Nonequilibrium phase transitions in perturbed particle systems

Paul Chleboun
Complexity DTC, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK
(Dated: November 10, 2008)

Condensation occurs in stochastic interacting particle systems when a finite fraction of the particles occupy a single lattice site. We consider a one dimensional zero range process that is well known to exhibit a condensation transition. We present a detailed analysis of the effect of quenched disorder in the particle interactions on the critical behaviour of the system. The focus is on finite lattices because of their application to real systems. Recent theoretical results on the change of phase diagram under perturbation are supported by Monte Carlo simulations in the canonical ensemble. A detailed numerical analysis of the systems in the canonical ensemble is given and we show that finite size effects can be described by an extension of the grand canonical ensemble and single condensate site. We show that the condensation transition may be unstable in the thermodynamic limit and present a method to investigate this further.

PACS numbers:

I. INTRODUCTION

The statistical mechanics of nonequilibrium systems can be used to understand processes that occur across physical and social sciences. These systems are generally driven in a manner so that their dynamics are not time reversible. Examples include traffic flow on a highway and charged particles moving in an electromagnetic field. Considerable understanding of how microscopic interactions can influence the collective behaviour of non equilibrium systems can be gained by studying interacting particle systems.

We focus on a particular interacting particle system, the zero-range process (ZRP). In the ZRP, particles hop randomly on a lattice at a rate that depends only on the local particle number at the departure site and position on the lattice. It was first introduced as a mathematical model for interacting diffusing particles [1]. Since its introduction the model has stimulated considerable interest and has been applied to a variety of physical phenomena, for a review see [2]. The model is simple enough for its steady state to factories allowing for detailed analysis. However it can still exhibit an interesting condensation transition when the particle density exceeds some critical value, whereby the system phase separates into a homogeneous background at the critical density and all the excess particles concentrate on a single lattice site. This effect has been observed and studied in detail in experiments on shaken granular media [3, 4]. It is also a generic mechanism for phase separation in single-file diffusion [5] and condensation phenomena in many complex systems [6, 7].

In the present work we study the effect of a random perturbation to a homogeneous system known to exhibit a condensation transition. Condensation can arise in homogeneous systems in which the jump rates decay as the on-site occupancy increases (in analogy with shaken granular media and traffic models). Such a model with a generic power law decay has been introduced in [9]. For this model the condensation transition is understood on a mathematically rigorous level in the context of equivalence of ensembles [10, 11]. Variants of the model have been well studied and widely applied [12–16]. However the assumption of strict spatial homogeneity is not desirable for applications to real complex systems in which there may be some disorder due to local imperfections. It has been shown recently that a small perturbation of the jump rates has a drastic effect on the critical behvaiour [17].

We begin by confirming recent theoretical results on the change of critical behaviour under perturbation using Monte Carlo methods [17]. It is seen that that finite systems show significant deviation from the theoretical results. In particular, the low density, fluid phase appears to continue to higher densities than predicted resulting in an overshoot in current above the critical density. Related finite size effects have been reported in traffic models and have been studied in detail for a zero-range process with a single defect site [18]. We proceed with a detailed study of the finite size effect by a steady state analysis of the finite system.

Initial theoretical results suggest that the condensation transition in the perturbed system may not be robust in the thermodynamic limit, as the system size increases the critical density may diverge. We address this issue by attempting to bound the critical density so that we may find sufficient conditions for convergence.

The paper is organized as follows: in section II we define the ZRP and discuss the condensation mechanism in both the thermodynamic limit and on a finite lattice; in section III we present the change of critical behaviour under a perturbation to a generic choice of jump rates and confirmation by Monte Carlo methods; in section IV we give a detailed analysis of the finite size effect by studying exact numerics in the finite systems steady state. We study the critical behaviour of the perturbed system in the thermodynamic limit in section V and outline some related open questions; and we conclude in section VI.

II. THE ZRP AND CONDENSATION

A. Definition and steady states

We consider a one dimensional zero-range process where indistinguishable particles move on a one dimensional lattice Λ_L, which we take to be periodic and of finite size $|\Lambda_L| = L$. A configuration is denoted $\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda_L}$ where $\eta_x \in \mathbb{N}$ is the occupation number at site x. The process is defined as a continuous time Markov Chain with state space $X_{\Lambda_L} = \mathbb{N}^{\Lambda_L}$ in which particles jump off site $x \in \Lambda_L$ with a rate given by $g_x(\eta_x)$ and move to a target site y according to some finite range x-independent probability distribution $p(y-x)$. We assume p is irreducible to exclude hidden conservation laws and also the jump rates satisfy $g_x(n) = 0$ if and only if $n = 0$. For example in one dimension with nearest neighbour hoping, particles move to the right with probability p and to the left with $1-p$. Note that the jump rate $g_x(\eta_x)$ depends only on the departure site, x, and the number of particles on the site, η_x.

The formal generator of the process is given by

$$(\mathcal{L}f)(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} g(\eta_x)p(y)\left[f(\boldsymbol{\eta}^{x\to x+y}) - f(\boldsymbol{\eta})\right]$$

where $\boldsymbol{\eta}^{x\to x+y}$ is the configuration resulting from $\boldsymbol{\eta}$ after a particle has jumped from site x to site $x+y$. The function f can be interpreted as an observable and is restricted to Lipshitz functions on X_{Λ_L} [20].

The dynamics clearly conserve the total number of particles in the system so the state space decomposes into invariant subsets $X_{\Lambda_L,N} = \{\boldsymbol{\eta} \in X_{\Lambda_L} | \sum_{x=1}^{L}\eta_x = N\}$. The process restricted to one of these sets is then an irreducible finite state continuous time Markov chain and so ergodic. The unique steady state distribution for fixed N is known as the *canonical ensemble* and it is straightforward to show (see e.g. [2]) that it is

$$\pi^{L,N}(\boldsymbol{\eta}) = \frac{1}{Z(\Lambda_L,N)}\prod_{x=1}^{L} w_x(\eta_x)\delta\left(\Sigma_x\eta_x - N\right) \quad (1)$$

where the stationary weights are given by

$$w_x(n) = \prod_{k=1}^{n} g_x(k)^{-1}. \quad (2)$$

The normalisation $Z(\Lambda_L,N)$ is analogous to the canonical partition function from equilibrium statistical mechanics,

$$Z(\Lambda_L,N) = \sum_{\boldsymbol{\eta}\in X_{\Lambda_L,N}}\prod_{x=1}^{L} w_x(\eta_x)\delta\left(\Sigma_x\eta_x - N\right). \quad (3)$$

It is also well known (see e.g. [2, 20]) that the process has a *grand-canonical* factorized steady state ν_ϕ^L with single-site marginal

$$\nu_{x,\phi}(\eta_x) = \frac{1}{z_x(\phi)} w_x(\eta_x)\phi^{\eta_x} \quad (4)$$

where the *fugacity* ϕ fixes the average on-site density. Clearly the grand canonical steady state is well defined for $\phi \in [0,\phi_c)$ where ϕ_c is the radius of convergence of the grand canonical (one site) partition function

$$z_x(\phi) = \sum_{n=0}^{\infty} w_x(n)\phi^n, \quad (5)$$

which is strictly increasing and convex in ϕ. Genrally the radius of convergence is given by $\phi_c^{-1} = \limsup_{k\to\infty} w_x(k)^{\frac{1}{k}}$. We consider models om which this is independent of site x. Site dependence of ϕ_c can also lead to a condensation transition [8], but this is not studied here. If $\lim_{k\to\infty} g_x(k)$ exists then it follows from Eqn (2) that,

$$\phi_c = \lim_{k\to\infty} g_x(k). \quad (6)$$

The average local particle density $\langle\eta_x\rangle_{\nu_{x,\phi}} = \rho^x(\phi)$ is a function of the fugacity ϕ

$$\rho^x(\phi) = \phi\partial_\phi \log z_x(\phi) \quad (7)$$

and the average total system density is defined

$$\rho(\phi) = \frac{1}{L}\sum_{x=0}^{L}\rho^x(\phi) \quad (8)$$

and is strictly increasing with ϕ.

B. Condensation

Depending on the chosen jump rates $g_x(n)$ the system can exhibit a condensation transition in the thermodynamic limit as $N, L \to \infty$ with the density of particles $\rho = N/L$ fixed. In this case, at low densities the system is in a *fluid* phase where all sites contain on average N/L particles. However, at high densities the system is in a condensed phase wherein a single site holds a finite fraction of the total number of particles. The condensation transition in the thermodynamic limit described bellow is well understood in terms of equivalence of ensembles [10, 11].

We define the critical density, as seen on Figure 1, as

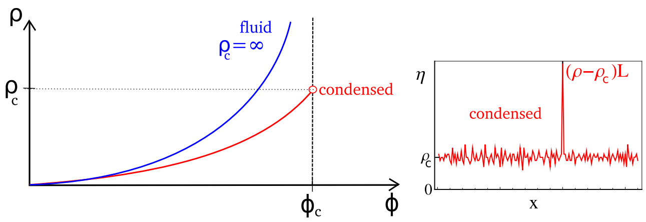
$$\rho_c = \lim_{\phi\to\phi_c}\rho(\phi)$$

Depending on the jump rates ρ_c may be finite and the system can undergo a condensation transition. If $\rho_c = \infty$ then the system will remain fluid in the thermodynamic limit for all values of the density ρ.

Since $\rho(\phi)$ is strictly increasing on $[0,\phi_c)$ it is invertible, and we denote its inverse $\phi(\rho)$. Then

$$\rho < \rho_c \Longrightarrow \pi^{[\rho L],L} \xrightarrow{w} \nu^{\infty}_{\phi(\rho)} \quad \text{as} \quad L\to\infty, \quad (9)$$

$$\rho \geq \rho_c \Longrightarrow \pi^{[\rho L],L} \xrightarrow{w} \nu^{\infty}_{\phi_c} \quad \text{as} \quad L\to\infty, \quad (10)$$



(a) (b)

FIG. 1: (a): Fundamental diagram showing system density in the grand canonical steady state as a strictly increasing function $\rho(\phi)$ converging to a finite ρ_c or diverging as $\phi \to \phi_c$. (b): Typical stationary profile for a condensed system with total number of particles ρL.

that is if $\rho < \rho_c$ then the canonical measure converges to the grand canonical measure at fugacity $\phi(\rho)$ and the system is interpreted as being in a fluid phase. If $\rho \geq \rho_c$ then the canonical measure converges to the grand canonical measure at the critical fugacity, so almost all sites are distributed according to $\nu^{\infty}_{\phi_c}$ and the remaining particles condense on a single lattice site. Convergence is pointwise in the n-point marginals of the measures and ensures weak convergence for all bounded cylinder functions, for details see [10].

The condensation transition can also be described on a finite lattice where the following has been proved rigorously for a homogeneous system [22]. If the critical density, $\rho_c < \infty$, then for fixed L as $N \to \infty$ by removing the site containing the most particles the canonical ensemble converges to the grand canonical ensemble at the critical fugacity. That is all but one site are distributed according to ν_{x,ϕ_c} with an average density of ρ_c and the remaining particles condense on a single site.

III. CHANGE OF THE PHASE DIAGRAM UNDER PERTURBATION

A. The perturbed ZRP

A generic homogeneous model with decaying jump rates was introduced in [9]. The jump rates are

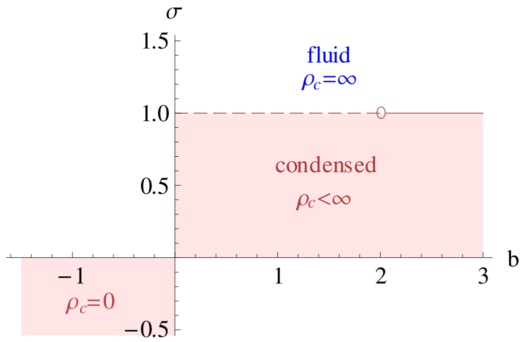
$$g_x(n) = 1 + b/n^{\sigma} \quad \text{for} \quad n \geq 1, \quad g_x(0) = 0. \tag{11}$$

Substituting these rates into Eqn (6) we find the critical fugacity $\phi_c = 1$. For such jump rates there is a finite critical density (and so condensation can occur) if the interaction exponent $0 < \sigma < 1$ and $b > 0$ or $\sigma = 1$, $b > 2$ (Figure 2(a)).

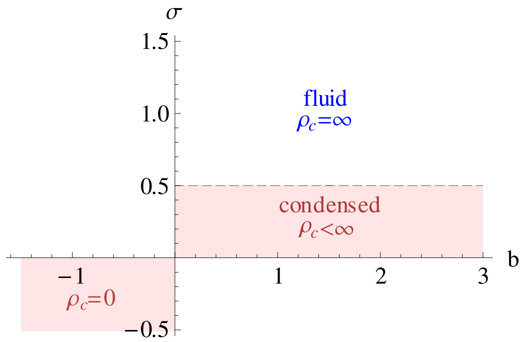
We consider a perturbation to this model whereby the jump rates are of the form

$$g_x(n) = \exp(e_x(n) + b/n^{\sigma}) \quad \text{for} \quad n \geq 1, \quad g_x(0) = 0 \tag{12}$$

where $e_x(n)$ are iid random variables with respect to x and n with $\mathbb{E}(e_x(n)) = 0$ and variance $\delta^2 > 0$. Note if



(a)Homogeneous system



(b)Perturbed system

FIG. 2: Change of phase diagram under perturbation of the jump rates. In side the red-shaded region condensation occurs on a finite system. (a) In the homogeneous system condensation is robust in the thermodynamic limit if $0 < \sigma < 1$ and $b > 0$ or $\sigma = 1$, $b > 2$. (b) Disorder changes the critical interaction exponent from 1 to 1/2.

$\delta^2 = 0$ it is clear from a power series expansion of the exponential that the asymptotic behaviour for large n is given by Eqn (11). We therefore interpret this model as a perturbation of (11).

Note that for $\sigma < 0$ and $b > 0$ or for $\sigma > 0$ and $b < 0$ the jump rates are increasing with n and hence there is no condensation. For $\sigma < 0$ and $b < 0$ the rates tend to zero, it is clear that the critical density $\rho_c = 0$, since $\phi_c = 0$, and the system always exhibits condensation. We therefore focus on positive interaction exponent and $b > 0$.

To access the conditions under which a condensation transition occurs in the perturbed system we examine the grand canonical partition function. A necessary condition for a condensation transition is that $z_x(\phi_c) < \infty$, since if $z_x(\mu) = \infty$ the local critical density $\rho^x_c = \rho^x(\phi_c) = \infty$ (see e.g. [23], Lemma I.3.3).

Firstly we must calculate the critical fugacity ϕ_c for the perturbed model. It is convenient to write the fugacity in terms of a chemical potential μ, so that $\phi = e^{\mu}$ and $\phi_c = e^{\mu_c}$. The one site grand canonical partition function, Eqn (5), for the perturbed system can then be written,

$$z_x(\mu) = \sum_{n=0}^{\infty} \exp(n\mu - \sum_{k=1}^{n} (e_x(n) + b/n^{\sigma})). \tag{13}$$

(The following summarizes relevant results from [17]). The convergence properties of $z_x(\mu)$ are given by the asymptotic behaviour of the exponent in Eqn (13). To leading order in n as $n\to\infty$ we have

$$\sum_{k=1}^{n}(e_x(n)+b/n^\sigma)\simeq\delta\sqrt{n}\,\xi_x(n)+\begin{cases}\frac{b}{1-\sigma}n^{1-\sigma} & ,\ \sigma\neq 1\\ b\ln n & ,\ \sigma=1\end{cases}\tag{14}$$

where by the central limit theorem

$$\xi_x(n)=\frac{1}{\delta\sqrt{n}}\sum_{k=1}^{n}e_x(k)\overset{n\to\infty}{\longrightarrow}N(0,1)\tag{15}$$

converges to a standard Gaussian. Moreover, the process $\big(\sqrt{n}\,\xi_x(n):n\in\mathbb{N}\big)$ is a random walk with increments of mean zero and variance 1. Since the fluctuations of such a process are of order $\sqrt{n}$ we have for all $C\in\mathbb{R}$

$$\mathbb{P}\big(\xi_x(n)\leqslant C\text{ for infinitely many }n\big)=1\ ,\tag{16}$$

and for all $\gamma>0,\ C>0$

$$\mathbb{P}\big(|\xi_x(n)|>Cn^\gamma\text{ for infinitely many }n\big)=0\ .\tag{17}$$

This is a direct consequence of the law of the iterated logarithm (see e.g. [21], Cor 14.8). It can be seen from Eqn (13) and (14) this implies that for almost all realizations of disorder $e_x(k)$ if $\mu<0$ then $z_x(\mu)<\infty$ and if $\mu>0$ then $z_x(\mu)=\infty$. So the critical chemical potential $\mu_c=0$ and $\phi_c=1$ (with probability one independent of site, the same as the homogeneous critical fugacity), and we have

$$z_x(\mu_c)=\sum_{n=0}^{\infty}\exp\Big(-\sum_{k=1}^{n}\Big(e_x(k)+\frac{b}{k^\sigma}\Big)\Big)\tag{18}$$

For $b>0$ the convergence properties of $z_x(\mu_c)$, depend on the value of the interaction exponent $\sigma>0$.

- For $\sigma>1/2$, $n^{1-\sigma}\lll\sqrt{n}$ and (14) is dominated by $\delta\sqrt{n}\,\xi_x(n)$. Applying (16) with $C=1$ we see that infinitely many terms in the series grows as $\sqrt{n}$ so $z_x(\mu_c)=\infty$ with probability one.
- For $\sigma=1/2$ both terms in (14) are of the same order since $n^{1-\sigma}\sim\sqrt{n}$ and

$$-\sum_{k=1}^{n}\Big(e_x(k)+\frac{b}{k^\sigma}\Big)\simeq-\delta\sqrt{n}\big(\xi_x(n)+\tfrac{2b}{\delta}\big)\ .\tag{19}$$

  Again applying (16), this time with $C=2b/\delta$, implies $z_x(\mu_c)=\infty$ with probability one.
- For $0<\sigma<1/2$ we have $n^{1-\sigma}\ggg\sqrt{n}$ and (14) is dominated by $\frac{b}{1-\sigma}n^{1-\sigma}$. We apply (17) for $\gamma=1-\sigma-1/2>0$ to see that

$$\delta\sqrt{n}|\xi_x(n)|>\frac{b}{2(1-\sigma)}n^{1-\sigma}$$

  for infinitely many n with probability zero and so $z_x(\mu_c)$ can be bounded above by a convergent series. Therefore $z_x(\mu_c)<\infty$ with probability one.

By the same argument as above it follows that when $z_x(\mu_c)<\infty$ then

$$\rho_c^x=\frac{1}{z_x(\mu_c)}\sum_{n=0}^{\infty}n\,e^{-\sum_{k=1}^{n}(e_x(k)+\frac{b}{k^\sigma})}<\infty\tag{20}$$

with probability one, since the factor n in the sum only gives a logarithmic correction in the exponent. It follows that the phase diagram under perturbation is as in Figure 2(b), where the critical interaction exponent has changed from 1 to 1/2.

B. Results 1: Monte Carlo

In this section we will present results obtained from Monte Carlo simulations that support the change of the phase diagram predicted theoretically. The theoretical results on the stationary distribution given in section III A were independent of the target probability distribution p (under the conditions stated in section II). To simplify the simulation the process was taken to be totally asymmetric so that $p(1)=1$. In the disordered system steady state quantities such as the critical density,

$$\rho_c=\frac{1}{L}\sum_{x=1}^{L}\rho_c^x,\tag{21}$$

are random variables fixed by the realisation of the system, so for each simulation, we fix the perturbation on a system with lattice Λ_L. Therefore we find quenched averages of observables with respect to a single realisation of $e_x(n)$. Particles were initially distributed uniformly at random on the lattice.

The continuous time dynamics were approximated using a standard Monte Carlo method of random sequential update (using the SIMD-oriented Fast Mersenne Twister pseudo random number generator [25]). Each time step a site x was chosen at random on the lattice and a particle on the site moved to the right with probability given by

$$p_{jump}(x,\eta_x)=\frac{g_x(\eta_x)}{g_{max}}\quad\text{where}\quad g_{max}=\max_{\substack{\hat{x}\in\Lambda_L\\0\leq k\leq N}}g_{\hat{x}}(k).\tag{22}$$

Regardless of the particle having moved or not the time was incremented by $(Lg_{max})^{-1}$. The number of Monte Carlo steps between samples was 4×10^8. Since the dynamics are ergodic the average of an observable taken over samples from the simulation converges to the expectation in the canonical distribution. Convergence was checked heuristically by examining the distribution of observables across the samples and also against exact numerics in the canonical ensemble (see section IV).

In accordance with the rigorous results for condensation on a finite lattice (section II B), we define the background density, ρ_{bg}, as the average number of particles

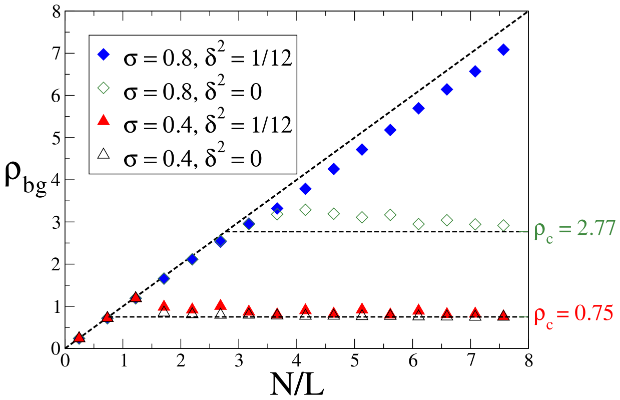


FIG. 3: Monte Carlo results demonstrating the change in phase diagram under perturbation. Background density ρ_{bg} as a function of N/L for $b = 1.2$ and $L = 1024$. For $\sigma = 0.4$ both systems demonstrate condensation. For $\sigma = 0.8$ the homogeneous system shows condensation and there is no condensation in the perturbed system.

per lattice site (in the canonical ensemble) after removing the site with highest occupation,

$$\rho_{bg} = \frac{1}{L-1}\left(N - \langle \max_{y\in\Lambda_L} \eta_y \rangle_{\pi^{L,N}}\right). \tag{23}$$

If the system has a finite critical density, $\rho_c < \infty$, then the background density converges to ρ_c as the total particle number N increases. This forms the so called critical-background containing on average ρ_c particles on each lattice site and a single condensed site contains the remaining $O(N)$ particles. If $\rho_c = \infty$ then the system will not undergo a condensation transition and ρ_{bg} will increase approximately linearly with N.

For the disordered system, where ρ_c^x is a random variable, we can define the slowest site, say y, by $\rho_c^y > \rho_c^x$ for all $x \neq y$. In this case the condensate is observed to reside on the slowest site and Eqn (23) simplifies to $\frac{1}{L-1}(N - \langle \eta_y \rangle_{\pi^{L,N}})$.

For $\sigma = 0.4$, ρ_{bg} converges to the system critical density in both the homogeneous and perturbed model. However for $\sigma = 0.8$, ρ_{bg} converges to ρ_c for the homogeneous system and increases approximately linearly with N/L for the perturbed system. This demonstrates the change in the critical exponent, in the perturbed system there is no-longer a condensation for $\sigma > 0.5$ These results are summarized in Figure 3. The deviation of ρ_{bg} from the identity when there is no condensation is due to the fluctuations in background density and is characterized by the tail of the distribution. Note also the overshoot of ρ_{bg} above the critical and convergence to ρ_c from above as N/L increases. This is a finite size effect due to sampling from the canonical ensemble and is studied in detail in the next section.

IV. RESULTS 2: STEADY STATE ANALYSIS OF FINITE SYSTEMS

In this section we study the finite size effects observed when sampling from the canonical steady state distribution. These include an overshoot in the background density (section III B) with a corresponding overshoot in the stationary current. Such effects have been observed in physical systems and studied for systems with a single defect site [18].

We focus on fixed lattice Λ_L and realisation of $e_x(n)$ for the same reasons discussed previously (section III B). Since we are interested in the behaviour of a typical system rather than an average over the disorder.

A. Exact numerics

The canonical partition function can be computed exactly by a recursion relation [19] which we extend to the heterogeneous system as follows,

$$\begin{aligned} Z(\Lambda_L, N) &= \sum_{\boldsymbol{\eta}\in X_L} \prod_{x\in\Lambda_L} w_x(\eta_x)\delta\left(\Sigma_x\eta_x - N\right) \\ &= \sum_{k=0}^{N} w_y(k) \sum_{\boldsymbol{\eta}\in\bar{X}} \prod_{x\in\Lambda_L\setminus\{y\}} w_x(\eta_x)\delta\left(\Sigma_x\eta_x-(N-k)\right) \\ &= \sum_{k=0}^{N} w_y(k) Z(\Lambda_L\setminus\{y\}, N-k), \end{aligned} \tag{24}$$

where $\bar{X} = X_{\Lambda_L\setminus\{y\}}$. This result is usefully employed to calculate exact numerics on a computer.

We define the stationary current as being the average jump rate $\langle g_x(\eta_x)\rangle$ (for a totally asymmetric process this is exactly the number of particles past an edge in unit time). In a stationary distribution this is clearly site independent. In the grand canonical distribution it is straight forward to show

$$j_{g.c.} = \langle g_x(\eta_x)\rangle_{\nu_\phi^L} = \phi \tag{25}$$

and so is bounded by ϕ_c. In the canonical distribution the stationary current can be related to the partition function by [9]

$$j_{can} = \langle g_x(\eta_x)\rangle_{\pi^{L,N}} = \frac{Z(\Lambda_L, N)}{Z(\Lambda_L, N-1)} \tag{26}$$

Another useful exact result is the single site canonical distibution given as a product of a single site weights and partition functions

$$\pi_x^{L,N}(k) = w_x(k)\frac{Z(\Lambda_L\setminus\{x\}, N-k)}{Z(\Lambda_L, N)}. \tag{27}$$

We define the canonical entropy density as

$$S_{can}^{\Lambda_L}(\rho) = \frac{1}{L}\log Z(\Lambda_L, [\rho L]). \tag{28}$$

The grand-canonical entropy density is defined by the negative Legendre transform

$$S_{g.c.}^{\Lambda_L}(\rho) = -\sup_{\phi\in[0,\phi_c)}\left(\rho\log\phi - \frac{1}{L}\log\prod_{x=1}^{L} z_x(\phi)\right)$$
$$= \begin{cases} \frac{1}{L}\log\prod_{x=1}^{L} z_x(\phi(\rho)) - \rho\log\phi(\rho) & \rho \le \rho_c \\ S_{g.c.}^{\Lambda_L}(\rho_c) & \rho > \rho_c \end{cases} \quad (29)$$

It has been proven that for the homogeneous system $S_{can}^{\Lambda_L}$ converges to $S_{g.c}^{\Lambda_L}$ as $L \to \infty$ [10].

We apply the above methods to studying the canonical steady state distribution for both the homogeneous and disordered sytem with focus on understanding the finite size effects.

B. Homogeneous system

We begin by studying the homogeneous case where the jump rates are given by Eqn (11) with the aim of extending these results to the perturbed system.

It was observed from the Monte Carlo results that there is a finite size over shoot in the background density. However it is more convenient to study the steady state current defined above since it can be calculated exactly in both the canonical and grand canonical ensembles. The canonical current is plotted against particle density $\rho = N/L$ for a fixed system size, $L = 1024$, interaction exponent $\sigma = 0.2$ and $b = 1.2$, in Figure 4(a). For these parameter values the system exhibits a condensation transition and $\rho_c \simeq 0.52$.

For low densities the current increases monotonically with the system particle density $\rho = N/L$ and up to a current of $\phi_c = 1$ matches very closely the grand canonical stationary current $\rho(\phi)$. The system is thus interpreted as being in a 'fluid' phase. Note that for an infinite system the canonical current would increase to a maximum at ϕ_c and then remain there. Also in the limit $N \to \infty$ the current will saturate to the critical fugacity ϕ_c. These are both direct results of the condensation transition in terms of equivalence of ensembles in section II B. For $\rho > \rho_c$ in the canonical ensemble we observe non monontonic behaviour and overshoot of the current above its saturation value. This effect decreases with system size L, and was also found to decrease with σ.

In the canonical ensemble clearly the maximum number of particles that can occupy a single site is N. The grand canonical distribution does not exist for fugacity $\phi > \phi_c$ since the partition function diverges. However the grand canonical distribution conditioned on maximum on site occupancy of N, $\nu_{x,\phi}(\cdot|\eta_x \le N)$, exists for all N and all $\phi \ge 0$. So we define

$$\nu_\phi^{\Lambda_L,N}(\boldsymbol{\eta}) := \prod_{x\in\Lambda_L} \nu_{x,\phi}(\eta_x|\eta_x \le N) \quad (30)$$

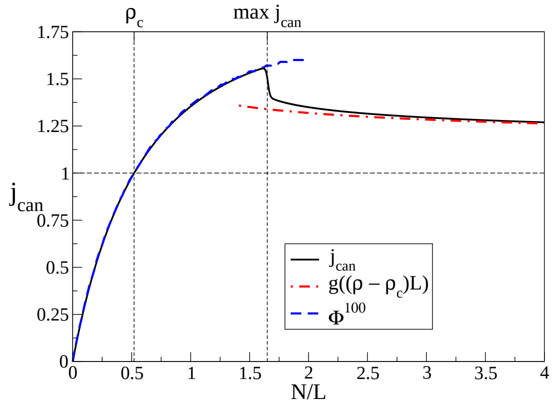
The distribution's partition function is just the grand canonical partition function with the infinite sum truncated at N, from Eqn (3) for a single site

$$z_x^N\phi = \sum_{n=0}^{N} w_x(n)\phi^n. \quad (31)$$

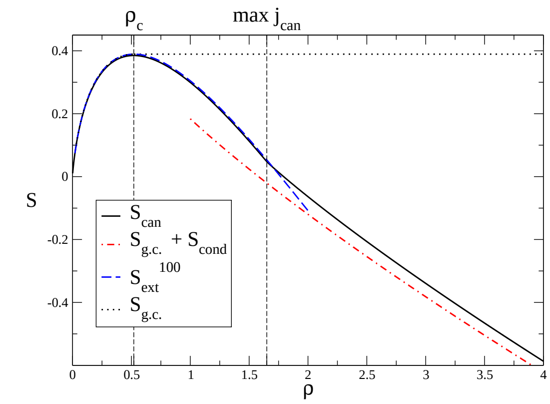
From this distribution we can define

$$\rho^{N,\Lambda_L}(\phi) = \frac{1}{L}\sum_{x\in\Lambda_L}\phi\partial_\phi \log z_x^N(\phi). \quad (32)$$

By monotonicity for fixed $\rho = N/L$ there is a unique ϕ such that $\rho^{[\rho L],\Lambda_L}(\phi) = \rho$, so we can extend the grand



(a)



(b)

FIG. 4: (a) Canonical current as a function of system density $\rho = N/L$ for homogeneous system with $L = 1024$, $b = 1.2$, $\sigma = 0.2$. The plot shows the match of the canonical current to the grand canonical extension Φ^{100} up to $\max j_{can}$ and then the rapid convergence of the canonical current to a single site condensate current. (b) Canonical entropy density for the same system as a function of system density $\rho = N/L$. Canonical entropy density matches closely the extension of the grand canonical entropy density S^{100} up to a density that correspond to $\max j_{can}$ after which it appears to converge slowly to $S_{g.c.} + S_{cond}$

canonical current to higher densities by defining,

$$\Phi^{\Lambda_L}(\rho) = \phi \quad \text{such that} \quad \rho^{[\rho L],\Lambda_L}(\phi) = \rho. \tag{33}$$

We interpret this as an extension of the fluid phase current since sites are restricted so that they can not contain infinitely many particles. This interpretation fits with the Monte Carlo results in which no condensate persisted for densities in the region of increasing current. The extension of the grand canonical current matches very closely the overshoot in current up to $\max j_{can}$, see Figure 4(a).

For high densities the canonical current converges to the critical fugacity, ϕ_c, from above and even for 'large' system sizes the current can be significantly above this value for densities well above ρ_c (Figure 4(a)). In order to understand this phenomenon we appeal to thermodynamic results [10], that suggest for large system sizes and densities above the critical densities the system is expected to contain a single site condensate. Monte Carlo results also support this interpretation (section III B). If the background density converges to ρ_c then for a system containing N particles the total number of particles in the condensate will be of order $N - \rho_c L = (\rho - \rho_c)L$ where $\rho = N/L$ is the system density. Since the current is site independent in the steady state then for a system containing a single condensate we can estimate the stationary current by the jump rate of a site containing $(\rho - \rho_c)L$ particles. Assuming the system is homogeneous then

$$g_x\left((\rho - \rho_c)L\right) = 1 + b/((\rho - \rho_c)L)^{\sigma}, \tag{34}$$

plotted in Figure 4(a). This matches very closely the exact numerics in the canonical ensemble which is compelling evidence for this interpretation of condensation in a finite system.

In order to support these results and to understand and predict the point of crossover from 'fluid' to 'condensed' on a finite system we study the entropy density. $S^{\Lambda_L}_{g.c.}(\rho)$, Eqn (29), is concave and increases to a constant at ρ_c. In the thermodynamic limit $S^{\Lambda_L}_{can}$ converges to $S^{\Lambda_L}_{g.c}$ [10], however for finite systems they deviate significantly, Figure 4(b).

We can extend the grand canonical entropy to fluid systems above ρ_c by restricting the on site occupancy again and using $\Phi^{\Lambda_L}(\rho)$, Eqn (33), we get

$$S^{\Lambda_L}_{ext}(\rho) = \frac{1}{L}\log\prod_{x=1}^{L} z_x(\Phi^{\Lambda_L}(\rho)) - \rho\log\Phi^{\Lambda_L}(\rho). \tag{35}$$

Figure 4(b) demonstrates how closely this matches the canonical entropy up to a density corresponding to $\max j_{can}$. We approximate the canonical entropy density beyond this point by assuming one randomly located site contains a condensate and each site in the critical background is distributed with ν_{x,ϕ_c}. In the homogeneous system the condensate is located on a site uniformly at random so we estimate the canonical partition function,

$$Z(\Lambda_L, N) \simeq L w((\rho - \rho_c)L) \sum_{\boldsymbol{\eta}\in X_{\Lambda_{L-1}}} w^{L-1}(\eta_x)\delta\left(\Sigma_x \eta_x - \rho_c L\right). \tag{36}$$

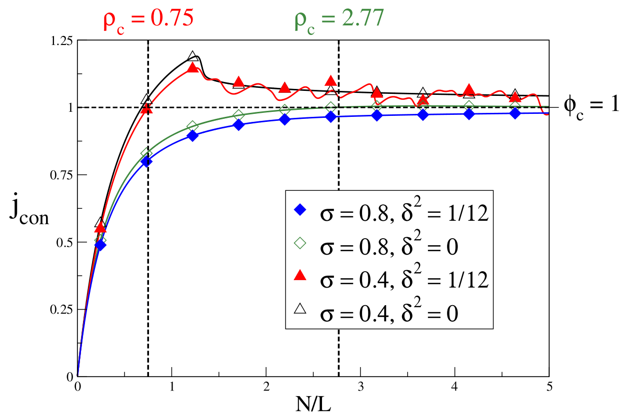


FIG. 5: Canonical currents for both the disordered, $\delta^2 = 1/2$, and homogeneous system as a function of system density $\rho = N/L$. For $\sigma = 0.4$ both the currents increase above ϕ_c, for $\sigma = 0.8$ with disorder the current increases monotonically and is interpreted as remaining in a fluid state. Lines show exact numerics and points are from Monte Carlo simulations. Note the nonmonotonic behaviour of the current after $\max j_{can}$ for the disordered system and $\sigma = 0.4$.

where we drop the subscript x because sites are identical. It follows immediately by assuming that the critical background is distributed with ν_{x,ϕ_c} that for $\rho \gg \rho_c$

$$S^{\Lambda_L}_{can}(\rho) \simeq S_{cond}(\rho) + S^{\Lambda_{L-1}}_{g.c}(\rho_c) \tag{37}$$

where we define,

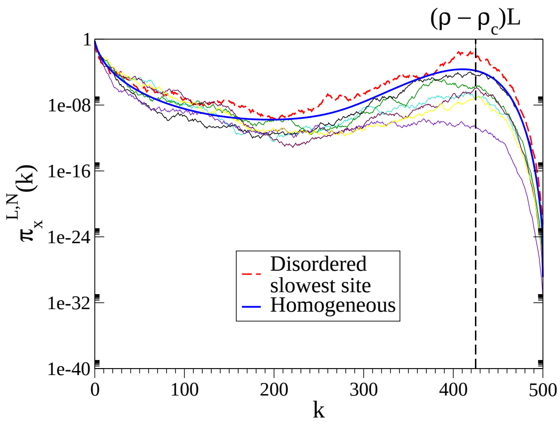
$$S_{cond}(\rho) = \frac{1}{L}\log w((\rho - \rho_c)L) + \frac{1}{L}\log L. \tag{38}$$

This is plotted in Figure 4(b).

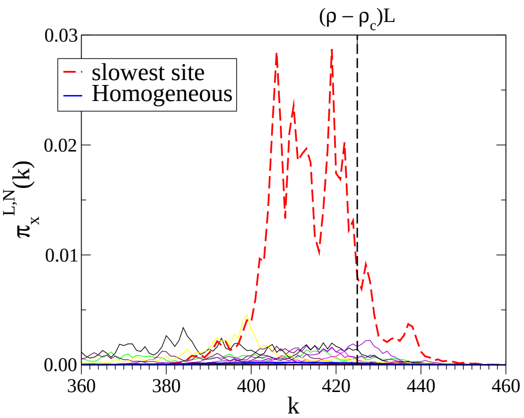
This result can be used to predict the point at which the canonical system 'condenses' (reaches $\max j_{can}$) as the point at which it becomes entropically favorable the excess, $(\rho - \rho_c)L$, particles to condense on a single lattice site over the system remaining fluid.

C. Heterogeneous/perturbed system

The same method as discussed in the previous section IV B for extending the grand canonical 'fluid' phase to densities above ρ_c fits the numerical results in the canonical ensemble for the perturbed system also. So this method shows itself to be robust to the perturbation. However beyond $\max j_{can}$ the number of particles in the condensate site fluctuates significantly leading to non monotonic behaviour of j_{can} within the 'condensed' regime Figure 5. This means that the method used for homogeneous systems to predict the current and entropy in this region no longer give as close an estimate. For each x, $w_x(N - \rho_c L)$ varies significantly for each N due to the nature of the perturbation and fluctuates faster and with



(a)



(b)

FIG. 6: Single site distribution of 10 sites for a system with $L = 100$, $N = 500$, $b = 1.2$ and $\sigma = 0.4$ within the 'condensed' region for both homogeneous and perturbed systems. (a) log linear plot showing a maximum in the distribution for both systems at low densities and then a lower local maximum at densities close to $N - \rho_c L$ (b) Linear plot zoomed in region close to $N - \rho_c L$ showing much higher probability of the slowest site in the perturbed system containing a significant fraction of the particles compared to the homogeneous system (line close to the axis)

greater magnitude than j_{can} as N varies. An average over all sites significantly over estimates the canonical current.

In order to explain this effect we examine the single site distribution in the canonical ensemble, Eqn (27), above the critical density in Figure 6. In the homogeneous system there is a large probability of a site containing around ρ_c particles and then another peak close to $(\rho - \rho_c)L$, where the area under the second peak is $1/L$ clearly representing the probability of a given site containing the condensate. However we observed from Monte Carlo simulations that in the perturbed system the condensate typically resides on the slowest site. This is supported by the single site distributions where the probability of the condensate residing on the slowest site is much greater than it being on any other site Figure 6(b). The complex form of the distribution close to $N - \rho_c L$ also explains why the previous methods of estimating the current in the 'condensed' regime will not directly work.

V. RESULTS 3: THERMODYNAMIC LIMIT IN PERTURBED SYSTEMS

Numerical estimates of the distribution of the one site critical density, ρ_c^x, for the perturbed system suggest that at least for some values of the system parameters σ, δ^2 and b it may be that $\mathbb{E}(\rho_c^x) = \infty$, see Figure 7. In this case it is clear from Eqn (21) that as the system size increase $L \to \infty$ the system's critical density $\rho_c \to \infty$ for each realisation. This in turn would imply that the condensation transition is not robust in the thermodynamic limit for the perturbed system, representing a significant change in the critical behaviour compared to the homogeneous system.

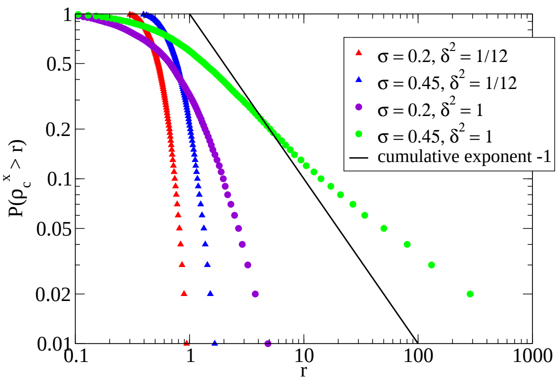


FIG. 7: (a) Cumulative tail of the distribution of ρ_c^x from 10^4 independent samples of Eqn (20) where the infinite sum was truncated at $N = 5 \times 10^5$ for a system where $b = 1.2$. The straight line corresponds to a cumulative exponent of -1. The plot demonstrates the heavy tails of the distribution suggesting for at least some parameter values ρ_c^x has infinite mean.

For large n we can calculate the expectation of a single term in the series for $z_x(\mu)$ [17] from Eqn (18) and Eqn (14)

$$\mathbb{E}(e^{\delta\sqrt{n}\,\xi_x(n)+\mu n}) = e^{(\mu+\delta^2/2)n} \tag{39}$$

which diverges for all $\mu > -\delta^2/2$. This also suggest that the critical density may have infinite mean. However for applications we are interested in the quenched average

$$\begin{aligned}\mathbb{E}(\rho_c^x) &= \mathbb{E}(\phi_c \partial_\phi \log z_x(\phi_c)) \\ &= \mathbb{E}(\partial_\mu \log z_x(\mu_c)),\end{aligned} \tag{40}$$

which may still be finite even though $\log \mathbb{E}(z_x(\mu))$ diverges for $\mu = \mu_c = 0$.

In order to investigate the convergence properties of ρ_c we study the $\mathbb{E}(\log z_x(\mu))$ since under the assumption that the expectation and differential operator commute we have

$$\mathbb{E}(\rho^x(\mu)) = \partial_\mu \mathbb{E}(\log z_x(\mu)). \tag{41}$$

Since $\mathbb{E}(\log z_x(\mu))$ is strictly increasing on $(-\infty, \mu_c]$, $\mathbb{E}(\rho_c^x)$ diverges if and there exists a $\mu \in (-\infty, \mu_c]$ such that $\mathbb{E}(\log z_x(\mu)) = \infty$. We proceeded by estimating the distribution of $\log z_x(\mu)$. Under the assumption that the random variables $e_x(n)$ are normally distributed then for a given realisation we can estimate $z_x(\mu)$ by an integral upper bound.

$$z_x(\mu) \lesssim 1 + \int_0^\infty \exp(\delta B_t + \mu t)\,\mathrm{d}t \tag{42}$$

where the integral is equal in distribution to $2/(\delta^2 Y_{-2\mu/\delta^2})$ where $Y_{-2\mu/\delta^2}$ is a gamma variable with index $-2\mu/\delta^2$ [26]. So by a change of variables we can estimate the distribution of $\log(z_x(\mu))$ and find that as $\mu \to 0$ the expectation diverges. However for all $\mu < 0$ we have that the expectation is bounded. It may be possible to use similar methods to find a lower bound of $\log(z_x(\mu))$ and so find sufficient conditions for the critical density to diverge.

VI. DISCUSSION

In this work we have studied the effect of a random perturbation to the ZRP that has been widely studied and applied. We have focused mainly on finite systems because of their physical relevance. The Monte Carlo results confirm recent theoretical work on the change of the critical exponent under perturbation. This could have implications for applications to real systems such as traffic models where there will be some inhomogeneities and suggests that predicted phase boundaries may be significantly effected by this disorder.

Simulations and numerics in the canonical ensemble demonstrated significant finite size effects also relevant to real systems. For both the homogeneous and perturbed system we can describe the initial overshoot in current in terms of an extension of the fluid phase due to finite maximum one site occupation. For the homogeneous system we can also describe the overshoot in the 'condensed' region in terms of a single condensate site. We can apply a similar method to the entropy density which could be useful in predicting the scaling of the crossover density with system parameters such as L. The results on the entropy density are also interesting in terms of a more rigours understanding of condensation on a finite system that could be extended to non homogeneous systems.

It would be interesting to further understand stationary properties of the perturbed system in the 'condensed' regime on a finite lattice. Whether or not the condensation transition in the perturbed system is robust in the thermodynamic limit also remains an interesting question. In order to shed some light on this we have presented a potential approach to tackle the problem.

Acknowledgements

I thank Stefan Grosskinsky for suggesting the topic and his continued support throughout the project. I would also like to acknowledge Stefan and Gunter Schütz for the work that this project was based upon. Finally I thank those involved in the Complexity DTC for its' support and tuition and the EPSRC for funding.

[1] F. Spitzer, Adv. Math. **5**, 246–290 (1970).
[2] M.R. Evans, T. Hanney, J. Phys. A: Math. Gen. **38**, R195–R239 (2005).
[3] J. Eggers, Phys. Rev. Lett. **83**(25), 5322-5325 (1999). van der Meer, D., van der Weele, J.P., Lohse, D., Phys. Rev. Lett. **88**, 174302 (2002). F. Coppex, M. Droz, A. Lipowski, Phys. Rev. E **66**, 011305 (2002).
[4] van der Meer, D., van der Weele, K., Reimann, P., Lohse, D., J. Stat. Mech.: Theor. Exp. P07021 (2007).
[5] Y. Kafri, E. Levine, D. Mukamel, G.M. Schütz, and J. Török. Phys. Rev. Lett. **89**(3), 035702 (2002).
[6] A.G. Angel, M.R. Evans, E. Levine, D. Mukamel, Phys. Rev. E **72**, 046132 (2005). A.G. Angel, T. Hanney, M.R. Evans, Phys. Rev. E **73**, 016105 (2006).
[7] Kaupuzs J, Mahnke R, Harris RJ, Phys. Rev. E **72**(5), 056125 (2005).
[8] I. Benjamini, P.A. Ferrari, C. Landim, Stoch. Proc. Appl. **61**, 181–204 (1996).
[9] M.R. Evans, Braz. J. Phys. **30**(1), 42–57 (2000).
[10] S. Grosskinsky, G.M. Schütz, H. Spohn, J. Stat. Phys. **113**(3/4), 389–410 (2003).
[11] M.R. Evans, S.N. Majumdar, R.K.P. Zia, J. Stat. Phys. **123**, 357–390 (2006).
[12] M.R. Evans, T. Hanney, S.N. Majumdar, Phys. Rev. Lett. **97**, 010602 (2006).
[13] J.M. Luck, C. Godreche, J. Stat. Mech.: Theor. Exp. P08005 (2007).
[14] A.G. Angel, M.R. Evans, E. Levine, D. Mukamel, J. Stat. Mech.: Theor. Exp. P08017 (2007).
[15] S. Grosskinsky, G.M. Schütz, to appear in J. Stat. Phys. (arXiv:0801.1310)
[16] Y. Schwarzkopf, M.R. Evans, D. Mukamel, J. Phys. A: Math. Theor. **41**, 205001 (2008).
[17] S. Großkinsky, P. Chleboun and G.M. Schütz. (Preprint), Available from: `http://arxiv.org/abs/0805.2748v1`.

[18] A.G. Angel, M.R. Evans, D. Mukamel, J. Stat. Mech.: Theor. Exp. P04001 (2004).
[19] O.J. O'Loan, M.R. Evans, M.E. Cates Phys. Rev. E **58** 1404
[20] E. Andjel, Ann. Probability **10**(3), 525–547 (1982).
[21] O. Kallenberg, *Foundations of Modern Probability, 2nd edition* (Springer, New York, 2002).
[22] P.A. Ferrari, C. Landim, V.V. Sisko, J. Stat. Phys. **128**, 1153-1158 (2007).
[23] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, Berlin, 1999).
[24] A.G. Angel, M.R. Evans, D. Mukamel, J. Stat. Mech.: Theor. Exp. P04001 (2004).
[25] M. Saito, M. Matsumoto MCQMC, 607-622 (2008)
[26] M. Yor, J. App. Prob. **29** 202-208 (1992)