# Trees on Trees

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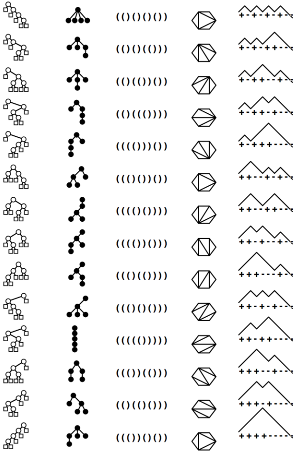
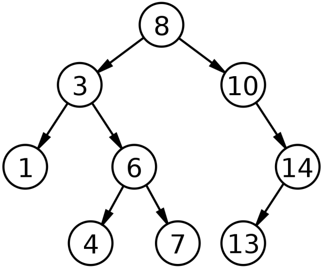
Stefan Langerman, and Pablo Pérez-Lantero (USACH)

# Binary Trees

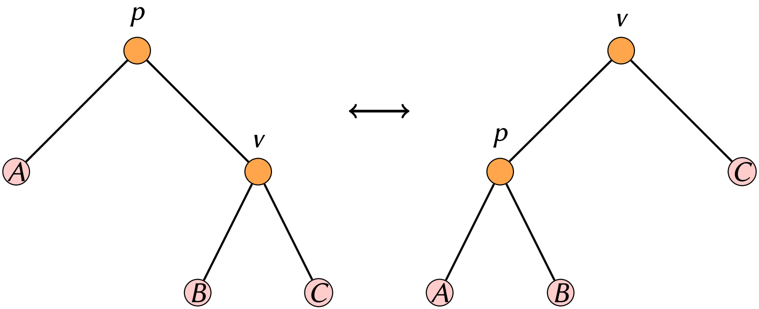


Combinatorics

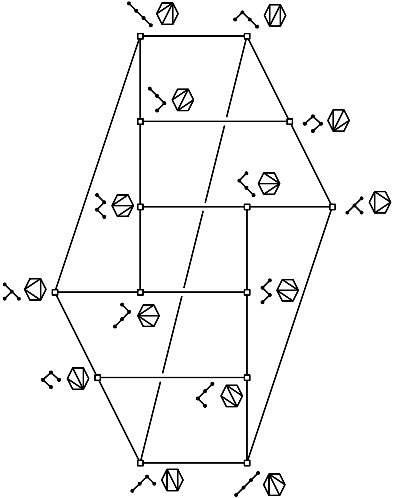



Data structures

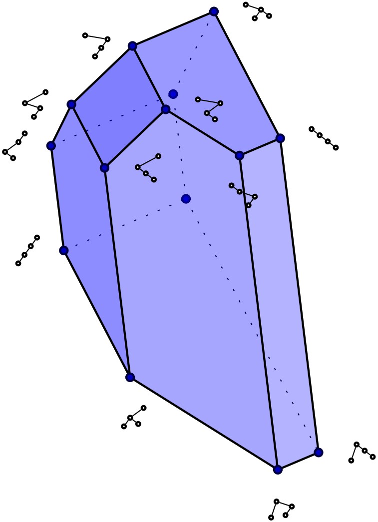
# Rotations



# Rotations and Flips



# Associahedra



Tamari 1951 – Stasheff 1963 – Lee 1989 – Loday 2004

# Diameter of Associahedra

What is the maximum number of rotations required to transform one binary tree on $n$ internal nodes into another?

The diameter of the associahedron is $2n - 6$.

Sleator, Tarjan, Thurston 1988

Pournin 2014

# Online Binary Search Trees

- BST search model: finger moves and rotations are unit-cost operations
- Given an *access sequence* of nodes, what is the minimum sequence of unit-cost operations that touch these nodes in order?
- Is there an $O(1)$-competitive online BST?

Sleator and Tarjan 1985

Demaine et al. 2009

- Dynamic optimality conjecture: *Splay trees* are $O(1)$-competitive.

Sleator and Tarjan 1985

- *Tango trees* are $O(\log \log n)$-competitive.

Demaine, Harmon, Iacono, Pătraşcu 2007

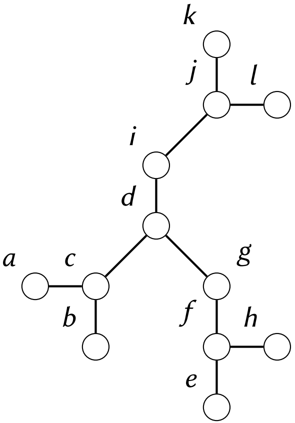
# Trees on Trees?

- Binary search trees deal with a linearly ordered search space.
- Generalize the notion of search trees to a *tree-structured* search space.
- Rotations and associahedra are well-defined.

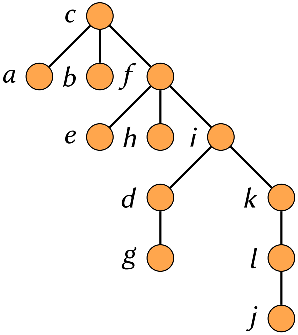
Two questions:

1. Diameter of tree associahedra?
2. Competitive online search trees on trees?

# Trees on Trees

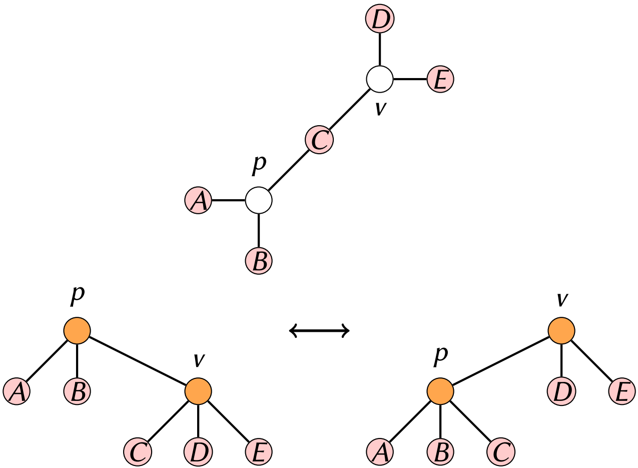



Tree $G$




Search tree $T$ on $G$

# Rotations



# Related Notions

- Vertex ranking

Schäffer 1989

Bodlaender et al. 1998

- Tree-depth

Nešetřil and Ossona de Mendez 2012

- Graph associahedra

Carr and Devadoss 2006

Postnikov 2009

- Optimal search in trees

Ben-Asher, Farchi, Newman 1999

Cicalese et al. 2011, 2014, 2016

Emamjomeh-Zadeh, Kempe, Singhal 2016

# Diameter of Tree Associahedra

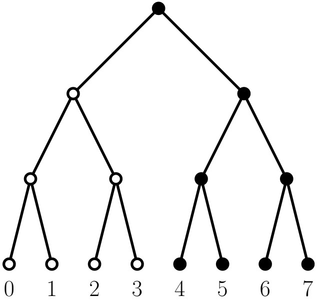
The diameter of graph associahedra is at least $|E|$ and at most $O(n^2)$. Is the diameter of tree associahedra $O(n)$?

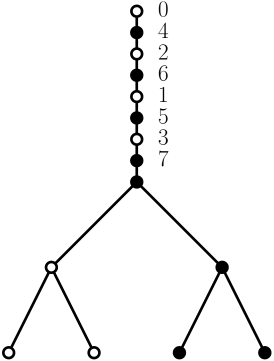
Manneville and Pilaud 2015

The diameter of tree associahedra is $\Theta(n \log n)$.

C., Langerman, Pérez-Lantero 2018

# Diametral Pair




*Bit-reversal* permutations

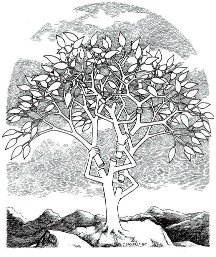
# Online Search Trees on Trees

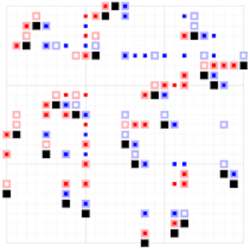
- GST search model: finger moves and rotations are unit-cost operations
- Given an *access sequence* of nodes of $G$, what is the minimum sequence of unit-cost operations that touch these nodes in order?
- Can we design competitive online search trees on trees?

# Tentative Generalizations

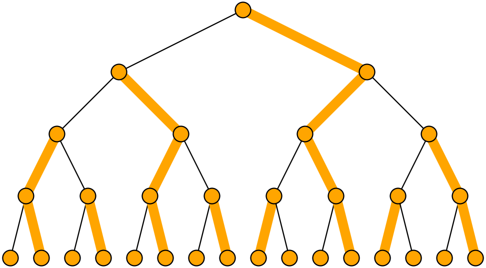
Many previous techniques / ideas do not generalize immediately:

- Splay trees
- Greedy algorithm
- Geometric view





# Preferred Paths and Lower Bound

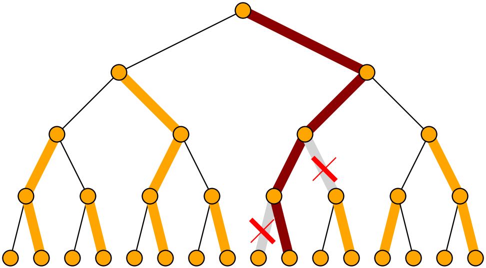


- Consider a fixed *balanced reference tree*.
- Preferred children keep track of last accesses.
- Interleave bound: total number of times some preferred child changes.

The interleave bound is a lower bound on the cost of an access sequence in the BST model.

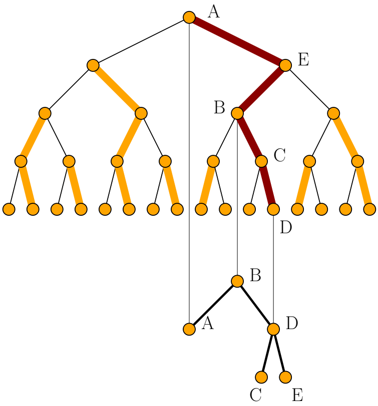
Demaine et al. 2007

# Tango Trees



- Changes in the preferred children are implemented via *splitting and merging* the preferred paths.

# Tango Trees




- Preferred paths are maintained using *red-black trees*, that can be split and merged in $O(\log \log n)$ time, since the reference tree has height $O(\log n)$.

# Tango Trees on Trees

The interleave bound is a lower bound on the cost of an access sequence in the GST model.

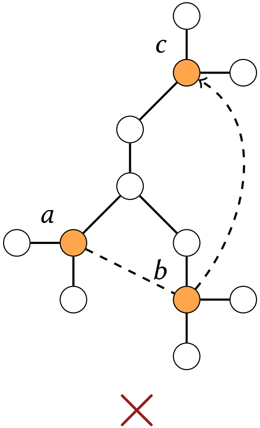
Bose et al. 2019

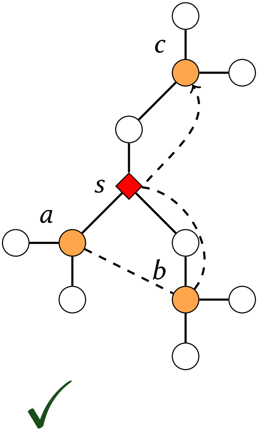
Generalize Tango trees:

- Balanced reference tree via centroid decomposition.
- The reference tree needs to be *Steiner-closed*.
- Splits and merges of paths can be handled by *link-cut trees*.

# Steiner-Closed Trees on Trees

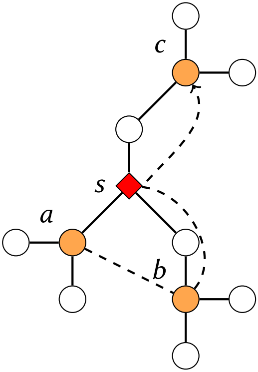
A path $S$ in a search tree is Steiner-closed if every vertex in $\mathrm{CH}(S) \setminus S$ has degree exactly two in $\mathrm{CH}(S)$.



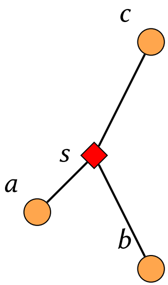



# Merges and Splits of Preferred Paths

- If every path $S$ in the reference tree is Steiner-closed, then it can be associated with a tree $G(S)$,
- and merges and splits of the preferred paths correspond to at most two *links* and *cuts* of the corresponding trees $G(S)$.




A path $S$




The tree $G(S)$

# A Swiss Army Knife

- These can be implemented with *link-cut* trees.

Sleator and Tarjan 1983

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*Printed in Belgium*

## A Data Structure for Dynamic Trees

DANIEL D. SLEATOR AND ROBERT ENDRE TARJAN

*Bell Laboratories, Murray Hill, New Jersey 07974*

Received May 8, 1982; revised October 18, 1982

A data structure is proposed to maintain a collection of vertex-disjoint trees under a sequence of two kinds of operations: a *link* operation that combines two trees into one by adding an edge, and a *cut* operation that divides one tree into two by deleting an edge. Each operation requires $O(\log n)$ time. Using this data structure, new fast algorithms are obtained

# Overview

- Construct a balanced Steiner-closed reference tree $P$ on $G$.
- First-level decomposition of $P$ into preferred paths $S$, each corresponding to an unrooted tree $G(S)$.
- As searches are performed, preferred paths are updated, and these updates correspond to linking and cutting trees $G(S)$.
- Link-cut trees use a second-level decomposition of the trees $G(S)$ into paths.
- Operations on those paths are eventually handled by splay trees. Together, they form a search tree on $G$.

# Summary and Perspectives

- The diameter of tree associahedra is $\Theta(n \log n)$.
- There exist $O(\log \log n)$-competitive online search trees on trees.

Dynamic optimality for online search trees on trees?

# Pointers

(click on the title)

- Jean Cardinal, Stefan Langerman, Pablo Pérez-Lantero: On the Diameter of Tree Associahedra. Electron. J. Comb. 25(4): P4.18 (2018).
- Prosenjit Bose, Jean Cardinal, John Iacono, Grigorios Koumoutsos, Stefan Langerman: Competitive Online Search Trees on Trees. SODA 2020: 1878-1891.