Reducing Path TSP to TSP

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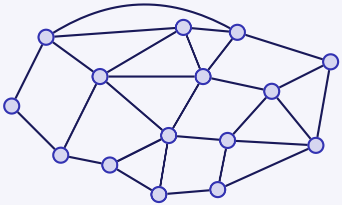
- $G = (V, E)$: undirected graph
- $\ell : E \to \mathbb{R}_{\geq 0}$: edge lengths

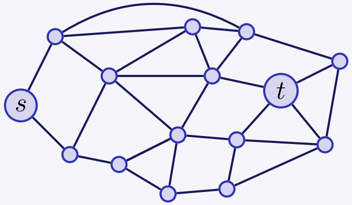
TSP: start = end



Find shortest connected sub(multi)graph s.t.:

$$\deg(v) \text{ is even } \quad \forall v \in V \ .$$

Path TSP: start $\neq$ end



Find shortest connected sub(multi)graph s.t.:

$$\deg(v) \text{ is } \begin{cases} \text{even} & \forall v \in V \setminus \{s,t\} \ , \\ \text{odd} & \forall v \in \{s,t\} \ . \end{cases}$$

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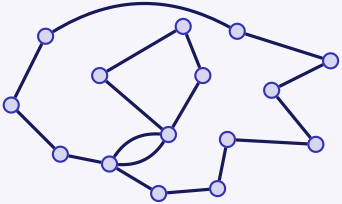

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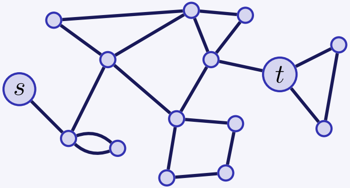

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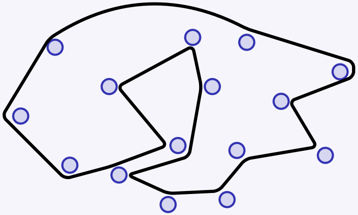

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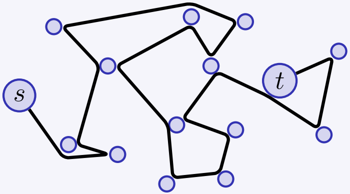

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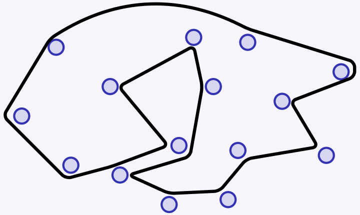
- $G = (V, E)$: undirected graph
- $\ell : E \to \mathbb{R}_{\geq 0}$: edge lengths → Graph (Path) TSP: $\ell \equiv 1$

TSP: start = end



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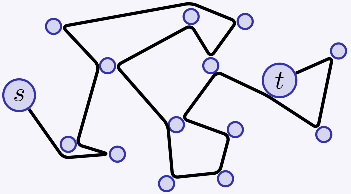

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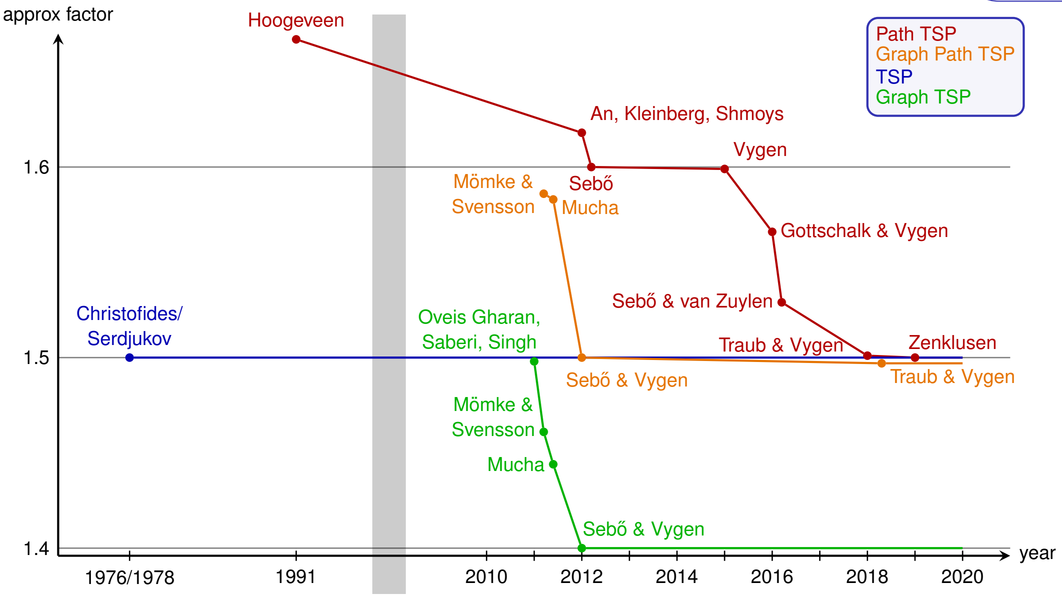
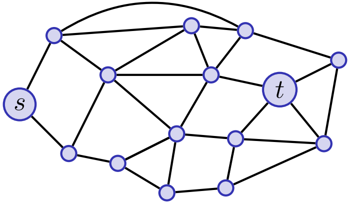
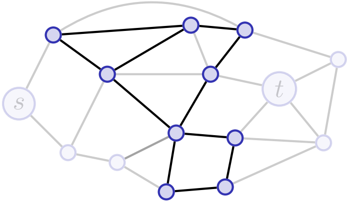

approx factor
Path TSP
Graph Path TSP
TSP
Graph TSP
Hoogeveen
An, Kleinberg, Shmoys
Vygen
Sebő
Mömke &
Svensson
Mucha
Gottschalk & Vygen
Sebő & van Zuylen
Christofides/
Serdjukov
Oveis Gharan,
Saberi, Singh
Traub & Vygen
Zenklusen
Sebő & Vygen
Traub & Vygen
Mömke &
Svensson
Mucha
Sebő & Vygen
1.6
1.5
1.4
year
1976/1978
1991
2010
2012
2014
2016
2018
2020

Theorem

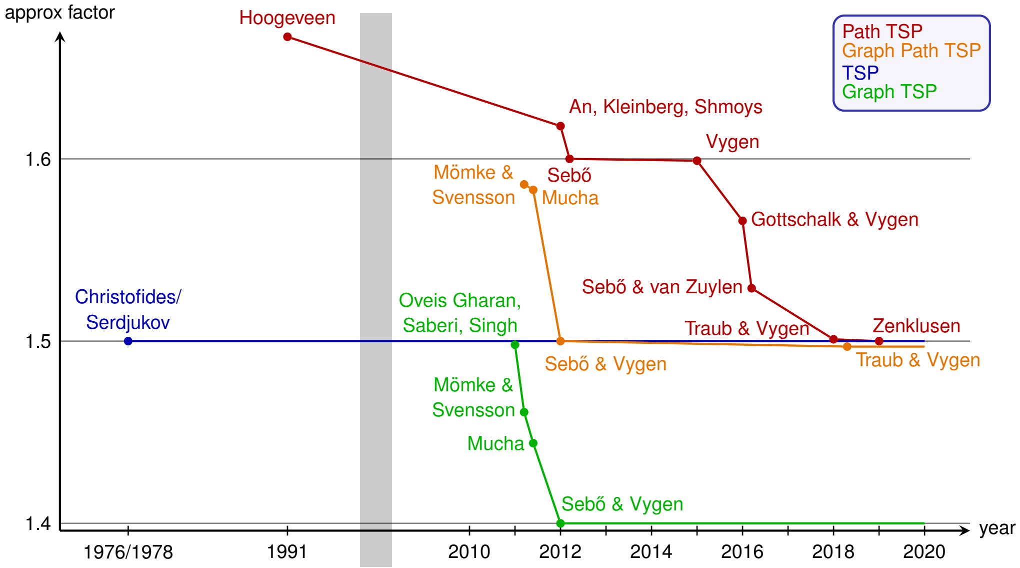
Let $\mathcal{A}$ be α-approx for TSP and let $\varepsilon > 0$. $\implies$ $\exists\ (\alpha + \varepsilon)$-approx $\overline{\mathcal{A}}$ for Path TSP.

- $\overline{\mathcal{A}}$ performs strongly polynomially many elementary operations and oracle calls to $\mathcal{A}$.
- $\overline{\mathcal{A}}$ calls $\mathcal{A}$ only on instances that are subgraphs of original instance.





One implication: $(1.4 + \varepsilon)$-approx for Graph Path TSP



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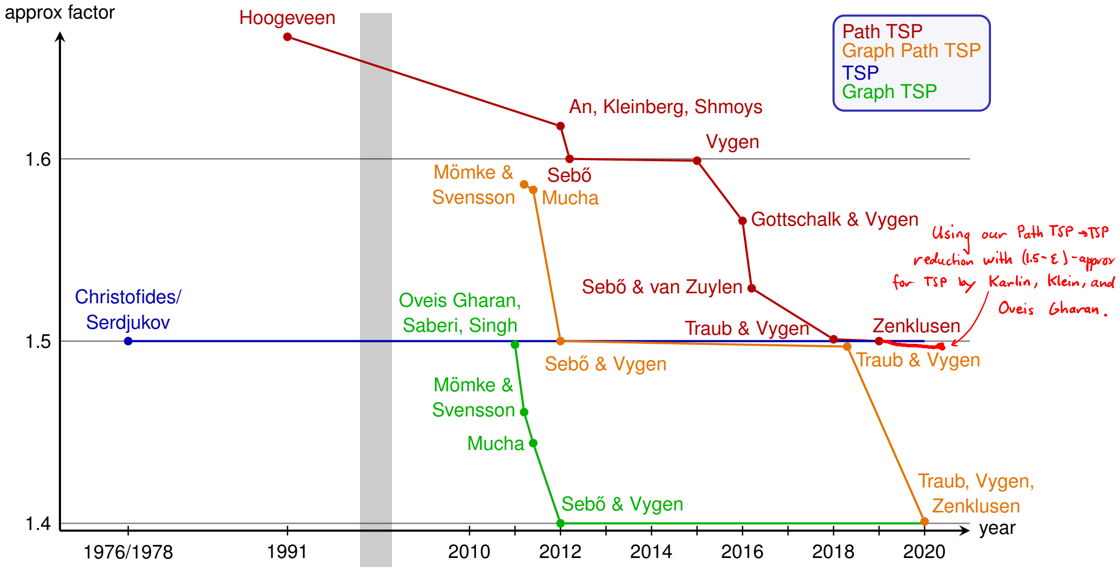
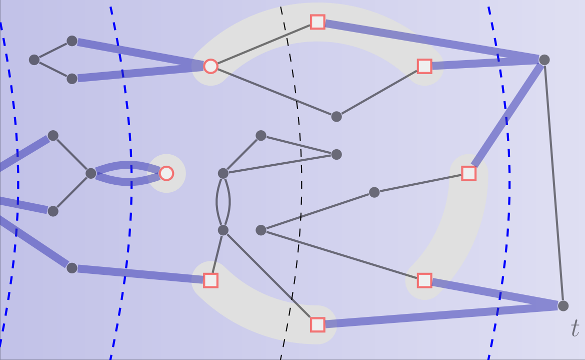
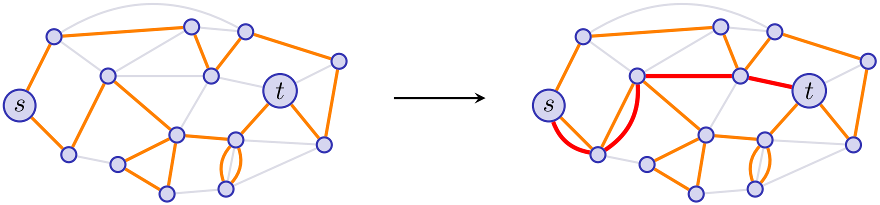




The reduction

Compute α-approx $\mathcal{A}(G)$ for TSP and add shortest s-t path $P_{s,t}$.



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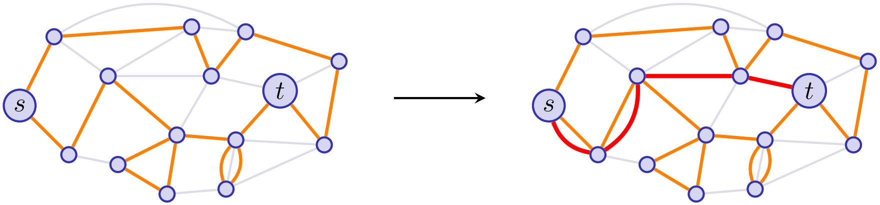

Analysis

Basic observation: $|\ell(\mathrm{OPT}_{\mathrm{TSP}}) - \ell(\mathrm{OPT})| \leq \mathrm{dist}(s,t)$.

$$\Longrightarrow \quad \ell\left(\mathcal{A}(G) \cup P_{s,t}\right) \leq \alpha \cdot \ell(\mathrm{OPT}_{\mathrm{TSP}}) + \mathrm{dist}(s,t) \leq \alpha \cdot \ell(\mathrm{OPT}) + (1+\alpha) \cdot \mathrm{dist}(s,t)$$
$$\leq \alpha \cdot \ell(\mathrm{OPT}) + 2\alpha \cdot \mathrm{dist}(s,t) \ .$$

Gives $(1+\varepsilon)\alpha$-approx for Path TSP if $\mathrm{dist}(s,t) \leq \frac{\varepsilon}{2} \cdot \ell(\mathrm{OPT})$.

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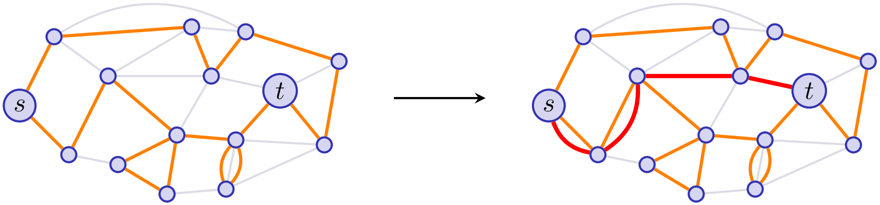

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Gives $(1+\varepsilon)\alpha$-approx for Path TSP if $\mathrm{dist}(s,t) \leq \frac{\varepsilon}{2} \cdot \ell(\mathrm{OPT})$.

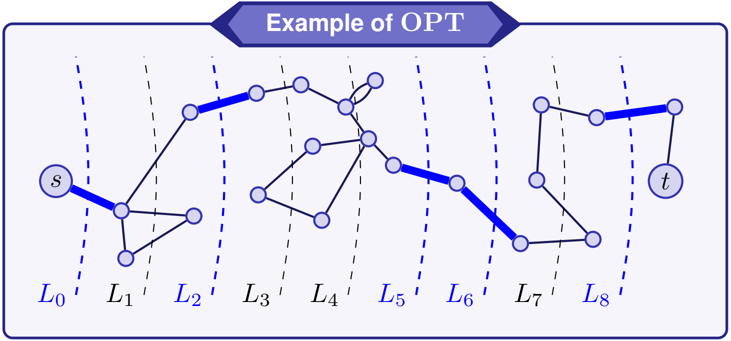
What if $\mathrm{dist}(s,t)$ is large?

Warm-up: Reducing to $\mathrm{dist}(s,t) \leq (\frac{1}{3}+\delta)\ell(\mathrm{OPT})$

[Blum, Chawla, Karger, Lane, Meyerson, Minkoff, 2007], [Traub & Vygen, 2018]

Assume $\mathrm{dist}(s,t) > \left(\frac{1}{3}+\delta\right)\ell(\mathrm{OPT})$.

Warm-up phase I: Assume instance is graphic ($\ell \equiv 1$).



$L_i := \{v \in V : \mathrm{dist}(s,v) \leq i\}$

s-t cuts contain odd number of OPT-edges.

$\implies |\delta(L_i) \cap \mathrm{OPT}| = 1$ or $|\delta(L_i) \cap \mathrm{OPT}| \geq 3$.

$\implies$ #L_i with 1 OPT-edge

$\geq \frac{1}{2}(3\,\mathrm{dist}(s,t) - |\mathrm{OPT}|)$

$> \frac{3}{2}\delta|\mathrm{OPT}|$.

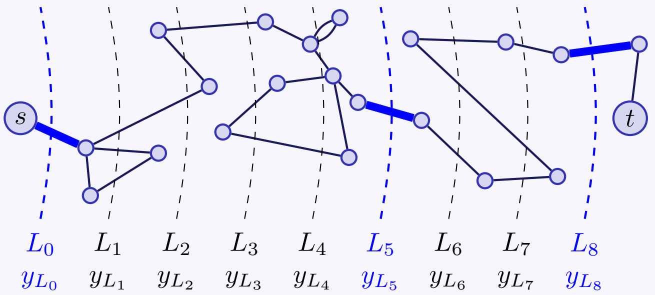
Can use dynamic program (DP) to "guess" ▬ from left to right and recurse on resulting subinstances.

[Blum, Chawla, Karger, Lane, Meyerson, Minkoff, 2007], [Traub & Vygen, 2018]

Assume $\mathrm{dist}(s,t) > \left(\frac{1}{3}+\delta\right)\ell(\mathrm{OPT})$.

Warm-up phase II: General instance ($\ell \in \mathbb{R}^{E}_{\ge 0}$).

Example of OPT

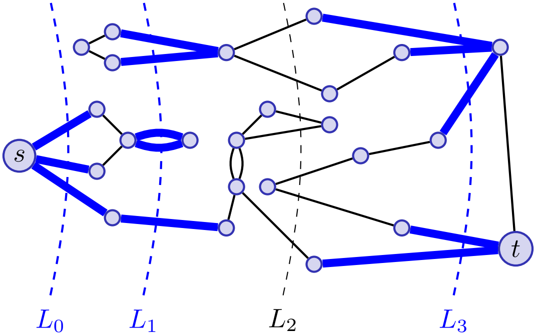


$(L_i, y_i)_i$: optimal dual sol to shortest s-t path problem:

- $(L_i)_i$: chain of s-t cuts,
- $y_{L_i} \in \mathbb{R}_{\ge 0}\ \forall i$,
- $\sum_i y_{L_i} = \mathrm{dist}(s,t)$,
- $\sum_{i: e\in\delta(L_i)} y_i \le \ell(e)\ \forall e \in E$.

$\Longrightarrow \ell(\rule{1em}{2pt}) \ge \frac{3}{2}\delta\ell(\mathrm{OPT})$.

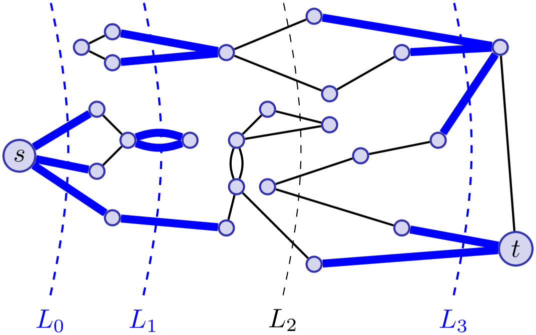
Again, recursive DP to “guess” $\rule{1em}{2pt}$ guesses $\frac{3}{2}\delta$-fraction of $\ell(\mathrm{OPT})$ of Path TSP in each iteration.



Consider OPT-edges in cuts L_i with $|\delta(L_i) \cap \mathrm{OPT}| \leq \frac{4}{\varepsilon}$:

$\rightarrow$ if $\mathrm{dist}(s,t) > \frac{\varepsilon}{2} \cdot \ell(\mathrm{OPT}) \implies \ell($▬$) \geq \frac{\varepsilon}{4} \cdot \ell(\mathrm{OPT})$.

We want to use again DP to “guess” ▬.



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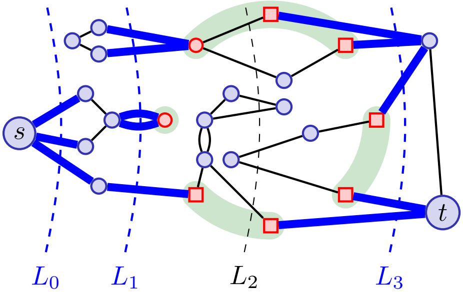
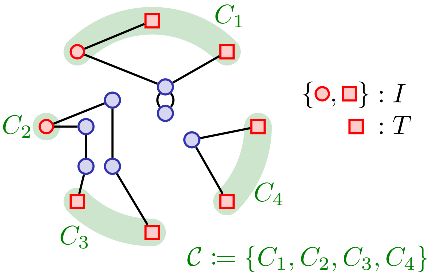
Key challenge

Subproblems are not Path TSP problems anymore!





Definition: Interface $\Phi = (I, T, \mathcal{C})$

- $T \subseteq I \subseteq V$ with $|T|$ even,
- $\mathcal{C} \subseteq 2^I$: partition of I.

We call $|I|$ the size of Φ.

Definition: Φ-tour $F \subseteq E$

- $T = \mathrm{odd}(F)$, i.e., F is a T-join,
- F connects each vertex to I,
- $C \in \mathcal{C}$ lie in same conn. comp. of (V, F).

Theorem [TVZ, 2020]

Given:
- α-approx $\mathcal{A}$ for TSP on subgraphs of G,
- interface $\Phi = (I, \boldsymbol{T} = \boldsymbol{\emptyset}, \mathcal{C})$.
- $\varepsilon > 0$.

$\Longrightarrow$ $(1+\varepsilon)\alpha$-approx for Φ-TSP can be computed in $|V|^{O(\frac{|I|}{\varepsilon})}$ time.

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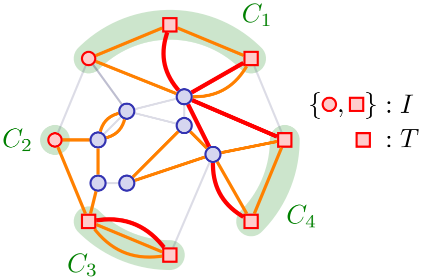
This leads to following algorithm for general Φ-tours.

Algorithm $\mathcal{A}_1$ for Φ-tours

1. Invoke Theorem wrt $\overline{\Phi} = (I, \emptyset, \mathcal{C}) \to \overline{\Phi}$-tour $\boldsymbol{F} \subseteq E$.
2. Compute shortest T-join $\boldsymbol{J} \subseteq E$.
3. Return $\boldsymbol{F} \cup \boldsymbol{J}$.



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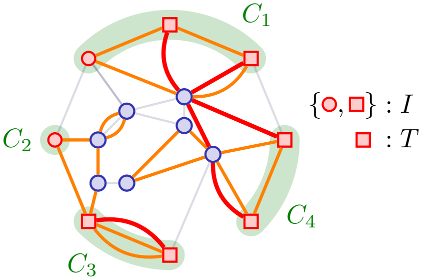
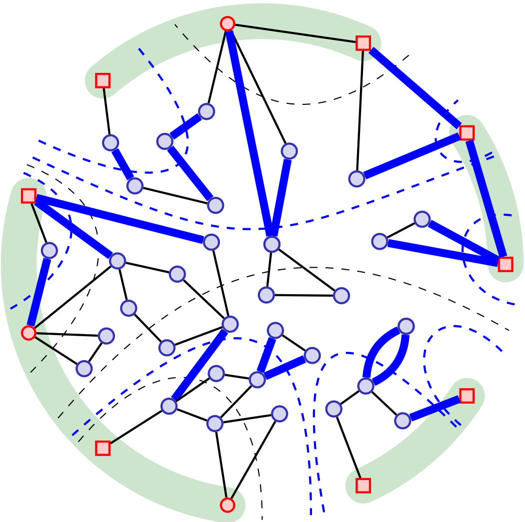


$$\rightarrow \quad \ell(F \cup J) \leq \left(1 + \tfrac{\varepsilon}{3}\right) \cdot \alpha\Big(\ell(\mathrm{OPT}_\Phi) + 2\ell(J)\Big) \Longrightarrow (1+\varepsilon)\alpha\text{-approx if } \ell(J) \leq \frac{\varepsilon}{4} \cdot \ell(\mathrm{OPT}_\Phi) \ .$$



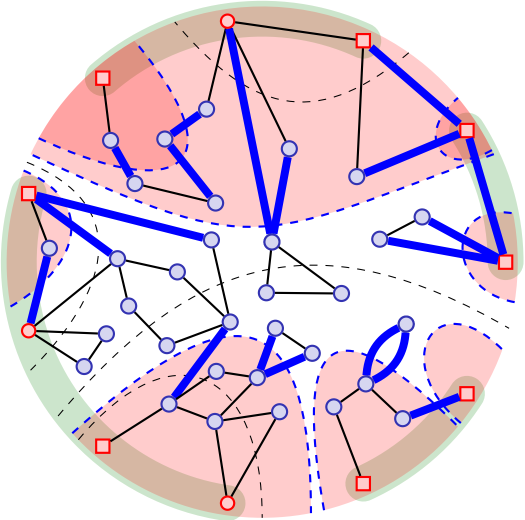
Opt. dual sol $(L_i, y_i)_i$ of min T-join problem satisfies:

- $(L_i)_i$: laminar family of T-cuts,
- $y_{L_i} \in \mathbb{R}_{\geq 0}\ \forall i$,
- $\sum_i y_{L_i} = \ell(J)$,
- $\sum_{i:e\in\delta(L_i)} y_{L_i} \leq \ell(e)\ \forall e \in E$.

We use DP to "guess" OPT-edges (and interfaces) in cuts L_i with $|\delta(L_i) \cap \mathrm{OPT}| \leq \frac{8}{\varepsilon}$.

$\implies \ell(\text{▬}) \geq \frac{\varepsilon}{8} \cdot \ell(\mathrm{OPT})$.

- DP "follows" laminarity of guessed cuts (- - -).
- Interface sizes of subproblems (when recursing) increase by factor $O(\frac{1}{\varepsilon})$.



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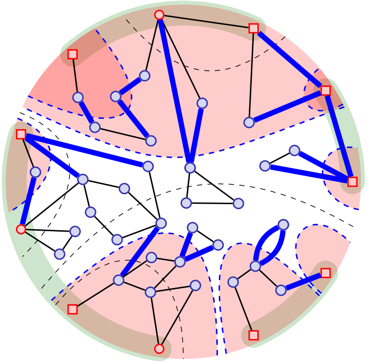


Compute shortest T-join J.

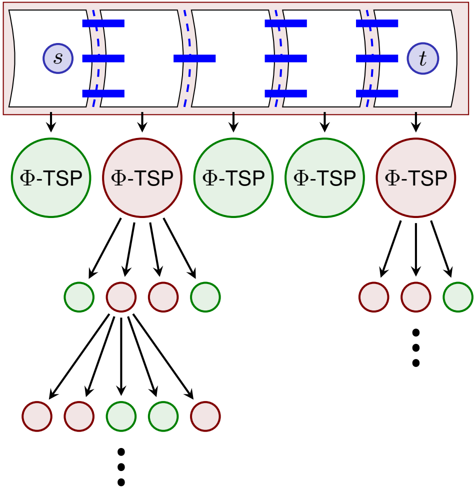
$\ell(J) \leq \frac{\varepsilon}{4} \cdot \ell(\mathrm{OPT})$?

yes: Get $(1+\varepsilon)\alpha$-approx using TSP algo and adding J.

no: Use DP to guess OPT-edges of total length $\geq \frac{\varepsilon}{8} \cdot \ell(\mathrm{OPT})$ and recurse on subproblems.

○ (green): min T-join shorter than $\frac{\varepsilon}{4}$-fraction of opt.

○ (red): min T-join longer than $\frac{\varepsilon}{4}$-fraction of opt.



interface size: $\lvert I \rvert$	# possible interfaces per subproblem	time to approx each ○-instance	total length of OPT-edges in subproblems
2	1	$n^{O(1)}$	$\ell(\mathrm{OPT})$
$O\left(\frac{1}{\varepsilon}\right)$	$n^{O\left(\frac{1}{\varepsilon}\right)}$	$n^{O\left(\frac{1}{\varepsilon^2}\right)}$	$\leq \left(1 - \frac{\varepsilon}{8}\right) \ell(\mathrm{OPT})$
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		⋮	

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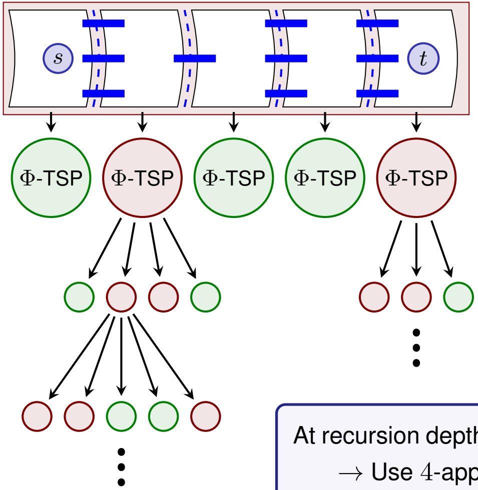

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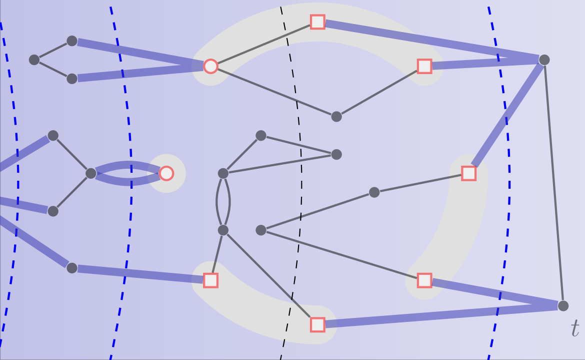
$O\left(\frac{1}{\varepsilon^2}\right)$	$n^{O\left(\frac{1}{\varepsilon^2}\right)}$	$n^{O\left(\frac{1}{\varepsilon^3}\right)}$	$\leq \left(1-\frac{\varepsilon}{8}\right)^2\ell(\mathrm{OPT})$
	⋮		

At recursion depth $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$, impact of ○-instances becomes negligible.

$\rightarrow$ Use 4-approx based on Jain's iter. rounding.

$\Longrightarrow$ Overall $(1+\varepsilon)\alpha$-approx for Path TSP.



Conclusions

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- Extension to Asymmetric Path TSP? (There is reduction to $(2\alpha + \varepsilon)$-approx by [Feige, Singh, 2007].)
- Extension to T-tours?