Edit Distance in Near Linear Time: $O(1)$ factor

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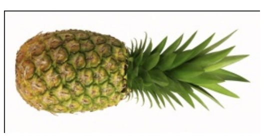
Edit distance (Levenstein)

- Strings $x, y \in \Sigma^n$
- $ed_n(x, y)$ = minimum number of insertions/deletions/substitutions to transform x into y

$ed_n($  ,

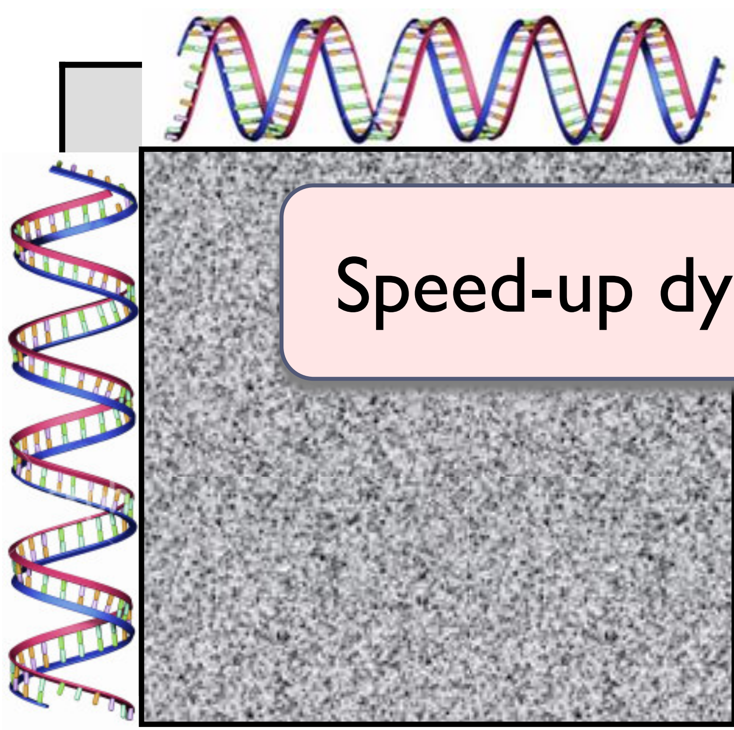
 $) = 2$

Applications:

- bioinformatics
- natural language processing

Crucially: A classic dynamic programming

- Computing $ed_n(x, y)$:
  - $O(n^2)$ time [Wagner-Fischer'74]



$D(i,j) = ed(x[1:i], y[1:j])$

Speed-up dynamic programming?

f $x[i] = y[j]$

$D(i,j) = \min \begin{cases} D(i-1,j)+1 \\ D(i,j-1)+1 \end{cases}$

Faster Algorithms?

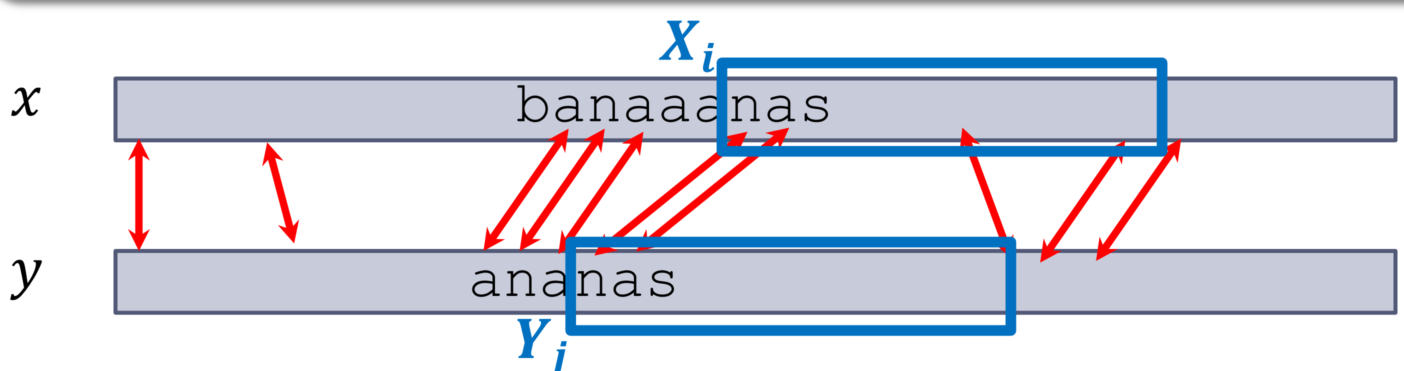
- Computing $ed_n(x,y)$:
  - $O(n^2)$ time [Wagner-Fischer'74]
  - $O(n^2/\log^2 n)$ [MP'80]
  - Better in special cases (small ed, average case, smoothed, etc): [U83,LV85,M86,GG88,GP89,UW90,CL90,CH98,LMS98,U85,CL92,N99,CPSV00,MS00,CM02,BCF08,AK08, K'19…]
  - FGC: $n^{2-o(1)}$ likely best possible!
    - assuming Strong Exponential Time Hypothesis [BI'15,AHWW'16,…]
- Approximation in near-linear time?
  - $\log^{1/\epsilon} n$ factor in $n^{1+O(\epsilon)}$ time [BEKMRRS'03, BJKK'04, BES'06, AO'09, AKO'10]
  - $O(1)$ factor in $O(n^{1.781})$ quantum time [BEGHS'18]
  - $O(1)$ factor in $O(n^{1.618})$ time [CDGKS'18]
  - $O_\epsilon(1)$ factor & $\pm n^{1-f(\epsilon)}$ additive in $O(n^{1+\epsilon})$ time [KS'20, BR'20]

Main result

Can compute $ed(x, y)$ with $O_\epsilon(1)$ approx. in $n^{1+\epsilon}$ time

- Approach setup:
  - $ed_n(x, y) \Leftrightarrow$ an optimal alignment π: $[n] \to [n] \cup \{\perp\}$
  - X_i, Y_j : substrings starting at i/j of length w (think $w = n^{1-\delta}$)
  - Then $\sum_i \frac{ed_w\left(X_i, Y_{\pi(i)}\right)}{w} \approx ed_n(x, y)$

Goal: find near-optimal matching π between X_i's and Y_j's, using calls to $ed_w\left(X_i, Y_j\right)$ (possibly recursive)



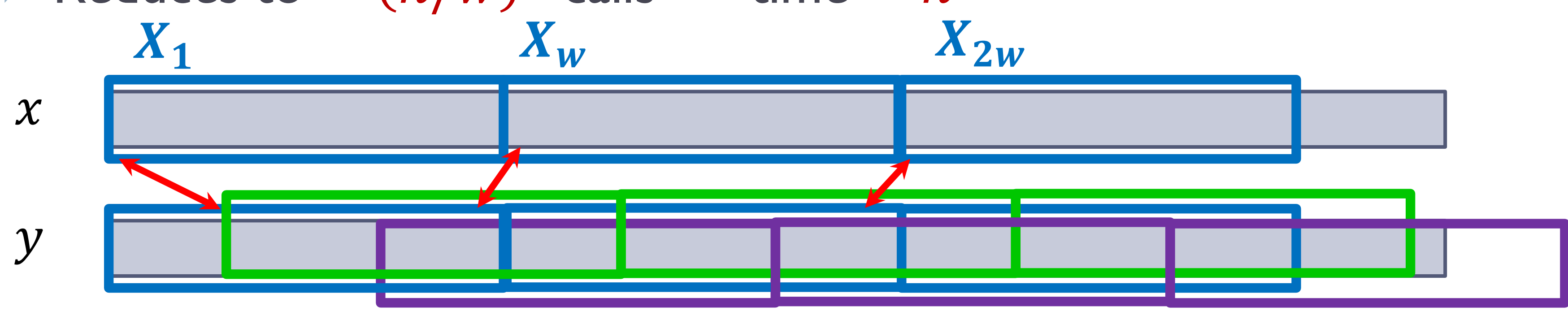
Past approaches

Goal: find near-optimal matching π between X_i's and Y_j's, using calls to $ed_w(X_i, Y_j)$

X_i: interval of length w

Why should help? [BEGHS'18, CDGKS'18]

- Naive compute-all: n^2 calls to ed_w => time $n^2 w^2$
  - Finding actual π: $(n/w)^{O(1)}$ time (~standard DP)
- Idea 0: enough to consider i be multiple of w
  - Issue: $j = \pi(i)$ may not be w-multiple
  - Can round j to δw, at the cost of additive δn error
  - Reduces to $\approx (n/w)^2$ calls => time $\approx n^2$

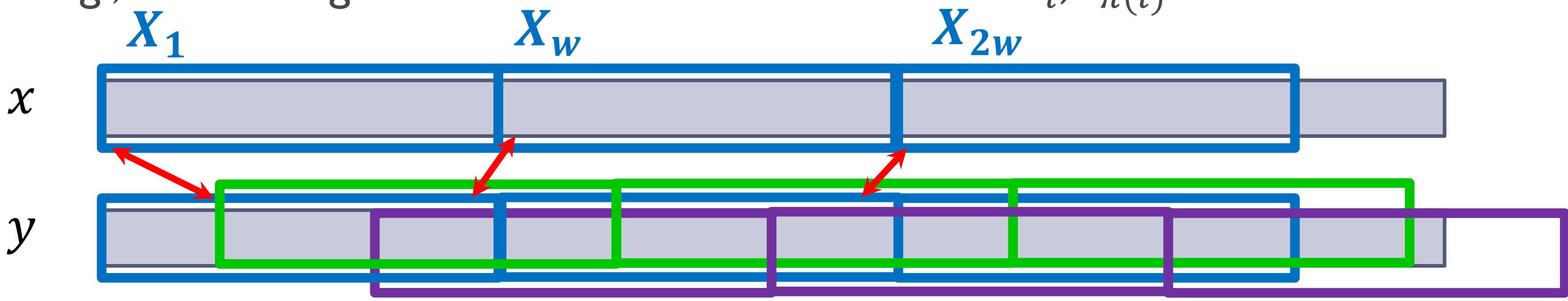


Reducing # of calls to $ed_w(X_i, Y_j)$

Goal: find near-optimal matching π between X_i's and Y_j's, using calls to $ed_w(X_i, Y_j)$

- Idea 1: use triangle inequality to deduce $ed_w(X_i, Y_j)$
  - If X_i is "close" to $X_{i_1}, \ldots X_{i_m}$ and Y_j "close" to $Y_{j_1}, \ldots Y_{j_m}$ => so are all of them, up to factor 2
  - Reduces # of ed_w computations from m^2 to ~$2m$ (if ideal) !
- Idea 2: for $\pi(iw) = j$, most likely $\pi\big((i+1)w\big) \approx j + w$
- +Idea 1,2 [CDGKS'18]: $(n/w)^{1.5}$ computations of ed_w!
  - Total time: $(n/w)^{1.5} \cdot w^2 + (n/w)^{O(1)}$
- [KS'20, BR'20]: $(n/w)^{1+\epsilon}$ computations of ed_w
  - Extra $n^{1-f(\epsilon)}$ error term
  - E.g., allows to ignore a $n^{-f(\epsilon)}$ fraction of matches $X_i, Y_{\pi(i)}$

Or ~$w^{1.5}$ if recursing on ed_w

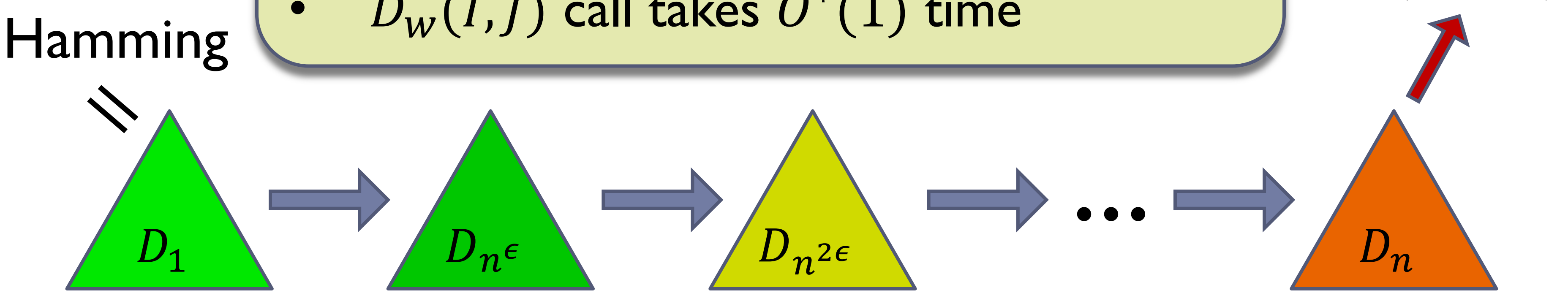


Our high-level approach

- For each $w = 1, n^{\epsilon}, n^{2\epsilon}, \dots n$,
- build a distance oracle D_w for the metric $(\mathfrak{I}_w, ed_w(\cdot,\cdot))$ where $\mathfrak{I}_w$ = all $2n$ substrings of length w

Oracle D_w: for $I, J \in \mathfrak{I}_w$

- $ed_w(I,J) \le D_w(I,J)$
- $D_w(I,J) \lesssim ed_w(I,J)$ where it "matters"
- $D_w(I,J)$ call takes $O^*(1)$ time

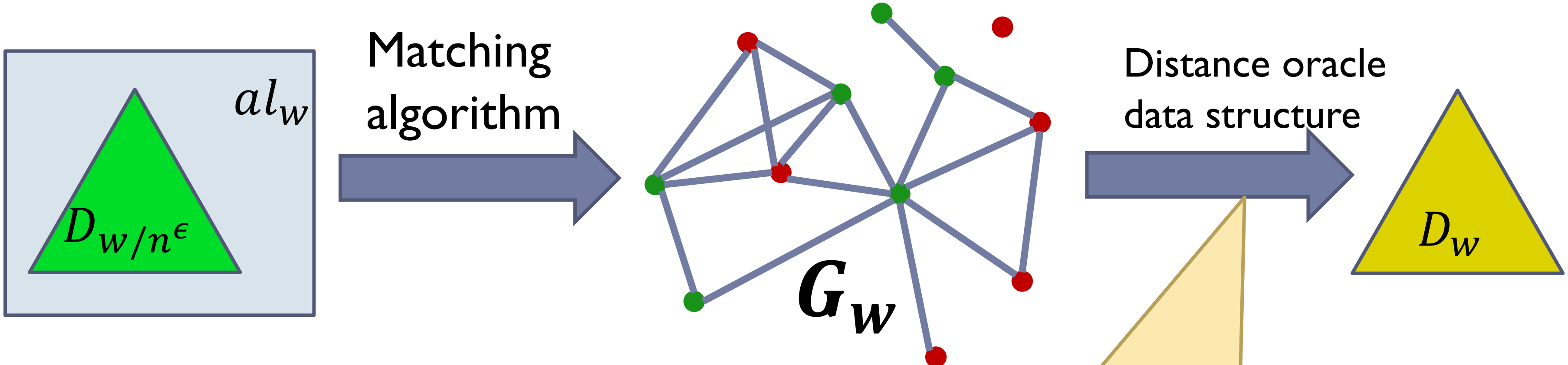


New goal: given $D_{w/n^{\epsilon}}$, compute D_w, in $n^{1+O(\epsilon)}$ time

2 components:

$ed_w(I,J) \le al_w(I,J) \lesssim ed_w(I,J)$
$al_w(I,J)$ uses $O^*(1)$ time & D_{w/n^ϵ} calls

- 1. Alignment algo: oracle $al_w(I,J)$
- 2. Matching algo: building D_w from al_w



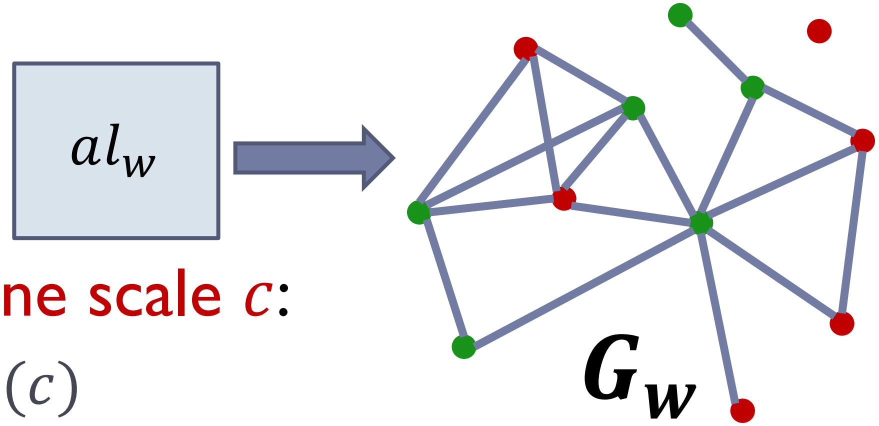
Distance G_w (shortest path): for $I,J \in \mathfrak{I}_w$

- $ed_w(I,J) \le G_w(I,J)$
- $G_w(I,J) \lesssim al_w(I,J)$ where it "matters"

$O_\epsilon(1)$ factor approx.
Any distance oracle OK,
as long as *metric output*

E.g., [Thorup-Zwick'05] not metric output
But [Matousek'96]: embed into ℓ_∞^d where
$d = n^\epsilon = O^*(1)$ for approximation $O(1/\epsilon)$

Matching Algorithm



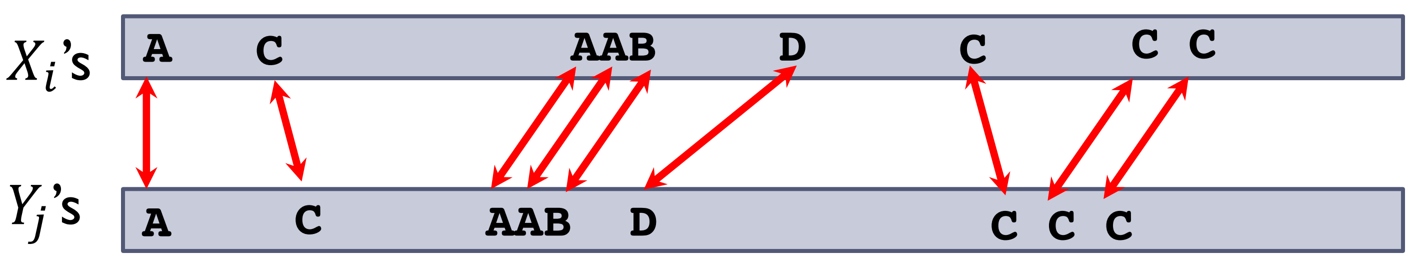
- Enough to build graph G for one scale c:
  - Edge (I,J) implies $al(I,J) \leq O_\epsilon(c)$
  - For any alignment π, for $i \in [n]$:
    - If $al(X_i, Y_{\pi(i)}) \leq c$, there is a 2-hop path in G
    - Can miss ... e $\lesssim$ number of pairs where $al(X_i, Y_{\pi(i)}) > c$
  - O^* ... calls to al

BIG simplification…

...ect Neighborhood Assumption:
- Either $al(I,J) \leq c$
- Or $al(I,J) \gg c$

Ie, $\mathfrak{T}_w$ composed of equivalence classes

Instead of triangle inequality

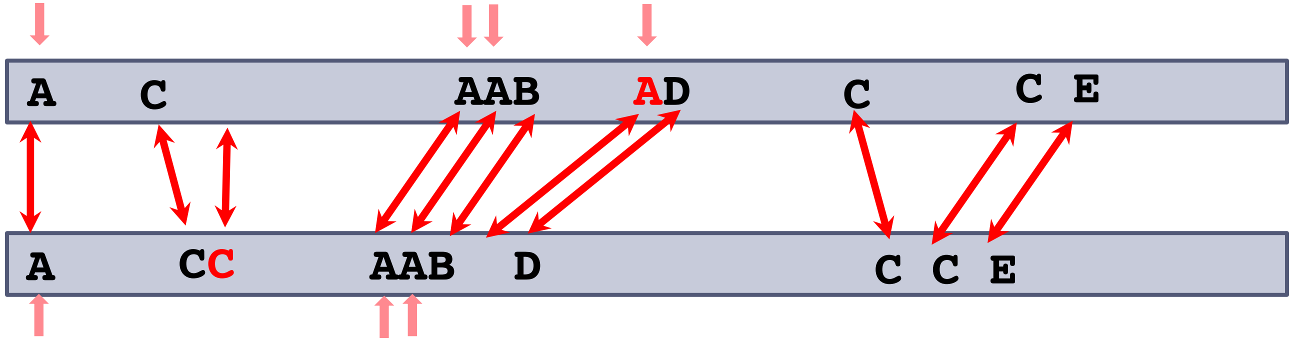


Main loop

1. Iteratively partition $\mathfrak{T}_w$ into finer parts
   - In step $t = 1 \ldots 1/\epsilon$, produce Π_t
   - $\approx \lambda^t$ parts of size $\approx n/\lambda^t$, for $\lambda = n^\epsilon$

Fix part $P \in \Pi_{t-1}$:
- Size $\lambda \cdot n/\lambda^t$
- About λ anchors inside
- Each should capture n/λ^t

- Construction in step t
  - Sample λ^t anchors $\in \mathfrak{T}_w$ (each will produce a part in Π_t)
  - For each anchor A, compare to all in $\Pi_{t-1}(A)$ using al oracle
  - Obtain set $E(A)$: all "equivalent" substrings (at distance $\leq c$)
  - Each such $I \in E$ is given credit $\phi_A(I) = \frac{n/\lambda^t}{|E(A)|}$



Partition via proximity

- Proximal extension of $I \in E(A)$:
  - Distribute $\phi_A(I)$ to "nearby" J's
  - R intervals $J \in P_{t-1}(I)$ to left/right
  - Defines $\psi_A(J)$
- New partition Π_t of $\mathfrak{T}_w$:
  - Consider vectors $\psi(J) \in \Re_+^{\lambda^t}$
  - Partition using (weighted) minhash $h: 2^{[\lambda^t]} \to [\lambda^t]$:
    - J assigned to part $h(\psi(J))$
    - $\Pr[X_i \text{ and } Y_{\pi(i)} \text{ separated}] \approx ||\psi(X_i) - \psi(Y_{\pi(i)})||_1 \approx$ "local error"

Fix part $P \in \Pi_{t-1}$:

- Size $\lambda \cdot n/\lambda^t$
- About λ anchors inside
- Each should capture n/λ^t
- $I \in E(A)$ is given credit $\phi_A(I) = \frac{n/\lambda^t}{|E(A)|}$

Issues:
1) Value of R?
2) Need to group non-proximal sparse substrings together

Intervals partitioned according to their proximity

Repeat for $R = n^{\epsilon l}$, for $l = 1 \ldots$

Each level l "takes care" of intervals I of density $E(I) \approx \Theta^*\left(\frac{n/\lambda^t}{n^{\epsilon l}}\right)$

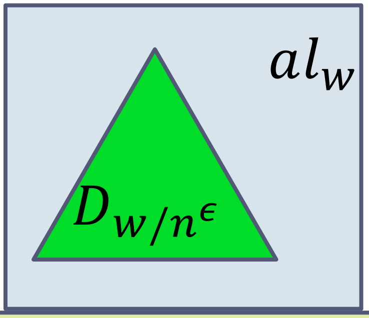
Use *thresholded* $\psi_A(J)$: zeroed-out if too small (to ensure no big parts)

Remove partitioned intervals from subsequent levels

A sample of the rest

Perfect Neighborhood Assumption:
- Either $al(I,J) \leq c$
- Or $al(I,J) \gg c$

- Beyond "Perfect Neighborhood Assumption":
  - Challenge: can't use usual ideas to reduce to PNA
    - E.g., if choose a random cut-off point c: constant probability to separate X_i from $Y_{\pi(i)}$ => like $ed \approx n$
    - Or FRT-like metric decomposition: Pr pair together $\approx n^{-\epsilon}$ not enough
    - Need a "for all" guarantee instead of "for each"
  1. Smooth out everything: "matching quantities" => up to $n^{O(\epsilon)}$
    - Eg, use *fractional* partitions (colorings): interval (logically) split b/w "parts"
    - New challenges to keep palettes sufficiently sparse
  2. Replace Jaccard (w-minhash) with "distortion resilient ℓ_1":
    - $dd_F(p,q) = \sum_i p_i \cdot \mathbb{I}[p_i > F \cdot q_i]$ for $F = n^{O(\epsilon)}$
- Alignment Algorithm al_w:
  - Challenge 1: D_{w/n^ϵ} arbitrary metric
  - Challenge 2: output of al_w needs to be a metric



$al_w(I,J)$ uses $O^*(1)$ time and D_{w/n^ϵ} calls

Finale

Can compute $ed(x,y)$ with $O_\epsilon(1)$ approx. in $n^{1+\epsilon}$ time

- Approximation ~doubly-exponential in $1/\epsilon$
- Open questions:
  - $poly(1/\epsilon)$ approximation?
    - Natural because using "dimension reduction" methods for metrics, where standard to have $2/\epsilon$ approx. vs n^ϵ dimension
  - Best runtime for $3+\epsilon$ approximation?
    - E.g., $\approx n^{1.5}$ natural: bottleneck is dynamic programming on substrings
    - Current best: $\approx n^{1.6}$ [A'18, RSSS'19, GRS'20]
  - < 3 approximation (beyond triangle ineq)? [RSSS'19]
  - Many other edit distance problems:
    - Text indexing [CDK'19, A'18], embedding/cutting modulus/NNS