Emergent phenomena are ubiquitous in systems with a large number of interacting components. Such systems arise across physical and social sciences. Examples include traffic behaviour and ribosomes moving along messenger RNA in protein synthesis. These systems can be studied mathematically in the context of non-equilibrium statistical mechanics applied to interacting particle systems [1,2].

The Zero Range Process

The ZRP is a continuous time Markov Chain with state space \( \mathbb{N} \) in which particles jump on a lattice \( \mathbb{Z} \) at rates which depend only on the departure site and the number of particles there. Configurations are given by \( \eta = (n_x)_{x \in \mathbb{Z}} \), where \( n_x \) is the occupation number of site \( x \). Particles leave site \( x \) at \( g_x(\eta) \) and jump to site \( y \) with probability \( \rho(\eta) \).

Motivation

The Bus route model

A totally asymmetric exclusion process is an example of a coarse-grained description, internal structures are integrated out leaving effective dynamics depending only on the distance between particles.

Steady states

Canonical ensemble: Fixed number of particles \( \sum_{x=1}^{L} n_x = N \),

\[
\rho^N(\eta) = \frac{1}{Z(N,L)} \prod_{x=1}^{L} w(n_x) (\Sigma_{x \in \eta} n_x - N),
\]

where the stationary weights are given by \( w(n) = \prod_{k=1}^{n} g(k)^{-1} \).

Grand canonical ensemble: Expected number of particles is fixed by the fugacity \( \phi \). Steady state \( \rho^\phi(\eta) \) is given by product of one site marginals,

\[
\rho^\phi(n_x) = \frac{1}{Z(\phi)} w(n_x) \phi^{n_x},
\]

where \( Z(\phi) = \sum_{n} w(n) \phi^{n} \) converges for \( \phi \in [0, \rho(\eta)] \).

The local particle density is \( \langle n_x \rangle = \rho^\phi(\eta) \). Total density \( \rho = \frac{1}{L} \sum_{x} \rho^\phi(\eta) \) is invertible on \( [0, \rho(\eta)] \).

Condensation

If the jump rates \( g_x(k) \) decrease with \( k \) then there can exist a critical density \( \rho_c = \rho(\eta_0) < \infty \). In the thermodynamic limit \( N \to \infty \) with \( \rho = \rho N \) fixed. For \( \rho > \rho_c \) excess particles condense on a single site (below). A similar result has also been proven rigorously for \( L \) fixed as \( N \to \infty \) for a homogeneous system [6].

Research Goals

- Examine stationary properties of the system in the canonical and grand canonical ensemble from numerical studies of the current and the total entropy, \( 1 / L \log Z(L,N) \).
- Explain finite size behaviour by studying the contribution of the condensate to the total entropy.
- Support theoretical results on the change of the stationary behaviour using Monte Carlo simulations [5].

Preliminary results

Monte Carlo (MC)

Analysis of a system with fixed disorder. For MC results \( L = 1024 \). For \( N / L > \rho_c \), \( \langle n_x \rangle_{\eta_0} = \rho_c^{\phi(\eta)} \) except for the "slowest" site, this is expected to contain the condensate. For the site with the next particle, as \( N \to \infty \) we expect \( \rho_{bg} \to \rho_c \) in accordance with [6].

Numerics

The stationary current \( j = \langle g_x(\eta) \rangle \) is independent of the site \( x \).

Canonical:

\[
J_{\text{can}}(N,L) = \frac{Z(L,N-1)}{Z(L,N)}
\]

Grand canonical:

\[
J_{\text{grand}}(\phi) = \frac{\partial}{\partial \rho} \rho \ln \rho = \frac{\phi}{\rho} = \frac{\phi}{\rho_0}.
\]

References