Triangle Factors in Randomly Perturbed Graphs

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The London School of Economics and Political Science
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Minimum degree thresholds in graphs

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Determine the minimum degree threshold that ensures a graph G contains a given (spanning) subgraph H.

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Perfect H-tiling (Kühn and Osthus, 2009)

- $\delta(G) \geq (1 - 1/\chi^\star(H))n + O(1)$,

  where $\chi^\star(H) \in \left\{\chi(H), \frac{(\chi(H)-1)|H|}{|H|-\sigma(H)}\right\}$ and $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours.

Thresholds in random graphs

The random graph $G(n,p)$ has vertex set $[n] := \{1, \dots, n\}$ and each pair is an edge with probability p, independently of all other choices.

$t(n)$ is a threshold for a property $\mathcal{A}$ if, for every $p(n)$,

$$\lim_{n\to\infty} \mathbb{P}\big(G(n,p) \in \mathcal{A}\big) = \begin{cases} 0 & \text{if } p(n) = o(t(n)), \\ 1 & \text{if } p(n) = \omega(t(n)). \end{cases}$$

Theorem (Bollobás – Thomason, 1987)

Every non-trivial monotone property $\mathcal{A}$ has a threshold.

$\mathcal{A}$ is monotone if it is closed under addition of edges
(containing H as subgraph vs. containing H as induced subgraph)

Thresholds in random graphs

There has been a lot of work in this area.

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Perfect K_r-tiling (Johansson, Kahn and Vu, 2008)

- $t(n) = n^{-2/r} (\log n)^{2/(r^2 - r)}$
- Conjectured the thresholds for perfect H-tiling for every H; resolved the case when H is a strictly balanced graph;
- Gerke and McDowell (2015) gave a proof when H is a non vertex-balanced graph.
  The problem is still open for some graphs H.

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Starting from a (dense) graph, determine how many random edges need to be added to ensure that the resulting graph a.a.s. contains a given spanning subgraph H.

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Definition (Bohman, Frieze and Martin, 2003)

Let $\alpha, p \in [0,1]$, $n \in \mathbb{N}$ and G_α be a graph on n vertices with minimum degree at least αn. We call $G_\alpha \cup G(n,p)$ a randomly perturbed graph.

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Given α, determine the threshold at which $G_\alpha \cup G(n, p)$ contains a.a.s. a given spanning subgraph H.

- The dense graph 'helps' $G(n, p)$ to have the spanning structure. (small α)
- Random edges 'help' the dense graph to have the spanning structure. (small p)

Randomly perturbed graphs: Hamiltonicity

Problem revised

Given α, determine the threshold $t(n)$ at which $G_\alpha \cup G(n,p)$ contains a.a.s. a given spanning subgraph H:

- 0-s: If $p(n) = o(t(n))$, then, for all n, there is an n-vertex G_α such that $G_\alpha \cup G(n,p)$ a.a.s. does not contain a perfect H-tiling.
- 1-s: If $p(n) = \omega(t(n))$, then, for all n-vertex G_α, we have that $G_\alpha \cup G(n,p)$ a.a.s. contains a perfect H-tiling.

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Theorem (Bohman, Frieze and Martin, 2003)

For every $\alpha > 0$, there is a $C = C(\alpha)$ such that with $p \geq C/n$, a.a.s. $G_\alpha \cup G(n,p)$ is Hamiltonian.

α	$\alpha = 0$	$0 < \alpha < 1/2$	$1/2 \leq \alpha$
$t(n)$	$n^{-1} \log n$	n^{-1}	0

One sees a decrease in the probability threshold (by a logarithmic factor).

Randomly perturbed graphs: Perfect tilings

Theorem (Balogh, Treglown and Wagner, 2019)

Let $r \geq 2$. For every $\alpha > 0$, there is a $C = C(\alpha, r)$ such that with $p \geq Cn^{-2/r}$, a.a.s. $G_\alpha \cup G(n,p)$ contains a perfect K_r-tiling.

α	$\alpha = 0$	$0 < \alpha < 1 - \frac{1}{r}$	$1 - \frac{1}{r} \leq \alpha$
$t(n)$	$n^{-2/r}(\log n)^{2/(r^2-r)}$	$\leq n^{-2/r}$	0

They give bounds for perfect H-tilings, for every graph H (when $\alpha > 0$).

- When $H = K_r$, their result is optimal for $0 < \alpha < 1/r$.
- What more can be said if $\alpha \geq 1/r$?

Randomly perturbed graphs: Perfect tilings (ctd.)

Theorem (Han, Morris and Treglown, 2020+)

Let $2 \leq k \leq r$ and $1 - \frac{k}{r} < \alpha < 1 - \frac{k-1}{r}$.

There is a $C = C(\alpha, r)$ such that with $p \geq Cn^{-2/k}$, a.a.s. $G_\alpha \cup G(n,p)$ contains a perfect K_r-tiling. Moreover, this is indeed the threshold.

- The threshold exhibits a 'jumping' behaviour.

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Example for perfect K_3-tiling:

$\alpha = 0$	$0 < \alpha < 1/3$	$1/3 < \alpha < 2/3$	$2/3 \leq \alpha$
$n^{-2/3}(\log n)^{1/3}$	$n^{-2/3}$	n^{-1}	0

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Question

What about the boundary cases: $\alpha = 1/r, 2/r, \ldots, (r-2)/r$?

- For perfect K_3-tiling, the only left case is $\alpha = 1/3$.

Perfect K_3-tiling: The boundary case

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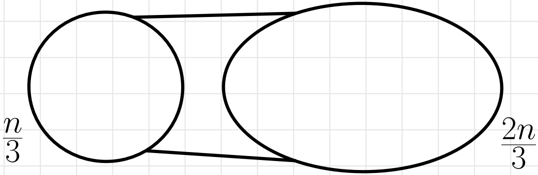
For $\alpha = 1/3$, $\omega(1/n)$ is not enough and $\omega(\log n/n)$ is needed:

$K_{\frac{n}{3},\frac{2n}{3}}$



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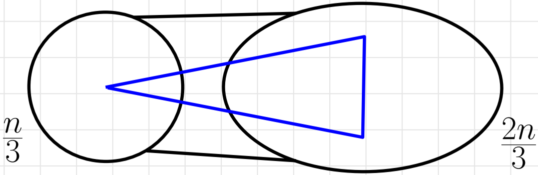

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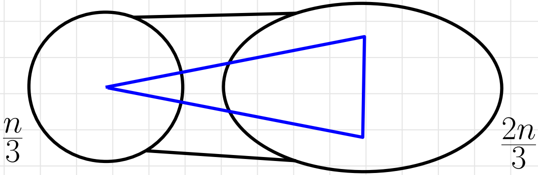

Theorem (Böttcher, Parczyk, Sgueglia and S., 2020+)

There exists $C > 0$ such that for $p \geq C \log n/n$ the following holds. $G_{1/3} \cup G(n,p)$ a.a.s. contains a triangle factor.

Perfect K_3-tiling: Boundary case (a bit more)

Theorem (Böttcher, Parczyk, Sgueglia and S., 2020+)

For any $1/12 > \beta > 0$ there exist $\gamma > 0$ and $C > 0$ such that for $4\beta \leq \alpha \leq 1/3$ and $p \geq C/n$ the following holds. If G

- has minimum degree at least $(\alpha - \gamma)\, n$ and
- is not 'β-close to the extremal graph',

then a.a.s. $G \cup G(n, p)$ contains $\min\{n/3, \alpha n\}$ disjoint triangles.

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Theorem (Böttcher, Parczyk, Sgueglia and S., 2020+)

There exists $C > 0$ such that for $p \geq C \log n/n$ and any n-vertex G the following holds. $G \cup G(n,p)$ a.a.s. contains at least $\min\{n/3, \delta(G)\}$ disjoint triangles.

- $\omega(1/n)$ is enough unless the deterministic graph is close to the (unique) extremal graph.

A glimpse into the proof: Extremal case

Embedding Lemma

For all $d \in (0,1)$ there are $\varepsilon > 0$ and C so that: Let U, V, W be of size n. If

- (V, U) and (V, W) are (ε, d)-super-regular pairs,
- $G(U, W, p)$ is a random bipartite graph with $p \geq C \log n / n$,

then a.a.s. there is a triangle factor in $G[U \cup V \cup W]$.

A glimpse into the proof: Non-extremal case

Stability tool concerning matchings

If $\delta(G) \geq (\frac{1}{3} - \gamma)n$ and G is not β-close to the extremal graph, then its reduced graph R has a matching with $(\frac{1}{3} + 4\gamma)v(R)$ edges.

Larger Cliques

Boundary cases for larger cliques

For a perfect K_3-factor:

$\alpha = 0$	$0 < \alpha < 1/3$	$\alpha = 1/3$	$1/3 < \alpha < 2/3$	$2/3 \leq \alpha$
$n^{-2/3}(\log n)^{1/3}$	$n^{-2/3}$	$n^{-1}(\log n)$	n^{-1}	0

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False!

However, we know that if the deterministic graph is not extremal, then $n^{-2/3}$ is the right threshold. Is this true in general?

A similar behaviour happens in the case of K_r-tiling for all $\alpha = 1 - \frac{k}{r}$ with $2 < k < r$ and $r \geq 4$.

K_4-tiling at $n/4$: More complicated than expected.

Let $\log^3 n \leq m \leq n^{1/7}$ and $p = n^{-2/3}(\log n)^{1/3}$.

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- Construct G : take $A \cup B$ with $|A| = n/4 - m$ and $|B| = 3n/4 + m$, A is an independent set, $G[B]$ is given by $|B|/(2m)$ disjoint copies of $K_{m,m}$ and $G[A, B]$ is complete.
- From the construction: $\delta(G) \geq n/4$.
- If $G \cup G(n, p)$ contains a K_4-factor, since A only contains $n/4 - m$ vertices, at least m copies of K_4 must lie within B.

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- We can build copies of K_4 using both edges from G and $G(n, p)$, but since $G[B]$ is bipartite, there are only seven possible types of K_4.
- First moment method shows that a.a.s. $G \cup G(n, p)[B]$ does not contain m K_4's, so a.a.s. $G \cup G(n, p)$ does not contain a K_4-factor.

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One can take small $\varepsilon > 0$, $n^{7\varepsilon} \leq m \leq n^{1/7}$ and $p = n^{-2/3+\varepsilon}$.

Ongoing work: Universality

2-universality

- $\mathcal{F}(n, \Delta) := \{F : |V(F)| = n \text{ and } \Delta(F) \leq \Delta\}$

A graph is 2-universal if it contains every element of $\mathcal{F}(n, 2)$ as a subgraph.

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The hardest graph to embed is the K_3-factor:

In progress (Böttcher, Parczyk, Sgueglia and S.)

When $\alpha = 1/3$, the threshold for 2-universality is $\log n/n$.

- We know $\omega(1/n)$ suffices if G is not close to the extremal graph.

More open problems

- Balogh, Treglown and Wagner gave optimal thresholds for perfect H tiling in $G_\alpha \cup G(n,p)$ for $\alpha < 1/|H|$. The problem is still wide open for larger values of α.

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In progress (Böttcher, Parczyk, Sgueglia and S.)

When $\alpha = 1/3$, $\omega(\log n/n)$ is the correct threshold for 2-universality.

- Find the correct threshold for 3-universality in $G_\alpha \cup G(n,p)$.

This is much harder because $\mathcal{F}(n,3)$ contains expanders while $\mathcal{F}(n,2)$ only unions of cycles and paths.

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Thank you!