How Does Probability Theory Generalize Logic?

David Miller
Department of Philosophy, University of Warwick
COVENTRY CV4 7AL UK
e-mail: d.w.miller@warwick.ac.uk

Rolando Chuaqui

Rolando Chuaqui y yo, nos encontramos una única vez, en Bahía Blanca en agosto 1992, en el Simposio Latino-Americano de Lógica Matemática. Lamentablemente, Chuaqui murió antes de mi próxima visita a América del Sur, igual que otro gran lógico latinoamericano, Carlos Alchourrón.
Chuaqui estuvo en Bahía Blanca juntos con varios alumnos que hablaron sobre aspectos de la lógica algebraica. Desde entonces, he tenido muchos deseos de visitar Chile, y fue con mucho gusto que recibí en septiembre del año pasado una invitacin del Profesor Quezada para dictar una conferencia en estas jornadas. Agradezco al comité de organización por esta invitación, y además a la Academia Británica, quien ha auspiciado mi vuelo transatlántico.
Por que estas jornadas son en memoria de Rolando Chuaqui, parece apropiado de hablar sobre un tópico de lógica algebraica. He escogido el vínculo entre la lógica deductiva y la probabilidad, un tema muy caro a los sentimientos de Chuaqui. Lamento que no tuve nunca la oportunidad de discutir personalmente estas ideas con él.

Summary: Nearly half a century ago POPPER [1959], appendices $*$iv and $*$v, presented a number of related axiomatizations of the theory of probability in each of which $p(x \mid z)$ is defined for all x and z, even $z = yy'$ (where concatenation turns out to represent meet, and the accent complementation). These systems are too little known amongst mathematicians. Popper went on to claim that his systems provide a context within which it is possible to give fully correct definitions of the relation of derivability between sentences (in the sense of the classical sentential calculus), and of the property of sentential demonstrability, by means of the formulas

$$z \vdash x \ =_{\mathrm{Df}} \ p(x \mid zx') = 1$$
$$\vdash x \ =_{\mathrm{Df}} \ p(x \mid x') = 1.$$

This claim has been challenged by STALNAKER [1970], HARPER [1975], and by LEBLANC & VAN FRAASSEN [1979]. The challenge has never been properly answered (though a start was made in POPPER & MILLER [1994], §4). The aim of this talk is to answer it, and to contrast Popper's enterprise with what is known as *probabilistic semantics*.

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The System $\mathbb{B}+$

(nontrivial)	$\exists x \exists z \;\; p(x \mid z)$	$\neq$	$p(z \mid z)$
(sub2)	$x \simeq z$	$\Longrightarrow$	$\forall y[p(y \mid x) = p(y \mid z)]$
(identity)	$p(x \mid x)$	$\leq$	$p(z \mid z)$
(monotony)	$p(xz \mid y)$	$\leq$	$p(x \mid y)$
(product)	$p(xz \mid y)$	$=$	$p(x \mid zy)p(z \mid y)$
(sum)	$p(x \mid y) + p(z \mid y)$	$=$	$p(xz \mid y) + p(x \vee z \mid y)$
(negation)	$p(y \mid z) \neq p(z \mid z)$	$\Longrightarrow$	$p(x \mid z) + p(x' \mid z) = p(z \mid z)$

Here the symbol $\simeq$ stands for the relation of **probabilistic indistinguishability**, defined by

(0) $$x \simeq z \;\; =_{\text{Df}} \; \forall y[p(x \mid y) = p(z \mid y)]$$

It is helpful to define in the same way

(1) $$z \preceq x \;\; =_{\text{Df}} \; \forall y[p(z \mid y) \leq p(x \mid y)]$$

The axiom system $\mathbb{B}+$ (Popper & Miller [1994], §4) is an adaptation of that given in Popper [1959], appendix $*$v. The principal difference is the presence of the axiom (sum). In Popper's system the symbol $\vee$ is introduced by means of the explicit definition

$$(2) \qquad x \vee z \quad =_{\mathrm{Df}} \; (x'z')'$$

It is not hard to show that the two systems are equivalent. Note that no other assumptions are made about the elements $x, y, z, \ldots$, or about the operations represented by concatentation, $\vee$, and $'$.

The properties of the axiomatic system $\mathbb{B}+$ are illuminatingly analysed through a series of weaker systems. Each system is formulated in a language with denumerably many variables $x, y, z, \ldots$, a (real-valued) functor p, and one or more operators (concatentation, and perhaps also $\vee$, $'$). The letters $\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \ldots$ are used for terms composed from variables and operators. We shall suppose that each system is supplemented by the definitions (0) and (1).

The System $\mathbb{M}$–

(downbound)	0	$\leq$	$p(x \mid z)$
(upbound)	$p(x \mid z)$	$\leq$	$p(y \mid y)$
(monotony)	$p(xz \mid y)$	$\leq$	$p(x \mid y)$
(product)	$p(xz \mid y)$	$=$	$p(x \mid zy)p(z \mid y)$

$\mathbb{M}$– is an extremely weak system, and has models in which p is interpreted by a function μ that is everywhere 0. Yet it may be shown that the concatenation operation has all the properties of a semilattice operation, in the sense that each of the following indistinguishabilities holds in $\mathbb{M}$–.

(3)	xx	$\simeq$	x
(4)	xz	$\simeq$	zx
(5)	$x(yz)$	$\simeq$	$(xy)z$

The following result shows that probabilistically indistinguishable elements are mutually exchangeable in the first argument of the functor p.

Suppose that X is a subterm of Y. Let Y° be any result of replacing or not replacing occurrences of X in Y by Z. In $\mathbb{M}$– the following substitution principle holds.

$$(6) \qquad \mathsf{X} \simeq \mathsf{Z} \implies p(\mathsf{Y} \,|\, y) = p(\mathsf{Y}^\circ \,|\, y)$$

Proof: By induction on the length of the term Y. If Y is a variable then $\mathsf{Y} = \mathsf{Y}^\circ$ and there is nothing to prove. For the induction step suppose that (6) holds for terms U and W, and that $\mathsf{Y} = \mathsf{UW}$.

Then

$$\begin{array}{llcll}
(7) & p(\mathsf{UW} \,|\, y) & = & p(\mathsf{U} \,|\, \mathsf{W}y)p(\mathsf{W} \,|\, y) & \text{by (product)} \\
(8) & & = & p(\mathsf{U}^\circ \,|\, \mathsf{W}y)p(\mathsf{W} \,|\, y) & \text{by hypothesis} \\
(9) & & = & p(\mathsf{U}^\circ\mathsf{W} \,|\, y) & \text{by (product)} \\
(10) & & = & p(\mathsf{W}\mathsf{U}^\circ \,|\, y) & \text{by (4)} \\
(11) & & = & p(\mathsf{W} \,|\, \mathsf{U}^\circ y)p(\mathsf{U}^\circ \,|\, y) & \text{by (product)} \\
(12) & & = & p(\mathsf{W}^\circ \,|\, \mathsf{U}^\circ y)p(\mathsf{U}^\circ \,|\, y) & \text{by hypothesis} \\
(13) & & = & p(\mathsf{W}^\circ\mathsf{U}^\circ \,|\, y) & \text{by (product)} \\
(14) & & = & p(\mathsf{U}^\circ\mathsf{W}^\circ \,|\, y) & \text{by (4)}
\end{array}$$

Since $\mathsf{Y}^\circ = (\mathsf{UW})^\circ = \mathsf{U}^\circ\mathsf{W}^\circ$, the proof is complete. ∎

It is well known that this result cannot be immediately extended to the second argument of p. It is the purpose of the axiom (sub2) in system $\mathbb{B}+$ above to enable the proof that probabilistic indistinguishability is a *congruence*.

The system composed of $\mathbb{M}-$ together with (sub2) may be called $\mathbb{M}\mp$. It is plain that each structure $\mathfrak{M}$ that satisfies the axioms of $\mathbb{M}\mp$ is reducible to a lower semilattice (with concatenation interpreted as meet) by factoring by the equivalence relation in $\mathfrak{M}$ that is the interpretation of $\simeq$.

If the interpretation of p in $\mathfrak{M}$ is a function taking only a single value, the quotient semilattice will consist only of a single element. Adding to $\mathbb{M}\mp$ the axiom

$$\text{(nonzero)} \qquad \exists x \exists z \;\; p(x \,|\, z) \quad \neq \quad 0$$

yielding the system $\mathbb{M}+$, ensures that each such semilattice contains at least two elements. The addition of this axiom to $\mathbb{M}-$, yielding the system $\mathbb{M}$, is sufficient to ensure also the crucial identity

$$\text{(unity)} \qquad p(y \,|\, y) \quad = \quad 1$$

In the case of (lower) semilattices it is straightforward to prove a converse to the representation theorem. Indeed if $\mathfrak{M} = \langle \mathcal{M}, \sqsubseteq \rangle$ is a lower semilattice, and $\bullet$ its meet operation (so that for all $a, c \in \mathcal{M}$, $c \sqsubseteq a$ if & only if $c = a \bullet c$), then the function μ defined on $\mathcal{M} \times \mathcal{M}$ by

$$\mu(a, c) \quad = \quad \begin{cases} 1 & \text{if } c \sqsubseteq a \\ 0 & \text{otherwise} \end{cases}$$

satisfies the axioms of the system $\mathbb{M}+$.

The System $\mathbb{D}$

(nonzero)	$\exists x \exists z \ \ p(x \mid z)$	$\neq$	0
(downbound)	0	$\leq$	$p(x \mid z)$
(upbound)	$p(x \mid z)$	$\leq$	$p(y \mid y)$
(monotony)	$p(xz \mid y)$	$\leq$	$p(x \mid y)$
(product)	$p(xz \mid y)$	$=$	$p(x \mid zy)p(z \mid y)$
(sum)	$p(x \mid y) + p(z \mid y)$	$=$	$p(xz \mid y) + p(x \vee z \mid y)$

The system $\mathbb{D}$– is what results from $\mathbb{D}$ by the deletion of axiom (nonzero).

Each of the following indistinguishabilities holds in $\mathbb{D}$–.

(15)	$x \vee x$	$\simeq$	x
(16)	$x \vee z$	$\simeq$	$z \vee x$
(17)	$x \vee (y \vee z)$	$\simeq$	$(x \vee y) \vee z$
(18)	$(x \vee z)x$	$\simeq$	x
(19)	x	$\simeq$	$(xz \vee x)$
(20)	$(x \vee z)y$	$\simeq$	$xy \vee zy.$

Here is a proof of the most important of these, the distributive law (20), using successively (product), J, (product), the semilattice laws ((3), (4), (5)), and finally J again.

$$\begin{aligned} p((x \vee z)y \,|\, w) &= p(x \vee z \,|\, yw)p(y \,|\, w) \\ &= [p(x \,|\, yw) + p(z \,|\, yw) - p(xz \,|\, w)]p(y \,|\, w) \\ &= p(xy \,|\, w) + p(zy \,|\, w) - p((xz)y \,|\, w) \\ &= p(xy \,|\, w) + p(zy \,|\, w) - p((xy)(zy) \,|\, w) \\ &= p(xy \vee zy \,|\, w) \end{aligned}$$

The system $\mathbb{D}$ may also be augmented by

(sub2) $$x \simeq z \quad \Longrightarrow \quad \forall y[p(y \,|\, x) = p(y \,|\, z)]$$

to yield a system $\mathbb{D}+$ in which the relation of probabilisitic indistinguishability is a congruence. Each structure $\mathfrak{D}$ that satisfies the axioms of $\mathbb{D}+$ is reducible to a distributive lattice (with concatenation interpreted as meet and $\vee$ as join) by factoring by the equivalence relation in $\mathfrak{D}$ that is the interpretation of $\simeq$.

We now return to the system $\mathbb{B}+$, where similar results may be obtained. In particular we may establish the following identities, inequalities, and indistinguishabilities.

$$
\begin{array}{llcl}
(21) & p(x \mid y) + p(x' \mid y) & = & p(z \mid y) + p(z' \mid y) \\
(22) & p(x'' \mid z) & = & p(x \mid z) \\
(23) & 0 = p(zz' \mid y) & \leq & p(x \mid y) \\
(24) & p(x \mid y) & \leq & p(z \vee z' \mid y) = 1 \\
(25) & x \vee z & \simeq & (x'z')'
\end{array}
$$

In the light of (23) and (24) we may choose some variable y and adopt the definitions

$$
\begin{array}{llcl}
(26) & s & =_{\mathrm{Df}} & yy' \\
(27) & t & =_{\mathrm{Df}} & y \vee y'
\end{array}
$$

of selfcontradictory elements s and tautological elements t. These of course depend on y, but any two selfcontradictory elements are indistinguishable, as are any two tautological elements.

The following are a selection from the many striking results that we can prove about s and t.

$$
\begin{array}{lrcl}
(28) & p(y \,|\, s) & = & p(t \,|\, y) \\
(29) & z \simeq s \iff p(s \,|\, z) & \neq & 0 \\
(30) & z \simeq s \iff \forall x\, p(x \,|\, z) & = & 1 \\
(31) & z \simeq s \iff p(z' \,|\, z) & \neq & 0 \\
(32) & z \simeq s \iff p(z' \,|\, z) & = & p(z \,|\, z) \\
(33) & x \simeq t \iff \forall z\, p(x \,|\, z) & = & 1 \\
(34) & x \simeq t \iff p(x \,|\, x') & \neq & 0 \\
(35) & x \simeq t \iff p(x \,|\, x') & = & p(x \,|\, x) \\
(36) & z \preceq x \iff p(x \,|\, zx') & = & 1 \\
(37) & z \preceq x \iff p(x \,|\, zx') & \neq & 0
\end{array}
$$

Each structure $\mathfrak{B}$ that satisfies the axioms of $\mathbb{B}+$ is reducible to a Boolean algebra (with concatenation interpreted as meet, $\vee$ as join, $'$ as complement, s as zero, and t as unit) by factoring by the equivalence relation in $\mathfrak{B}$ that is the interpretation of $\simeq$.

In the case of $\mathbb{B}+$ it is possible to prove a weak converse (which transfers also to $\mathbb{D}+$) to the representation theorem. Indeed if $\mathfrak{B}$ $= \langle \mathcal{B}, \sqsubseteq \rangle$ is a Boolean lattice, and Δ a maximal filter on it, then the function μ defined on $\mathcal{B} \times \mathcal{B}$ by

$$
\mu(a, c) \quad = \quad \begin{cases} 1 & \text{if } a \in \Delta \text{ or } c \notin \Delta \\ 0 & \text{otherwise} \end{cases}
$$

satisfies the axioms of the system $\mathbb{B}+$.

The Central Problem

It is obvious from these results that there is a close connection between the theory $\mathbb{B}+$ and classical sentential logic (and parallel connections between $\mathbb{M}+$ and $\mathbb{D}+$ and what we might call conjunctive and distributive logic). How are we to articulate this connection?

Popper claims that classical sentential derivability and demonstrability are actually definable within $\mathbb{B}+$, and offers the definitions

$$(38) \qquad z \vdash x \quad =_{\mathrm{Df}} \quad z \preceq x$$

$$(39) \qquad \vdash x \quad =_{\mathrm{Df}} \quad \forall z \;\; z \vdash x$$

In view of (36) and (unity), more immediate equivalences are

$$(40) \qquad z \vdash x \iff p(x \,|\, zx') = 1$$

$$(41) \qquad \vdash x \iff p(x \,|\, x') = 1$$

The correctness of (41) — and implicitly of (38)–(40) — has been challenged by LEBLANC & VAN FRAASSEN [1979], p. 369 (see also their note 5), on the grounds that we can easily construct a function μ, satisfying the axioms of $\mathbb{B}+$, under which $\mu(x, x') = 1$ if & only if x is true. Hence, they say, the formula $p(x \,|\, x') = 1$ cannot be a correct rendering of 'x is demonstrable [or necessary]'.

That such a function μ can indeed be constructed is the content of the weak converse above to the representation theorem for $\mathbb{B}+$. What does it show? According to LEBLANC & VAN FRAASSEN [1979], *loc.cit.*, it shows that the system $\mathbb{B}+$ must be strengthened by an additional assumption called B7.

> By a *state description of* PC [propositional calculus] understand any wff of PC of the form
>
> $$(\cdots(\pm P_1 \;\&\; \pm P_2 \;\&\; \cdots) \;\&\; \pm P_p,$$
>
> where (i) p is a positive integer, (ii) $P_1, P_2, \ldots P_p$ are distinct propositional variables, and (ii) for each i from 1 through [p], $\pm P_i$ is either P_i or $\sim P_i$. [footnote] Our requirement will then run:
>
> > B7. *If* A *is a state description of* PC, *then* $Pr(\sim A, A) = 0$.

A Popper function Pr is a function for which $\mathbb{B}+$ holds. LEBLANC & VAN FRAASSEN announce the theorem

> *If* Pr *is a Popper function that meets requirement B7 … and* $Pr(A, \sim A) = 1$, *then* A *is a tautology.*

It must be said that B7 is a most unattractive postulate to have to introduce into an axiomatic system such as $\mathbb{B}+$. Unlike any of the other axioms it makes explicit reference to propositional variables. It is fortunately not necessary.

In FIELD [1977], LEBLANC [1979], LEBLANC [1983], and many related works alternative ways have been given of characterizing various logical relations. For example,

¶ A is logically true if & only if $Pr(A, B) = 1$ for all B and for all probability functions Pr [FIELD, LEBLANC [1979]]

¶ A is entailed by C if & only if $Pr(C, B) \leq Pr(A, B)$ for all B and for all probability functions Pr [FIELD]

¶ A is entailed by C if & only if, for all B and for all probability functions Pr, $Pr(A, B) = 1$ if $Pr(C, B) = 1$ [LEBLANC [1979]]

Each of these characterizations quantifies over all probabiltiy functions Pr, and is in an obvious sense external to the axiomatic system. This is not perhaps surprising, since these authors see probability theory (in particular $\mathbb{B}+$) as a generalization of, perhaps even an improvement on, traditional truth-table semantics. LEBLANC [1983], p. 264 writes of the result that every model of Popper's axioms is reducible to a Boolean algebra:

> The earliest theorem that probabilistic semantics boasts of is in Popper (1959), Appendix ∗v. A soundness theorem, it is roughly to the effect that if a boolean identity $A = B$ ... is provable by means of the 'fourth set' of Huntington (1933), then $P(A/C) = P(B/C)$ for any statement C and any binary probability function P meeting Popper's constraints.

This is by no means my view. I see Popper's result not as proving **soundness** for a **new** semantics for an **old** formulation of propositional logic, but as proving **completeness** for an **old** semantics for a **new** formulation of propositional (meta)logic: indeed, it establishes that the axiomatic system is strong enough to yield $\mathsf{X} \simeq \mathsf{Z}$, the probabilistic indistinguishability of the terms X and Z, whenever $\mathsf{X} = \mathsf{Z}$ is an identity of Boolean algebra.

This is not to say that there is anything incorrect about probabilistic semantics; only that there is not anything incorrect about probabilistic **syntax** either.

Where Leblanc & van Fraassen have gone wrong, perhaps understandably given how little Popper said about this matter, is in not seeing that the terms 'x', 'y', 'z', ... that appear in the second argument of the functor p may be understood not only as names of sentences but as names of sentence forms. More explicitly we may use the letters 'X', 'Y', 'Z', ... to stand for sentence forms, and assign the value r to $p(\mathsf{X} \mid \mathsf{Z})$ if & only if for every (uniform) substitution instance of terms to the variables in X and Z, the outcome $p(\mathsf{X}^{\circ} \mid \mathsf{Z}^{\circ})$ has the value r. (The only values of r for which this will be of interest are 0 and 1.)

In these terms, let us rewrite (41), which defines $\vdash x$ as $p(x \mid x') = 1$, as

$$\text{(42)} \qquad \vdash \mathsf{X} \iff p(\mathsf{X} \mid \mathsf{X}') = 1$$

Given the presence of the axiom (nontrivial), any function μ that interprets the functor p in $\mathbb{B}+$ will take more than one value. In these circumstances it is obvious that there is always a substitution instance of $p(\mathsf{X} \mid \mathsf{X}')$ that does not take the value 1; and so $\vdash \mathsf{X}$ does not hold. This is indeed the point of the criticism levelled by LEBLANC & VAN FRAASSEN. But if we write $(\mathsf{Z}\mathsf{Z}')'$, for example, for X in (42), the right side becomes $p((\mathsf{Z}\mathsf{Z}')' \mid (\mathsf{Z}\mathsf{Z}')'') = 1$, which indeed is a theorem of $\mathbb{B}+$. Thus we may conclude, quite correctly, that $(\mathsf{Z}\mathsf{Z}')'$ is demonstrable in classical logic. Exactly similar considerations prevail in the understanding of (40). If x and z are distinct variables, then the formula $z \preceq x$ is not generally true. But then $z \vdash x$ is not valid in classical (or any other non-trivial) logic.

All these results hold, to be sure, for each function μ that interprets p in $\mathbb{B}+$. But it is not necessary to consider every such μ, or even more than one, in order to determine whether or not $\mathsf{Z} \vdash \mathsf{X}$. If $\mathsf{Z} \nvdash \mathsf{X}$ then for each μ there will be substitution instances of X and Z that provide a suitable counterexample.

There is, therefore, a genuine probabilistic generalization of classical **meta**logic. But what would be a probabilistic generalization of logic?

We may take the matter a little further by considering how a probabilistic generalization of intuitionistic logic might be formulated. The following system $\mathbb{H}$+ of axioms ('H' for Heyting) is equivalent to one developed in 1981 and published (with several misprints) by Miller & Popper [1986]. The letter 's' is here a constant symbol, whose referent is eventually identified with the zero element of the resulting lattice. It will be recalled that $z \preceq x$ was defined in (1) by the formula $\forall y[p(z \mid y) \leq p(x \mid y)]$.

The System $\mathbb{H}$+

(nontrivial)	$\exists x \exists z \;\; p(x \mid z)$	$\neq$	$p(z \mid z)$
(sub2)	$x \simeq z$	$\Longrightarrow$	$\forall y[p(y \mid x) = p(y \mid z)]$
(identity)	$p(x \mid x)$	$\leq$	$p(z \mid z)$
(monotony)	$p(xz \mid y)$	$\leq$	$p(x \mid y)$
(product)	$p(xz \mid y)$	$=$	$p(x \mid zy)p(z \mid y)$
(sum)	$p(x \mid y) + p(z \mid y)$	$=$	$p(xz \mid y) + p(x \vee z \mid y)$
(zero†)	$p(x \mid z)$	$\leq$	$p(z \mid s)$
(zero‡)	$p(x \mid z) \neq p(x \mid x)$	$\Longrightarrow$	$p(x \mid s) + p(s \mid z) = p(x \mid x)$
(conditional†)	$p(x \to y \mid xz)$	$\leq$	$p(y \mid xz)$
(conditional‡)	$xy \preceq z$	$\Longrightarrow$	$x \preceq y \to z$

Each model of these axioms can be reduced, in the same way as before, to a Heyting algebra. If the two axioms (conditional†) and (conditional‡) for the conditional are omitted, the residual system characterizes the class of distributive lattices with zero.

Other system of probability axioms for intuitionistic logic have been given in

VAN FRAASSEN, B.C. [1981]. 'Probabilistic Semantics Objectified: I. Postulates and Logics'. *Journal of Philosophical Logic* **10**, pp. 371-394

MORGAN, C.G. & LEBLANC, H. [1983a]. 'Probabilistic Semantics for Intuitionistic Logic'. *Notre Dame Journal for Formal Logic* **XXIV**, pp. 161-80

MORGAN, C.G. & LEBLANC, H. [1983b]. 'Probability Theory, Intuitionism, Semantics, and the Dutch Book Argument'. *Notre Dame Journal for Formal Logic* **XXIV**, pp. 289-304

None of these sytems, including our own, is entirely satisfactory. The system of MORGAN & LEBLANC contains the axiom

$$(43) \qquad p(x \to y \,|\, z) \quad = \quad p(y \,|\, xz)$$

which does not hold generally for the classical conditional (that is, it fails in $\mathbb{B}+$ when we write $x \to y = x' \vee y$); though one half of it,

$$(44) \qquad p(x \to y \,|\, z) \quad \geq \quad p(y \,|\, xz)$$

is classically correct. Indeed, a classic result of D.K. LEWIS, extended to $\mathbb{B}+$ by LEBLANC & ROEPER, shows that (43) does not hold for any operation $\to$. The system of VAN FRAASSEN, on the other hand, from the outset simply excludes the possibility that probabilistically indistinguishable elements might not be intersubstitutable in the second argument of p. In this way it manages to miss some of the more interesting problems.

The main bother with the system $\mathbb{H}+$ is the axiom (conditional‡), which more or less postulates outright that probabilistic indistinguishability is a congruence, and comes close to postulating outright the principal relative pseudocomplement properties for the operation $\to$. To be sure, it is not as strong as

$$(45)\qquad p(xy \mid w) \leq p(z \mid w) \implies p(x \mid w) \leq p(y \to z \mid w)$$

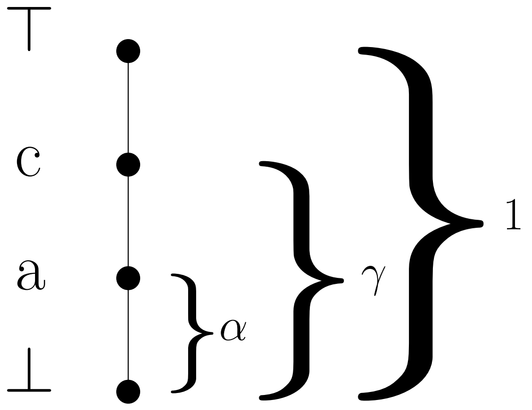
but it is still an unattractive axiom.

It may be asked whether this axiom (conditional‡) can not be replaced by the converse of (conditional†), so that we assume about the conditional $\to$ only the identity

$$\text{(conditional}\star\text{)}\qquad p(x \to y \mid xz) \quad = \quad p(y \mid xz)$$

(which is classically correct). The answer is **No**. Unlike (43), the identity (conditional$\star$) is insufficient to ensure that probabilistic indistinguishability is a congruence. We show this by constructing a model in which hold all the axioms of the system $\mathbb{H}+$, except (conditional‡), and also (conditional$\star$) holds.

Consider the 4-element Heyting algebra depicted below, where ∂ is a (pseudo)metric operation and $\alpha \leq \gamma < 1$.



Define

$$\mu(x, z) = \frac{\partial(xz, \bot)}{\partial(z, \bot)} \quad \text{if } z \neq \bot$$

$$\mu(x, \bot) = 1$$

The following tables identify for each row x and column z the element $z \longrightarrow x$ and the value of $\mu(x, z)$.

$z \longrightarrow x$	$\top$	**c**	**a**	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$
c	**c**	$\top$	$\top$	$\top$
a	**a**	**a**	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$

$\mu(x, z)$	$\top$	**c**	**a**	$\bot$
$\top$	1	1	1	1
c	γ	1	1	1
a	α	α/γ	1	1
$\bot$	0	0	0	1

All the axioms of $\mathbb{H}+$ hold in this model if $\alpha < \gamma$. If $\alpha = \gamma < 1$, however, the controversial axiom (conditional‡) fails, and probabilistic indistinguishability is no longer a congruence. For **a** and **c** are probabilistically indistinguishable, yet **c** $\rightarrow$ **a**, which is identical with **a**, is not indistinguishable from **c** $\rightarrow$ **c**, which is identical with $\top$.

Indeed

$$\forall y[\mu(\top \mathrm{c}, y) \leq \mu(\mathrm{a}, y)]$$

(since $\top\mathrm{c} = \mathrm{c}$), yet

$$\mu(\top, \top) = 1 > \alpha = \mu(\mathrm{a}, \top) = \mu(\mathrm{c} \rightarrow \mathrm{a}, \top).$$

It is worth noting that when $\alpha < \gamma < 1$, so that the Heyting algebra is a model for the whole of $\mathbb{H}+$, the classically correct inequality (44) fails. For

$$\mu(\mathrm{c} \rightarrow \mathrm{a}, \top) = \mu(\mathrm{a}, t) = \alpha < \alpha/\gamma = \mu(\mathrm{a}, \mathrm{c}) = \mu(\mathrm{a}, \mathrm{c}\top)$$

(44) therefore plays a role in the theory of probability somewhat similar to that played by Peirce's law in logic. It would be interesting if it turned out that the addition of this axiom to $\mathbb{H}+$ were to produce a system logically equivalent to $\mathbb{B}+$.

What is clear, is this: in any class of structures in which there exists an operation, in this case the conditional, that is not **continuous**, a strong axiom will be needed to ensure that probabilistic indistinguishability is a congruence. The conditional is not continuous in any Heyting algebra (unless it is a Boolean algebra), since there may be two adjacent elements x, z of the algebra, such as a and c in our example, such that $x \rightarrow y$ is not adjacent to $z \rightarrow y$, or $y \rightarrow x$ is not adjacent to $y \rightarrow z$. (Neither $\mathrm{a} \rightarrow \mathrm{a}$ nor $\mathrm{c} \rightarrow \mathrm{c}$, each of which is $\top$, is adjacent to $\mathrm{c} \rightarrow \mathrm{a}$, which is a.)

It is intriguing too that those varieties of algebras that we have succeeded in giving decent probabilistic characterizations of — lower semilattices, distributive lattices (with or without zero), Boolean algebras — these were, with Heyting algebras, almost exactly the varieties singled out also by van Fraassen [1981] for probabilistic treatment — are also those varieties that are, in the terminology of Kalicki & Scott [1955], **equationally complete**. What this means is that it is impossible to add to the theory of the variety any equation that does not trivialize the theory. The variety of lattices is not equationally complete, since we can add the modular law or the distributive law. Heyting algebras are not equationally complete, for we may add the identity $x'' = x$, for example, and obtain the subvariety of Boolean algebras.

Otherwise put: if A is an equation and $\mathfrak{A}$ an algebra in which every substitution instance of A is true, then every substitution instance of A **is true in every other algebra in the variety**.

Equational completeness is crucial to the present response to Stalnaker, Harper, and Leblanc & van Fraassen. In $\mathbb{H}+$ it is not admissible to define $z \vdash x$ by the identity $p(x \mid zx') = 1$, for there are models of $\mathbb{H}+$ (Boolean algebras) in which every substitution instance of $x'' \simeq x$ holds , even though $x'' = x$ is not a law of intuitionistic logic.

Yet equationally complete varieties and varieties in which all operations are continuous are not the same. Both meet and join are continuous in modular lattices, but modular lattices do not comprise an equationally complete variety.

More work is needed on the problem of why it is so difficult to give a decent probabilistic characterization of intuitionistic logic . . . .