - \chi_i(\omega, s; \omega_0, \sigma_i)$$

for all $\omega \in$, all $s \in S$ and for all $i \in N$, where $\alpha(i) = \omega_0$ and $\chi_i(\cdot,\cdot;\omega_0,\sigma_i)$ is the individual weight function of player i given attribute function α and strategy vector σ. Let $W_{\alpha-\chi_i}$ denote the set of weight functions obtained when player i's contribution is not included.

2.2 The games $\Gamma(N,\alpha)$.

With the above definitions now in place, we can now define the games $\Gamma(N,\alpha)$. For any society (N,α), the game $\Gamma(N,\alpha)$ is given by

$$\Gamma(N, v) = \{S, h_\omega(\cdot,\cdot;\rho(N,\alpha)), \omega \in \alpha^{-1}(N)\}$$

where, for each $i \in N$, $h_{\alpha(i)}(\cdot,\cdot;\rho(N,\alpha))$ is a given *payoff function* mapping $\Delta S \times W_{\alpha-\chi_i}$ into $\mathcal{R}_+$.

Given an attribute function α, a strategy vector σ and the corresponding weight function $w(\cdot,\cdot;\alpha,\sigma) \in W_\alpha$, the payoff of player $i \in N$ is given by

$$h_{\alpha^{-1}(i)}(\sigma_i, w_{-i}(\omega, s_k; \alpha, \sigma), \rho(N,\alpha)) \in \mathcal{R}_+. \qquad (1)$$

The interpretation is that $h_\omega(\sigma_i, w_{-i}(\cdot,\cdot;\alpha,\sigma), \rho(N,\alpha))$ is the payoff to a player $i \in N$ with $\alpha(i) = \omega$, in the game $\Gamma(N,\alpha)$, from playing the (possibly) mixed strategy σ_i when the strategy choices of the remaining players are represented by $w_{-i}(\cdot,\cdot;\alpha,\sigma)$. Note that payoff functions are parameterized by the population profile $\rho(N,\alpha)$ since different population profiles correspond to different games. We make the standard assumption that the payoff to a mixed strategy is the expected payoff from pure strategies, that is,

$$h_\omega(p, w_{-i}(\cdot,\cdot;\alpha), \rho(N,\alpha)) = \sum_k p_k h_\omega(s_k, w_{-i}(\cdot,\cdot;\alpha), \rho(N,\alpha)). \qquad (2)$$

Note that implicit in the definition of the payoff function there is an anonymity assumption. For example, consider two players $i, j \in N$, where $a(i) = \alpha(j)$, and two alternative scenarios. In the first scenario player i plays pure strategy s_1 and player j plays pure strategy s_2. In the second scenario, roles are reversed so that player i plays s_2 and player j plays s_1. Then, assuming everything else remains the same, the payoff to a third player $i' \in N$ is indifferent to this switch between i and j. This example is a special case of a continuity assumption (continuity 1) below.

The standard definition of a Nash equilibrium applies. A strategy vector σ is a *Nash equilibrium* only if, for each $i \in N$ and for each pure strategy $s_k \in$*support*(σ_i), it holds that

$$h_{\alpha(i)}(s_k, w_{-i}(\cdot,\cdot;\alpha,\sigma), \rho(N,\alpha)) \geq h_{\alpha(i)}(t, w_{-i}(\cdot,\cdot;\alpha,\sigma), \rho(N,\alpha)) \text{ for all } t \in S.$$

2.3 Large anonymous games

We now introduce the following assumptions about growing sequences of games which together constitute a *large anonymous game property.* First, without loss of generality we can suppose that the furthest distance between any two points in   is less than one.

Observe that a profile f induces a probability measure $\frac{|1|}{\|f\|}f$ on   where each singleton set $\{\omega\}$ is assigned the probability

$$\frac{|f(\omega)|}{\|f\|};$$

let us call this probability measure $\mu(f)$. Similarly, a weight function w induces a probability measure $\frac{|\omega(\omega,s_k;\alpha)|}{\|w\|} = \mu(w(\cdot,\cdot;\alpha))$ on $\times S$ where probability of a singleton set $\{(\omega, s_k)\}$ is $\frac{|\omega(\omega,s_k;\alpha)|}{\|w\|}$. We will apply the notion of weak convergence in measure.[3]

[3] For discussion of the weak topology and the Prohorov metric, see, for example, Kirman (1981).

For two attribute functions α and α' satisfying $\sum_{\alpha^{-1}(\omega)\neq 0} |\alpha^{-1}(\omega)| = \sum_{\alpha'^{-1}(\omega)\neq 0} |\alpha'^{-1}(\omega)|$, define the metric *dist* as follows: First, let N be a finite set with $|N| = \sum_{\alpha^{-1}(\omega)\neq 0} |\alpha^{-1}(\omega)|$. List the points in the supports of α and α' with each point ω appearing as many times as its multiplicity (for α, this multiplicity is $|\alpha^{-1}(\omega)|$ and similarly for α'). Label the points in $N, i = 1, ..., |N|$, so that the supremum of the distances between $\alpha(i)$ and $\alpha'(i)$ is minimized (where these distances are, of course, with respect to the metric on). Define the distance between α and α' as this supremum. For any two attribute functions α and α' such that $\sum_{\alpha^{-1}(\omega)\neq 0} |\alpha^{-1}(\omega)| \neq \sum_{\alpha'^{-1}(\omega)\neq 0} |\alpha'^{-1}(\omega)|$ define[4]

$$dist(\alpha, \alpha') = \left| \sum_{\alpha^{-1}(\omega)\neq 0} |\alpha^{-1}(\omega)| - \sum_{\alpha'^{-1}(\omega)\neq 0} |\alpha'^{-1}(\omega)| \right|.$$

Then *dist* is a well-defined metric. In fact, for attribute functions α and α' satisfying $\sum_{\alpha^{-1}(\omega)\neq 0} |\alpha^{-1}(\omega)| = \sum_{\alpha'^{-1}(\omega)\neq 0} |\alpha'^{-1}(\omega)|$, *dist* corresponds to the Prohorov metric.[5] Note also that given a player set N and attribute functions α and α' defined on N, $dist(\alpha(i), \alpha'(i))$ is well defined and indicates how much the attributes of player i are perturbed by a change in the attribute function from α to α'.

Throughout the following, let $\{N^\nu\}$ be a sequence of player sets with $|N^\nu|$ becoming large as ν becomes large and let $\{\alpha^\nu\}$ be a sequence of attribute functions where α^ν: $N^\nu \to$. For ease in notation, let $\{f^\nu\}$ be the sequence of profiles where, for each ν, we have $f^\nu = \rho(N^\nu, \alpha^\nu)$.

We give two variants of a continuity property - the second implies the first. These continuity properties are both formulated as Lipshitz conditions on large games and are with respect to changes in attributes of players. Both conditions dictate that if we change attributes of players in large player sets only slightly, then for any given strategy vector, the change in payoffs of players is small. The second continuity condition states that in addition, if we change the attribute of a player only slightly then the change in his own payoff are small.

[4]Note that since $|\cdot|$ is commonly used for these two functions and since no ambiguity is likely to arise, we've used $|\cdot|$ for two different purposes, both for the cardinal number of a set and for absolute value.

[5]See, for example, Kirman in the *Handbook of Mathematical Economics*, pages 197-198.

Continuity with respect to attributes: Given $\varepsilon > 0$ there exists a *similarity parameter* $\delta(\varepsilon)$ such that:

for any sequence of attribute functions $\{\overline{\alpha}^{\nu}\}$, where $\overline{\alpha}^{\nu} : N^{\nu} \rightarrow$ and satisfies:

$$dist(\alpha^{\nu}(i), \overline{\alpha}^{\nu}(i)) < \delta(\varepsilon) \text{ for all } i \in N^{\nu} \text{ and for all } \nu,$$

and for any sequence of strategy vectors $\{\sigma^{\nu}\}$, weight functions $\{w^{\nu}(\cdot, \cdot; \alpha^{\nu}, \sigma^{\nu})\}$ and $\{w^{\nu}(\cdot, \cdot; \overline{\alpha}^{\nu}, \sigma^{\nu})\}$, and profiles $\{\overline{f}^{\nu}\}$ where $\overline{f}^{\nu} = \rho(N^{\nu}, \alpha^{\nu})$.:

Continuity 1 (with respect to the attributes of others):

$$\lim_{\nu \rightarrow \infty} \left| h_{\alpha^{\nu}(i)}(\sigma_i, w^{\nu}_{-i}(\cdot, \cdot; \alpha^{\nu}, \sigma^{\nu}), f^{\nu}) - h_{\alpha^{\nu}(i)}(\sigma_i, w^{\nu}_{-i}(\cdot, \cdot; \overline{\alpha}^{\nu}, \sigma^{\nu}), \overline{f}^{\nu}) \right| < \varepsilon$$

for all $\sigma_i \in \Delta(S)$ and for all i with $\alpha^{\nu}(i) = \overline{\alpha}^{\nu}(i)$.
(Note here that although the payoff functions of the players have been changed – for each player j, $\alpha^{\nu}(j)$ changes to $\overline{\alpha}^{\nu}(j)$, the *actions* of the players remain unchanged. This is possible since a strategy vector lists a strategy for each player $i \in N$ and the N remains unchanged – only the payoff functions of the players in N have possibly changed, not the set of players nor their strategies. Thus, if one finds it reasonable that the payoff functions of players are affected only by the *actions* of others and not their payoff functions, then this form of continuity is very mild.[6])

Continuity 2 (with respect to all attributes):

$$\lim_{\nu \rightarrow \infty} \left| h_{\alpha^{\nu}(i)}(\sigma_i, w^{\nu}_{-i}(\cdot, \cdot; \alpha^{\nu}, \sigma^{\nu}), f^{\nu}) - h_{\overline{\alpha}^{\nu}(i)}(\sigma_i, w^{\nu}_{-i}(\cdot, \cdot; \overline{\alpha}^{\nu}, \sigma^{\nu}), \overline{f}^{\nu}) \right| < \varepsilon$$

for all $\sigma_i \in \Delta(S)$ and all i.

[6] There are situations where individuals claim to be affected by the feelings, loyalties or thoughts of others, independent of their actions. In Arthur Miller's celebrated book, *The Crucible*, Rachel has been a pious woman, known for her good deeds and kind works, all through her long life. But the witch hunters of Salem interpreted Rachel's apparent goodness as just a clever disguise to hide her love of the devil. Rachel was put to death as a witch; for witch hunters, the private feelings of others and their thoughts are significant.

In continuity 1, we essentially only consider the changes in payoffs, from perturbing the attribute function, to players who keep the same attribute type in both societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$. In continuity 2, we consider the change in payoffs to players who themselves also have their attribute type slightly perturbed between societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$ and impose continuity.

Our next condition ensures that in large games, payoffs to individual players depend on aggregate of the strategies chosen by others.

Convergence: Assume that $\mu(f^v)$ converges weakly to f as $v \to \infty$ (in the sense of weak convergence in measure). Let $\{w^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)\}$ be a sequence of weight functions for some strategy vector $\sigma^\nu \in \Sigma$. Assume that $\lim_{\nu\to\infty} \mu(w^v) = w$ exists. Then, for all i,

$$h_{\alpha^\nu(i)}(p, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \text{ converges as } \nu \to \infty$$

for all $p \in \Delta(S)$.

Societies (N, α) and individual games $\Gamma(N, \alpha)$ derived from noncooperative pregames have an anonymity property as noted previously; the game $\Gamma(N, \alpha)$ and the payoff $h_{\alpha(i)}(\cdot, \cdot, \rho(N, \alpha))$ to an individual player $i \in N$ do not depend on the names of other players, only on the profile of the player set. To obtain our results we require some a further anonymity conditions on payoff functions as the numbers of players in the games becomes large. For our current results, we use the strong anonymity condition below. We also indicate that there are other interesting forms of anonymity that may suffice and present an example.

Anonymity: Assume that $\mu(f^v)$ converges weakly to $f \in P(\ \ )$, with support $\{\omega_1, ..., \omega_J\}$, as $v \to \infty$. Also assume that $\{\omega_1, ..., \omega_J\} \subset support(f^\nu)$ for each ν. Let $\{w^\nu(\cdot, \cdot; \alpha^\nu)\}$ and $\{g^\nu(\cdot, \cdot; \alpha^\nu)\}$ be sequences of weight functions where w^ν and, respectively, g^ν are relative to attribute function α^ν.

Strong anonymity: If, for each pure strategy $s_k \in S$, for some real numbers $\{\theta_{jk} : j = 1, ..., J\}$ it holds that:

$$\lim_{\nu\to\infty} \frac{w^\nu(\omega_j, s_k; \alpha^\nu)}{\|w^\nu\|} = \lim_{\nu\to\infty} \frac{g^\nu(\omega_j, s_k; \alpha^\nu)}{\|g^\nu\|} = \theta_{jk},[7] \tag{3}$$

then:

$$\lim_{\nu\to\infty} h_{\alpha^\nu(i)}(p, w^\nu_{-i}, f^\nu) = \lim_{\nu\to\infty} h_{\alpha^\nu(i)}(p, g^\nu_{-i}, f^\nu)$$

for all $p \in \Delta(S)$ and all i.[8]

Weak anonymity: If, for each pure strategy $s_k \in S$:

$$\lim_{\nu\to\infty} \frac{w^\nu(\omega, s_k; \alpha^\nu)}{f^\nu(\omega)} = \lim_{\nu\to\infty} \frac{g^\nu(\omega, s_k; \alpha^\nu)}{f^\nu(\omega)} = \theta_k(\omega)$$

for all $\omega \in$, where the limits are pointwise, then:

$$\lim_{\nu\to\infty} h_{\alpha^\nu(i)}(s, w^\nu_{-i}, f^\nu) = \lim_{\nu\to\infty} h_{\alpha^\nu(i)}(s, g^\nu_{-i}, f^\nu)$$

for all $s \in \Delta(S)$ and all $i \in N^\nu$.

If preferences satisfy strong anonymity then, for games with many players, the payoff to a player depends only on his own strategy choice and the *proportion*, relative to the total population of players, of each type playing each strategy. Weak anonymity refines the definition of anonymity to apply when players have scarce attributes; it requires that for all attributes, the proportion of players playing each strategy must remain the same to leave payoffs approximately the same.

Note that in the definition of strong anonymity, for each ν, it holds that $\{\omega_1, ..., \omega_J\} \subset$ *support*(f^ν) but in general it may be that the supports of the functions f^ν become infinite as ν becomes large. The definition of strong

[8]Note that we could have instead required that the measures $\mu(f^\nu)$ converged weakly to some measure f with finite support. We wish, however, to keep the requirement as narrow as possible. Suppose, for example, in each society (N^ν, α) all players are distinct in the sense that $f^\nu(\omega) \leq 1$ for all $\omega \in$ and for ν. Let the ν^{th} game have ν players. Suppose all the attributes of the players of the ν^{th} game, for $\nu > 2$, are contained in the interval $[\frac{1}{2} - \frac{1}{\nu}, \frac{1}{2} + \frac{1}{\nu}]$. Then $\lim \frac{f^\nu(\omega)}{\|f^\nu\|} = f(\omega) = 0$ for all $\omega \neq \frac{1}{2}$ and $f(\frac{1}{2}) = 1$ so the condition on the sequences of profiles $\{f^\nu\}$ in the definition of strong anonymity is not satisfied. Nevertheless, $\mu(f^\nu)$ converges weakly in measure to f where $f(\omega) = 0$ for all $\omega \neq 0$ and $f(\frac{1}{2}) = 1$.

Note also that we're slightly abusing notation. More precisely, we should write $f(S) = 1$ for all measurable subsets S containing $\omega = 1/2$ and $f(S) = 0$ otherwise. We believe, however, that our approach simplifies notation and makes reading easier.

anonymity requires, however, while the player sets tend to become infinitely large, 'in the limit' the number of attributes represented in the population tends towards a fixed, finite subset of the attributes represented in the individual games. The actions of players with attributes represented in the populations (N, α^ν) but in vanishingly small proportions do not have significant effects on the payoffs of other players. Strong anonymity does not imply that in our theorems we restrict attention to such sequences such as $\{f^\nu\}$; it only implies that for such sequences the conclusion of the definition of strong anonymity holds.

Given $\omega \in$, let the ball around ω with diameter δ be denoted $B(\omega, \sigma)$ and defined by

$$B(\omega, \delta) = \left\{ \omega^* \in \quad : dist(\omega, \omega^*) < \frac{\delta}{2} \right\}.$$

3 Results.

We first state two useful lemmas. The first lemma applies to any game and concerns approximation of mixed strategy vectors by degenerate strategy vectors. The second lemma concerns limiting approximations for sequences of games. With these two lemmas in hand, in the following subsection, we next prove our purification result and then, in the final subsection, we prove our social conformity result.

3.1 Two lemmas.

We firstly introduce some notation. We say a vector $a = (a_1, ..., a_N) \geq b = (b_1, ..., b_N)$ if and only if $a_i \geq b_i$ for all $i = 1, ..., N$. Let Z_+^K denote the set of K dimensional vectors for which every element is a non-negative integer.

For any strategy vector $\sigma = (\sigma_1, .., \sigma_N)$, let $\mathcal{M}(\sigma)$ denote the set of vectors $m = (m_1, .., m_N) \in Z_+^{KN}$ such that, for each $i \in N$:

(a) $m_i = (m_{i1}, ..., m_{iK}) \in Z_+^K$ for $i = 1, ..., N$,

(b) $\|m_i\| = 1$ for each i and,

(c) $m_{ik} = 1$ implies $\sigma_{ik} > 0$.

Informally, $\mathcal{M}(\sigma)$ denotes a strategy vector with the property that each player i is assigned, as a pure strategy, some strategy in the support of σ_i. If σ were a Nash equilibrium then in the strategy vector m, each player is assigned a strategy in his best response function for σ.

The lemma shows that given any choices of mixed strategies, $(\sigma_i,\ i = 1, ..., N)$, one for each player, we can select pure strategies m_i for each player so that each player's pure strategy is a best response to the initially given (possibly) mixed strategy choices and so that the total number of players assigned strategy s_k is within K of the total weight assigned to s_k by the initially given mixed strategy vector, that is,

$$\left| \sum_i \sigma_{ik} - \sum_{i:m_{ik}=1} m_{ik} \right| < K. \tag{4}$$

Actually, the result is somewhat stronger.

Lemma 1: For any strategy vector $\sigma \in \Delta^{KN}$ and for any vector $\overline{g} \in Z_+^K$ such that $\sum_i \sigma_i \geq \overline{g}$, there exists a pure strategy vector $m = (m_1, ..., m_{|N|}) \in \mathcal{M}(\sigma)$ such that:

$$\sum_i m_i \geq \overline{g}.$$

(To relate this to the interpretation above, choose $\overline{g}$ so that for each $k = 1, ..., K$, $\sum_i \sigma_{ik} - \overline{g}_k \leq 1$. Then, since $\sum_k \sum_i \sigma_{ik} = |N|$ it holds that $|N| - \sum_k \overline{g}_k \leq K$. Note also that $\sum_k \sum_i m_{ik} = |N|$. It follows that for each pure strategy s_k, (4) holds.)

Proof: Suppose the statement of the lemma is false. Then, there exists a strategy vector $\sigma = (\sigma_1, ..., \sigma_i, ..., \sigma_N)$ (where $\sigma_i = (\sigma_{i1}, ..., \sigma_{iK}) \in \Delta^K$ for $i = 1, ..., N$) and a vector $\overline{g} \in \mathcal{Z}^K$ such that, for any vector $m = (m_1, ..., m_{|N|}) \in \mathcal{M}(\sigma)$ there must be at least one $\widehat{k}$ for which $\sum_i m_{i\widehat{k}} < \overline{g}_{\widehat{k}}$. For each vector $m \in \mathcal{M}(\sigma)$ let L be defined as follows:

$$L(m) = \sum_{k:\overline{g}_k - \sum_i m_{ik} > 0} \left(\overline{g}_k - \sum_i m_{ik} \right)$$

Select $m^0 \in \mathcal{M}(\sigma)$ for which $L(m)$ attains its minimum value over all $m \in \mathcal{M}(\sigma)$. Pick a strategy $\widehat{k}$ such that $\overline{g}_{\widehat{k}} - \sum_i m^0_{i\widehat{k}} > 0$.

Consider the sets $I^n(\widehat{k})$, $n = 0, 1, ...$ defined as follows:

$$I^0(\widehat{k}) = \left\{ i \in (1, ..., N) : m^0_{i\widehat{k}} = 1 \right\},$$

$$I^{n+1}(\widehat{k}) = I^n(\widehat{k}) \cup \{ j \in (1, ..., N) : \text{for some } k \text{ such that } m^0_{ik} = 1 \\ \text{and } i \in I^n(\widehat{k}),\ \sigma_{jk} > 0 \text{ and } m^0_{jk} = 0 \}.$$

$I^0(\widehat{k})$ is the set of players who are assigned the pure strategy $s_{\widehat{k}}$ by m^0, that is, if $i_0 \in I^0(\widehat{k})$ then $m^0_{i_0\widehat{k}} = 1$. The set $I^1(\widehat{k}) \backslash I^0(\widehat{k})$ consists of those players who could have been assigned the strategy $\widehat{k}$ while following the assignment rule of the Lemma, but were not, that is, if $i_1 \in I^1(\widehat{k})$, then $\sigma_{i_1\widehat{k}} > 0$ but $m^0_{i_1\widehat{k}} = 0$. (Note that $I^1(\widehat{k})$ contains $I^0(\widehat{k})$ and is the set of all players i for whom $\sigma_{i\widehat{k}} > 0$. In defining $I^2(\widehat{k})$ values of $k \neq \widehat{k}$ can play a role. That is, there may be players in the set $I^1(\widehat{k})$ who are assigned some pure strategies $s_{k'}$ where $k' \neq \widehat{k}$. For such a player i_1, $\sigma_{i_1k'} > 0$, $\sigma_{i_1\widehat{k}} > 0$ and $m^0_{i_1k'} = 1$. A player i_2 is in $I^2(\widehat{k}) \backslash I^1(\widehat{k})$ if there is a player $i_1 \in I^1(\widehat{k})$ and some $s_{k'}$ such that $m^0_{i_1k'} = 1$, $m^0_{i_2k'} = 0$ and $\sigma_{i_2k'} > 0$. And so on.

A picture may be useful.

$i_0 \in I^0(\widehat{k})$	$i_1 \in I^1(\widehat{k}) \backslash I^0(\widehat{k})$	$i_2 \in I^2(\widehat{k}) \backslash I^1(\widehat{k})$	$i_3 \in I^3(\widehat{k}) \backslash I^2(\widehat{k})$	$\cdots$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	
$\sigma_{i_0\widehat{k}} > 0$ $m^0_{i_0\widehat{k}} = 1.$	$\sigma_{i_1\widehat{k}} > 0$ $m^0_{i_1\widehat{k}} = 0$	$\sigma_{i_2\widehat{k}} = 0$ $m^0_{i_2\widehat{k}} = 0.$	$\sigma_{i_3\widehat{k}} = 0$ $m^0_{i_3\widehat{k}} = 0.$	
	and for some k', $\sigma_{i_1k'} > 0$ $m^0_{i_1k'} = 1$.	For k', $\sigma_{i_2k'} > 0,$ $m^0_{i_2k'} = 0$	For k', $\sigma_{i_3k'} = 0,$ $m^0_{i_3k'} = 0.$	
		and for some k'' $\sigma_{i_2k''} > 0$ $m^0_{i_2k''} = 1$.	For k'' $\sigma_{i_3k''} > 0$ $m^0_{i_3k''} = 0$	
			and for some k''' $\sigma_{i_3k'''} > 0$ $m^0_{i_3k'''} = 1.$	

Figure 1

Any of the players in $I^1(\widehat{k})\backslash I^0(\widehat{k})$, in particular, i_1, could have been assigned the pure strategy $\widehat{k}$ (since $\sigma_{i_1\widehat{k}} > 0$) but instead i_1 was assigned the pure strategy k'. If we reassigned i_1 to $\widehat{k}$ this may leave us a "shortfall" with regard to the pure strategy k'. But we could re-assign i_2 to play k' (since $\sigma_{i_2k'} > 0$ and $m^0_{i_2k'} = 0$). But this may leave us a shortfall with regard to the pure strategy k''. And so on.

Ultimately, for some n^* we must have $I^{n^*+1}(\widehat{k}) = I^{n^*}(\widehat{k})$. For the purposes of illustration, suppose that $n^* = 2$. This means that $I^2(\widehat{k}) = I^3(\widehat{k})$. Thus, continuing to use Figure 1, there is no player $j \in N\backslash I^2(\widehat{k})$ for whom $\sigma_{jk''} > 0$ and $m_{jk''} = 0$; we essentially reach the 'end of the chain'. Note that there may be a player $j \in N\backslash I^2(\widehat{k})$ for whom $m_{jk''} = 1$. More generally, there may be players not belonging to the set $I^{n^*}(\widehat{k})$ but who are allocated a strategy represented in the set $I^{n^*}(\widehat{k})$. Formally, the set of strategies used in the construction of $I^{n^*}(\widehat{k})$ is given below.

Let the set $S^{n^*}(\widehat{k})$ be defined as follows:

$$S^{n^*}(\widehat{k}) = \left\{\widehat{k}\right\} \cup \left\{k \in (1, ..., K) : m^0_{ik} = 1 \text{ for some } i \text{ where } i \in I^{n^*}(\widehat{k})\right\}$$

For any $k = 1, ..., K$, $k \neq \widehat{k}$, if there exists an player $j \in I^{n^*}(\widehat{k})$ such that $m^0_{jk} = 1$ and

$$\sum_{i \in N} m^0_{ik} - m^0_{jk} \geq \overline{g}_k$$

we have the desired contradiction. In this case, there exists a set of vectors $\overline{m} \in \mathcal{M}(\sigma)$ such that $L(\overline{m}) = L(m^0) - 1$. To illustrate, suppose for example, that $\sum_{i \in N} m^0_{ik'''} - m^0_{jk'''} \geq \overline{g}_{k'''}$ in the situation depicted in Figure 1. Then, without violating the rules of the Lemma, we can reassign player i_3 to strategy k'', player i_2 to strategy k' and player i_1 to $\widehat{k}$. Thus, for all $k \in S^{n^*}(\widehat{k})$ it holds that:

$$\sum_{i \in N} m^0_{ik} \leq \overline{g}_k. \tag{5}$$

(Note that m^0_{ik} and $\overline{g}_k$ are integers for all k.)

Since $I^{n^*+1} = I^{n^*}$ (by the definition of I^n), however, there can exist no $j \in (1, ..., N) \backslash I^{n^*}(\widehat{k})$ such that $\sigma_{jk} > 0$ for some $k \in S^{n^*}(\widehat{k})$, unless $m^0_{jk} = 1$. This implies that:

$$\sum_{i \in N \backslash I^{n^*}(\widehat{k})} m^0_{ik} \geq \sum_{i \in N \backslash I^{n^*}(\widehat{k})} \sigma_{ik}$$

for all $k \in S^{n^*}(\widehat{k})$.

We also have, from the definitions of $I^n(\widehat{k})$ and $S^n(\widehat{k})$, that:

$$\sum_{k \in S^{n^*}(\widehat{k})} \sum_{i \in I^{n^*}(\widehat{k})} m^0_{ik} \geq \sum_{k \in S^{n^*}(\widehat{k})} \sum_{i \in I^{n^*}(\widehat{k})} \sigma_{ik}.$$

This implies that:

$$\sum_{k\in S^{n^*}(\widehat{k})}\sum_{i\in N} m_{ik} \geq \sum_{k\in S^{n^*}(\widehat{k})}\sum_{i\in N} \sigma_{ik} \geq \sum_{k\in S^{n^*}(\widehat{k})} \overline{g}_k$$

However:

$$\overline{g}_{\widehat{k}} > \sum_{i\in N} m^0_{i\widehat{k}}$$

and $\widehat{k} \in S^{n^*}(\widehat{k})$. Thus, there must exist at least one $k \in S^{n^*}(\widehat{k})$ such that:

$$\overline{g}_k < \sum_{i\in N} m^0_{ik}.$$

This contradicts 5 and completes the proof.♦

Now that the proof of Lemma 1 is complete, let us summarize its strategy. First, to appreciate how the sets $I^n(\widehat{k})$ work consider set $I^1(\widehat{k})$ given by:

$$I^1(\widehat{k}) = \left\{i \in (1, ..., N) : \sigma_{i\widehat{k}} > 0 \text{ and } m^0_{i\widehat{k}} = 0\right\}.$$

If there exists an integer $j \in I^1(\widehat{k})$ and k such that $m^0_{jk} = 1$ and where:

$$\sum_{i\in N} m^0_{ik} - m^0_{jk} \geq \overline{g}_k$$

then we can do the following: put m^0_{jk} equal to zero and set $m^0_{j\widehat{k}} = 1$. Essentially, the player j has been allocated to a strategy k where 'it is not needed' and so we can take j away from k allocate it to strategy $\widehat{k}$. This reduces L by one contradicting that we choose the minimum possible L.

If this does not work consider the set $I^2(\widehat{k})$ as follows:

$$I^2(\widehat{k}) = I^1(\widehat{k}) \cup \left\{ \begin{array}{l} j \in (1, ..., N) : \sigma_{j\widetilde{k}} > 0 \text{ and } m_{j\widetilde{k}} = 0 \text{ for some} \\ \widetilde{k} \text{ such that } m_{i\widetilde{k}} = 1 \text{ for some } i \in I^1(\widehat{k})\} \end{array} \right\}.$$

If there exists a player $j \in I^2(\widehat{k})$ and k such that $m_{jk} = 1$ and:

$$\sum_{i\in N} m_{ik} - m_{jk} \geq \overline{g}_k.$$

In this case, player j has been allocated to a strategy k where 'it is not needed'. However, $\sigma_{j\widehat{k}} = 0$ and so we cannot allocate j directly to strategy $\widehat{k}$. Instead we find an intermediary i such that j can be allocated the strategy of i and i can then be allocated the strategy $\widehat{k}$. Formally, there must exist a player $i \in I^1(\widehat{k})$ and a $\widetilde{k} \neq k \neq \widehat{k}$ such that $m^0_{i\widetilde{k}} = 1$, $\sigma_{j\widetilde{k}} > 0$ and $\sigma_{i\widehat{k}} > 0$. So, put m^0_{jk} to zero and $m^0_{j\widetilde{k}}$ to one. Put $m^0_{i\widetilde{k}}$ to zero and $m^0_{i\widehat{k}}$ to one. The number allocated to playing pure strategy s_k falls by one, but this is no problem, the number playing pure strategy $s_{\widetilde{k}}$ remains the same and the number playing pure strategy $s_{\widehat{k}}$ increases by one. This reduces L by one.

For a general $I^n(\widehat{k})$ the definition of $I^n(\widehat{k})$ directly implies that if there exists an integer $j \in I^n(\widehat{k})$ *and* pure strategy s_k where $m_{jk} = 1$ and:

$$\sum_{i \in N} m^0_{ik} - m^0_{jk} \geq \overline{g}_k$$

then the distance $\overline{g}_k - \sum_i m^0_{ik} = L > 0$ could have been reduced by at least 1. That is, there must exist an integer j allocated to a k where 'it is not needed' and a chain of at most n possible intermediaries whereby j can be reallocated from playing pure strategy s_k and some i allocated to playing $\widehat{k}$. Furthermore, along the chain players can replace one another to leave the total number allocated to all strategies, except k and $\widehat{k}$, constant.

Roughly, our next Lemma shows that, for any growing sequence of games, if there is only a finite number of types that appear in positive proportions in the limit, then in the limit, strategy vectors can be purified. Suppose, as is standard in papers showing purification of mixed strategy equilibria, we had a continuum of players with a finite number of types where type ω_a appears in the proportion θ_{ak}. Then the following result demonstrates that we can approach the continuum purification in large finite games.[9] But it shows more. The games considered in Lemma 2 could have vanishingly small percentages of players of some types. Our conditions ensure that these players cannot significantly effect payoffs to other players and are, in the continuum limit, negligible.

[9] In fact, such a result was obtained in Rashid (1983).

Lemma 2: Let $\{\overline{f}^{\upsilon}\}$ be a sequence of profiles satisfying the conditions given in the definition of strong anonymity. Let $\{\sigma^{\nu}\}$ a sequence of strategy vectors and let $\{w^{\nu}\}$ be a sequence of weight functions where w^{ν} is relative to strategy vector σ^{ν} and attribute function α^{ν}.

Assume the $\lim_{\nu\to\infty}\left(\frac{w^{\upsilon}(\omega_j,s_k;\alpha,\sigma)}{\|\overline{w}^{\upsilon}\|}\right)=\theta_{jk}$ exists for all $s_k\in S$. Then there exists a sequence $\{s^{\nu}\}$ of degenerate strategy vectors and a sequence $\{g^{\nu}\}$ of integer-valued weight functions, where g^{ν} is relative to strategy vector s^{ν} and attribute function α^{ν}, such that:

1. for all $s_k\in S$ and all $\omega_j\in$ *support*(f),

$$\lim_{\nu\to\infty}\frac{g^{\nu}(\omega_j,s_k;\alpha^{\nu},s^{\nu})}{\|g^{\nu}\|}=\lim_{\nu\to\infty}\frac{g^{\nu}(\omega_j,s_k;\alpha^{\nu},s^{\nu})-1}{\|g^{\nu}\|}=\theta_{jk}$$

2. the strategy vector s^{ν} is such that $s_i^{\nu}\in$*support*(σ_i^{ν}) for all $i\in N^{\nu}$ and for all ν.

Proof: Suppose the statement of the lemma is false. Then, there exists an $\varepsilon>0$ such that for any sequence of degenerate strategy vectors $\{s^{\nu}\}$,with $s_i^{\nu}\in$ *support*(σ_i^{ν}) for all $i\in N$ and all ν, any corresponding sequence of integer-valued weight functions $\{g^{\nu}\}$ and for any ν there exists a $\nu_0>\nu$ such that:

$$\left|\frac{g^{\nu_0}(\omega_j,s_k;\alpha^{\nu_0},s^{\nu_0})}{\|g^{\nu_0}\|}-\theta_{jk}\right|>\varepsilon$$

and / or:

$$\left|\frac{g^{\nu_0}(\omega_j,s_k;\alpha^{\nu_0},s^{\nu_0})-1}{\|g^{\nu_0}\|}-\theta_{jk}\right|>\varepsilon$$

for some $\omega_j\in$ *support*(f) and some $s_k\in S$.

We derive the desired contradiction as follows:

For each ν and for each $k=1,...,K$ define the integer $\overline{g}^{\nu}(\omega_j,s_k)$ as the largest integer less than or equal to $w^{\nu}(\omega_j,s_k;\alpha^{\nu},s^{\nu})$. Formally:

$$\overline{g}^{\nu}(\omega_j,s_k)=\{x\in Z_+:x=\min_{y\in Z_+}(y-w^{\nu}(\omega_j,s_k;\alpha^{\nu},s^{\nu}))\text{ and }x\leq w^{\nu}(\omega_j,s_k;\alpha^{\nu},s^{\nu})\}$$

The construction of $\overline{g}$ implies that:

$$1 \geq w^{\nu}(\omega_j, s_k; \alpha^{\nu}, s^{\nu}) - \overline{g}^{\nu}(\omega_j, s_k) \geq 0$$

for all $k = 1, ..., K$ and for all ν. This implies, by Lemma 1, that there must exist, for all ν, degenerate strategy vectors s^{ν} with corresponding integer valued weight functions g^{ν} such that:

1. $s_i^{\nu} \in support(\sigma_i^{\nu})$ for all $i \in N^{\nu}$.
2. $\overline{g}^{\nu}(\omega_j, s_k) + L^{\nu} \geq g^{\nu}(\omega_j, s_k) \geq \overline{g}^{\nu}(\omega_j, s_k)$ for all $k = 1, ..., K$, where $L^{\nu} \in \mathcal{Z}^{+}$ is defined for each ν as:

$$L^{\nu} = f^{\nu}(\omega_j) - \sum_{k=1}^{K} \overline{g}^{\nu}(\omega_j, s_k)$$

We note that $L^{\nu} \leq K$ for all ν. Thus:

$$\begin{aligned} |w^{\nu}(\omega_j, s_k; \alpha^{\nu}, s^{\nu}) - g^{\nu}(\omega_j, s_k)| &\leq \max\{1, K-1\} \\ |w^{\nu}(\omega_j, s_k; \alpha^{\nu}, s^{\nu}) - g^{\nu}(\omega_j, s_k) + 1| &\leq \max\{2, K-2\} \end{aligned}$$

Therefore, given that $\|f^{\nu}\| \to \infty$ as $\nu \to \infty$, for any ε_1 there exists a ν_1 such that for all $\nu > \nu_1$ we have that:

$$\max\left\{\frac{|w^{\nu}(\omega_j, s_k; \alpha^{\nu}, s^{\nu}) - g^{\nu}(\omega_j, s_k)|}{\|f^{\nu}\|}, \frac{|w^{\nu}(\omega_j, s_k; \alpha^{\nu}, s^{\nu}) - g^{\nu}(\omega_j, s_k) + 1|}{\|f^{\nu}\|}\right\} < \varepsilon_1$$

However, if we select $\varepsilon_1 \in (0, \varepsilon)$ then for all $\nu > \nu_1$:

$$\left|\frac{g^{\nu}(\omega_j, s_k)}{\|g^{\nu}\|} - \theta_{jk}\right| < \varepsilon$$

and

$$\left|\frac{g^{\nu}(\omega_j, s_k) - 1}{\|g^{\nu}\|} - \theta_{jk}\right| < \varepsilon$$

which gives the desired contradiction.♦

3.2 Existence of ε-equilibrium in pure strategies.

In the following Theorem, we demonstrate that, given $\varepsilon > 0$ there is an integer sufficientlyh large so that every game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies. To obtain this result, at a point in the proof we arbitrarily select a Nash equilibrium for each game in a sequence and show that if there are sufficiently many players, this Nash equilibrium can be used to construct an ε-equilibrium in pure strategies. Since the selection of the Nash equilibrium was arbitrary, our result can be viewed as a purification theorem – in sufficiently large games, every Nash equilibrium can be purified.

Theorem 1: Given a real number $\varepsilon > 0$ there exists a real number $\eta_0(\varepsilon) > 0$ such that for all societies (N, α), where preferences satisfying continuity 1, convergence and strong anonymity, and where $\|\rho(N, \alpha)\| > \eta(\varepsilon)$, the induced game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies. Moreover, for any mixed strategy equilibrium there exists an ε-purification.

Proof: Suppose that the statement of the Theorem is false. Then there is some $\varepsilon > 0$ such that, for each integer ν there is a society (N^v, α^v) and induced game $\Gamma(N^\nu, \alpha^\nu)$ with profile $\rho(N^\nu, \alpha^\nu) > \nu$ for which there does not exist an ε-equilibrium in pure strategies. For ease of notation, denote $\rho(N^\nu, \alpha^\nu)$ by f^ν. That is, for each induced game $\Gamma(N^\nu, \alpha^\nu)$ there does not exist a degenerate strategy vector s^ν, with corresponding integer-valued weight function $g^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$ such that:

$$h_{\alpha(i)}(s, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \geq h_{\alpha(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu)$$

for all $t \in S$ and for all $i \in N^\nu$ where $s \in support(\sigma^\nu_i)$.

Observe, however that the game $\Gamma(N^v, \alpha^v)$ has a mixed strategy Nash equilibrium; this is an immediate application of Nash's well known theorem. Denote a Nash equilibrium (NE) of the game $\Gamma(N^v, \alpha^v)$ by σ^v with the appropriate weight function $w^v(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$. Since σ^v is a Nash equilibrium, for each v and for each $i \in N^v$ we have:

$$h_{\alpha^\nu(i)}(\sigma^v_i, w^v_{-i}, f^v) \geq h_{\alpha^\nu(i)}(t, w^v_{-i}, f^v)$$

for all $s \in S$ and

$$h_{\alpha^\nu(i)}(s, w^v_{-i}, f^v) \geq h_{\alpha^\nu(i)}(t, w^v_{-i}, f^v)$$

for all $t \in S$ and for all $s \in support(\sigma_i^\nu)$.

Let $\delta\left(\frac{\varepsilon}{8}\right)$ be the similarity parameter as defined by (Lipshitz) continuity for a required payoff bound of $\frac{\varepsilon}{8}$. Use compactness of   to write   as the disjoint union of a finite number of non-empty subsets $_1, ..., _A$, each of diameter less than δ. For each a, choose and fix a point $\omega_a \in \ _a$.

We define the attribute function $\overline{\alpha}^\nu$ as follows, for all ν and for all $i \in N^\nu$:

$$\overline{\alpha}^\nu(i) = \omega_a \text{ if and only if } \alpha(i) \in \ _a$$

Given the weight function $w^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$ relative to society (N^ν, α^ν) and Nash equilibrium strategy vector σ^ν let $w^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu)$ denote the weight function relative to $\overline{\alpha}^\nu$ and σ^ν.

For each $a = 1, ..., A$ and for each $k = 1, .., K$ define θ_{ak}^ν as follows:

$$\theta_{ak}^\nu = \frac{w^\nu(\omega_a, s_k; \overline{\alpha}^\nu, \sigma^\nu)}{|N^\nu|}$$

By passing to a subsequence if necessary assume that the $\lim_{\nu \to \infty} \theta_{ak}^\nu = \theta_{ak}$ exists for all $a = 1, .., A$ and all $k = 1, ..., K$.

By Lemma 2 there exists a sequence $\{s^\nu\}$ of strategy vectors and a sequence $\{g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)\}$ of integer-valued weight functions relative to attribute function $\overline{\alpha}^\nu$ and the degenerate strategy vector s^ν, such that:

1. for all ν and for all $s_k \in S$ and all $\omega_a \in$,

$$\lim_{\nu \to \infty} \frac{g^\nu(\omega_a, s_k; \overline{\alpha}^\nu, s^\nu)}{\|g^\nu\|} = \lim_{\nu \to \infty} \frac{g^\nu(\omega_a, s_k; \overline{\alpha}^\nu, s^\nu) - 1}{\|g^\nu\|} = \theta_{ak} \qquad (6)$$

2. for all ν and for all $i \in N^\nu$, $s_i^\nu \in support(\sigma_i^\nu)$.

Given the weight function $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)$, let $g^\nu(\cdot, \cdot; \alpha^\nu, s^\nu)$ denote the integer valued weight function relative to society (N^ν, α^ν) and strategy vector s^ν.

Consider the payoff to player $i \in N^\nu$ from changing the strategy vector σ^ν to s^ν. We let $\overline{f}_{-i}^\nu$ denote the profile of a society in which the attribute of a player $j \in N^\nu \backslash \{i\}$ is given by $\overline{\alpha}(j)$ and the attribute of player i is given by $\alpha(i)$.

By continuity 1 and the choice of δ we have that there exists a ν_1 such that for all $\nu > \nu_1$:

$$\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) \right| < \tfrac{\varepsilon}{8}$$

for any $t \in S$.

Given Lemma 2 and strong anonymity, we have that for any $\varepsilon_2 > 0$ there exists a ν_2 such that for all $\nu > \nu_2$:

$$\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,s^\nu), \overline{f}^\nu_{-i}) \right| < \varepsilon_2$$

for any $t \in S$. Set $\varepsilon_2 \in (0, \frac{\varepsilon}{4})$.

Again, using continuity 1 and the choice of δ we have that there exists a ν_3 such that for all $\nu > \nu_3$:

$$\left| h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,s^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) \right| < \frac{\varepsilon}{8}$$

for any $t \in S$.

Thus, for any $\nu > \max\{\nu_1, \nu_2, \nu_3\}$ we have that:

$$\begin{aligned} &\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) \right| \\ \leq &\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) \right| \\ + &\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) \right| \\ + &\left| h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) \right| \\ &< \tfrac{\varepsilon}{8} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{8} = \tfrac{\varepsilon}{2} \end{aligned}$$

for any $t \in S$.

However, given that

$$h_{\alpha(i)^\nu}(s, w^v_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^v) - h_{\alpha(i)^\nu}(t, w^v_{-i}(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^v) \geq 0$$

for all $s \in support(\sigma_i^\nu)$, for all $i \in N$, for all $t \in S$ and for all ν, this implies that for $\nu > \max\{\nu_1, \nu_2, \nu_3\}$:

$$
\begin{aligned}
& h_{\alpha(i)^\nu}(s_i^\nu, g_{-i}^v(\cdot,\cdot;\alpha^\nu, s^\nu), f^v) - h_{\alpha(i)^\nu}(t, g_{-i}^v(\cdot,\cdot;\alpha^\nu, s^\nu), f^v) \\
\geq & - \left| h_{\alpha(i)^\nu}(t, w_{-i}^\nu(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, g_{-i}^\nu(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) \right| - \\
& \left| h_{\alpha(i)^\nu}(s_i^\nu, g_{-i}^\nu(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(s_i^\nu, w_{-i}^\nu(\cdot,\cdot;\alpha^\nu,\sigma^\nu), f^\nu) \right| \\
& \geq -\tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{2} = -\varepsilon
\end{aligned}
$$

which gives the desired contradiction

We note that the equilibrium mixed strategy vector with which we started our proof was arbitrary. Thus, any mixed strategy can be ε-purified. (See, for example, Aumann et. al. 1984).♦

3.3 Social conformity.

Besides permitting results such as Theorem 1 (and various extensions), our framework has the advantage that it allows us to address, and provide new formulations, of different questions than currently in the game-theoretic literature, as exemplified by the following result.

An important aspect of the following result is that the number of distinct cultures required to partition the total player set into connected intervals, with the property that all players in the same interval play the same pure strategy, is bounded by a constant, $J(\varepsilon)K$, which is independent of the size of the total player set.

Theorem 2: Assume that $\quad = [0, 1]$, the unit interval. Given a real number $\varepsilon > 0$, for all societies (N, α), where:

1. Preferences satisfy continuity 2, convergence and strong anonymity.

2. For some fixed number B, for all $\omega \in \quad$, $|\alpha(\omega)| \leq B$.

there exists a real number $\eta_1(\varepsilon) > 0$, an integer $J(\varepsilon)$ and a partition of   into $C \leq J(\varepsilon)K$ connected subsets $\{\omega_c\}_{c=1}^C$ such that if $N > \eta_1(\varepsilon)$ the induced game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies with the property that, for each $c = 1, ..., C$, all players in $\;_c$ choose the same pure strategy.

Proof: Suppose not. Then, there is some $\varepsilon_0 > 0$ such that for each integer ν there is a society (N^v, α^v) and induced game $\Gamma(N^v, \alpha^v)$ with profile f^ν, where $\|f^\nu\| > \nu$ and for which no ε-equilibrium satisfies the conditions of the lemma.

We begin by noting that, by Theorem 1, for any ε_0 there exists a number $\eta_0(\frac{8}{18}\varepsilon_0)$ and ν^* such that if $\|f^\nu\| > \left\|f^{\nu^*}\right\| \geq \eta(\frac{8}{18}\varepsilon_0)$ the society (N^ν, α^ν) has an $\frac{8}{18}\varepsilon_0$-equilibrium in pure strategies. Denote an $\frac{8}{18}\varepsilon_0$ equilibrium of society (N^ν, α^ν) by s^ν with corresponding weight function $g^\nu(\cdot, \cdot; \alpha^\nu, s^\nu)$.

Given ε_0, let $\delta\left(\frac{\varepsilon_0}{18}\right)$ be the similarity parameter as defined by (Lipshitz) continuity 2 for a required payoff bound of $\frac{\varepsilon_0}{18}$. Use compactness of to write as the disjoint union of a finite number of *connected* non-empty subsets $_1, ..., _{J(\varepsilon_0)}$, each of diameter less than δ.

Assume, without loss of generality that for all ν and for all $i, j \in N^\nu$, if $\alpha(i) < \alpha(j)$ then $i < j$.

For each ν we rearrange the strategy vector s^ν in two stages:

1. Let N^ν_{kl} denote the number of players i such that $\alpha(i) \in\ _l$ and $s_{ik} = 1$. That is N^ν_{kj} denotes the number of players with attributes in the set $_l$ playing pure strategy s_k with probability 1. Then for each $j = 1, ..., J(\varepsilon_0)$ starting with the minimum integer $i \in \mathcal{N}$ such that $\alpha^\nu(i) \in\ _j$ allocate players in ascending order to strategy 1 until N^ν_{1j} players are allocate to strategy 1. Then move onto strategies $2, ..., K$. This procedure will clearly reallocate the assignment of strategies within the partition $_j$ so that the weight within $_j$ to each pure strategy remains the same.

2. We have still, however, yet to create connected subsets in which all players use the same strategy. For example we may have B players with attribute type ω where the first player is allocated to pure strategy s_k and the next $B - 1$ players to pure strategy s_{k+1}. So, the second part of the reallocation is to allocate all those players with the same type to a unique pure strategy that at least one player previously used. It is relatively easy to see that the total number of people whose pure strategy we may have to change in this second part of the reallocation is less than or equal to $(K - 1)J(\varepsilon_0)(B - 1)$.

This reallocation can be used for all ν to partition into connected subsets $^\nu_1, ..., ^\nu_C$ such that any two players $i, j \in\ ^\nu_c$ use the same pure strategy. That is, given the $\frac{8}{18}\varepsilon_0$-equilibrium s^ν we create a new strategy vector $\overline{s}^\nu$ such that if $i, j \in\ ^\nu_c$ then $\overline{s}_{ik} = \overline{s}_{jk} = 1$ for some pure strategy $s_k \in S$. We now consider the change in payoffs from this reallocation.

The first part of the reallocation process can be seen as mathematically equivalent to changing the attribute types of players. That is, instead of thinking of swapping, say, the strategies that players i and j use, we can think of it as swapping the attribute types of players i and j while keeping their strategies unchanged. Thus, stage 1 is mathematically equivalent to creating a new society $(N^\nu, \overline{\alpha}^\nu)$ satisfying $dist(\alpha^\nu(i), \overline{\alpha}^\nu(i)) < \delta\left(\frac{\varepsilon_0}{18}\right)$ for all $i \in N^\nu$. The weight function relative to strategy vector s^ν and society $\overline{\alpha}^\nu$ is given by $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)$. (We note that the profile of societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$ are equivalent to f^ν.) This interpretation and the choice of δ allows us to make use of continuity 2 by arguing that there exists a ν_1 such that for all $\nu > \nu_1$:

$$\left|h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), f^\nu)\right| < \frac{1}{18}\varepsilon_0$$

for all $t \in S$ and for all $i \in N^\nu$.

Consider, now the second part of the reallocation in which at most finite number $(K-1)L(\varepsilon_0)(B-1)$ players change pure strategy. Assume this changes the strategy vector to $\overline{s}^\nu$ and the weight function to $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu)$. Because, only a finite number of players change strategy, it can be shown, using continuity 2 (or 1), strong anonymity and an argument analogous to that in Theorem 1, that there exists a ν_2 such that for all $\nu > \nu_2$:

$$\left|h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu)\right| < \frac{4}{18}\varepsilon_0$$

for all $t \in S$ and all $i \in N^\nu$. The intuition is clear - there are only a bounded and given number of players changing strategies - note that $J(\varepsilon_0)$ can be fixed at say $\frac{1}{\delta}+1$ - and so for large enough populations the strategies of these players are inconsequential to other players. Thus for $\nu > \max\{\nu_1, \nu_2\}$:

$$\begin{gathered}
\left|h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu)\right| \leq \\
\left|h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu)\right| \\
+ \left|h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu)\right| \\
< \tfrac{1}{18}\varepsilon_0 + \tfrac{4}{18}\varepsilon_0 = \tfrac{5}{18}\varepsilon_0
\end{gathered}$$

for all $t \in S$ and all $i \in N^\nu$.

We began by noting that there exists a finite ν^* such that for all $\nu > \nu^*$ there exits an $\frac{8}{18}\varepsilon_0$-equilibrium in pure strategies s^ν and corresponding weight function $g^\nu(\cdot,\cdot;\alpha^\nu,s^\nu)$, implying that:

$$h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) \geq h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) - \frac{8}{18}\varepsilon_0$$

for all $t \in S$ and all $s \in support(s_i^\nu)$ and for all $i \in N^\nu$.

This implies, for all $\nu > \max\{\nu_1,\nu_2,\nu^*\}$ that:

$$\begin{aligned}
&h_{\overline{\alpha}^\nu(i)}(s, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\overline{s}^\nu), \overline{f}^\nu) \geq && (7)\\
&-\left|h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(s, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\overline{s}^\nu), \overline{f}^\nu)\right| && (8)\\
&-\left|h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) - h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu)\right|\\
&-\left|h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot,\cdot;\alpha^\nu,s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot,\cdot;\overline{\alpha}^\nu,\overline{s}^\nu), \overline{f}^\nu)\right|\\
&\geq -\frac{5}{18}\varepsilon_0 - \frac{8}{18}\varepsilon_0 - \frac{5}{18}\varepsilon_0 = -\varepsilon_0
\end{aligned}$$

for all $s \in support(\overline{s}_i^\nu)$ and for all $i \in N^\nu$.

The above expression, however, gives the desired contradiction. To see this we make two observations. Firstly, we repeat the analogy that swapping the strategies of players is 'equivalent' to swapping their attribute types. Thus, if stage 1 of the reallocation swaps the strategy of players i and j the above shows that player j is at an ε_0-equilibrium. Secondly, we have to consider the players who were allocated a new and different strategy in stage 2 of the reallocation. We recall that, say $b \leq B$, players were of the same attribute type and at most $b-1$ were reallocated a different strategy. However, this implies that at least one player i did retain their original strategy s_i^ν and so remain at an ε_0-equilibrium. Given the other $b-1$ players have the same attribute type they must also be at an ε_0-equilibrium.⧫

4 Conclusions

The noncooperative framework promises to be fruitful. First, the techniques developed in this paper may be useful in other applications. One potential application, currently in progress, is to games with incomplete information. So far there appears to be no major obstacles to obtaining uniform large (but finite) analogues of the sorts of results of Aumann et al. (1983). In particular, note that Lemma 1 applies to *any* game and we conjecture that an extension of our model to incomplete information would be obtained using that Lemma similarly to how it is used in this paper. Indeed, it seems that as long as we restrict to compact metric spaces and to the appropriate Lipshitzian continuity conditions, analogues of Theorem 1 will continue to hold.

Other possible applications concern the so called "Equivalence Principle" of cooperative outcomes of large ("competitive") exchange economies. In exchange economies with many players, the set of equilibrium outcomes, represented by the induced utilities of members of the economy, coincides with the core of the core of the game generated by the economy and the value outcomes; see Debreu and Scarf (1963), Aumann (1963, 1985). We conjecture that when noncooperative games derived from pregames are required to satisfy the conditions of this paper (satisfied, in spirit, for exchange economies for which the Equivalence Principle holds) and, in addition, the condition of self-sufficiency – that what a coalition of players can achieve is independent of the society in which it is embedded – then analogues of the Equivalence Principle can be obtained for large noncooperative games. More precisely, we conjecture that under self sufficiency, (approximate) strong equilibrium outcomes are close to Pareto optimal and also treat similar individuals similarly – that is, strong equilibrium outcomes have the equal treatment property.

Comparing our model with those for cooperative pregames, in spirit the frameworks have significant similarities. The cooperative pregame framework, however, is not totally satisfactory. One shortcoming is that some of the results depend on the framework itself (cf. Wooders 1994, Theorem 4, relating small group effectiveness and boundedness of average or per capita payoff). This, the inability of the pregame framework to treat widespread externalities, and a desire to highlight what drives the results, led to the introduction of 'parameterized collections' of games (cf. Kovalenkov and Wooders 1999a, 1999b). We anticipate eventually that such a framework will be introduced for noncooperative pregames.

A different direction of research may lead to more insight into social norms and the difficulties of achieving economic efficiency. When individuals mimic similar individuals, the metric that they have on 'similarity' is crucial. If a bright and highly capable young woman, living in some rural area of Canada, for example, may aspire to occupations similar to those of the more successful women in her community – nurses, bank clerks, school teachers, for example – rather than occupations similar to those of successful males in her community with similar intellectual ability – doctors, bankers, school principals, for example. It may be that if the 'similarity metrics' that people use are biased to place too much weight on similarities of gender, race, color, or religion rather than on similarities of ability, interests, and so on, there may be (non-Nash) 'stable equilibrium' outcomes that are quite different than Nash outcomes. Some of the motivation for developing the current model is to explore such issues.

Related questions concern concepts of equilibrium based on imitation and learning. Typically, such equilibrium outcomes are not Nash. This is, in some senses, at odds with our Theorem 2 and its motivation. It may be fruitful to investigate what sorts of learning and imitation dynamics would lead to the sort of ε-equilibrium shown to exist in Theorem 2.

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