Mitigating harbour storms by enhancing nonlinear wave interactions

Paul Chleboun
Petr Denissenko, Colm Connaughton, Sergey Nazarenko, Sergei Lukaschuk.

Complexity Science Doctoral Training Centre
University of Warwick

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2. Theory
   - Surface gravity waves
   - Hamiltonian description
   - Wave turbulence
3. Experimental setup
4. Results
   - Initial surface elevation
   - Power spectra
   - 4-wave resonant interactions
5. Conclusion and Further work

Harbour storms

[www.hydrolance.net]

Motivation

- Waves of tens of meters pose risks in oceanic harbours.
- Waves driven by wind and waves from the ocean.
- Energy dissipation occurs at short wavelengths by wave breaking and white-capping.

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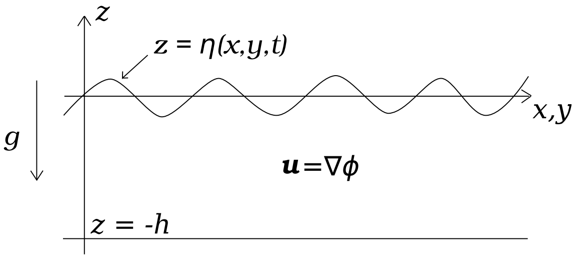
Aims

- Understand energy transfer by nonlinear wave interaction.
- Excite wave modes that enhance the energy transfer.

The model



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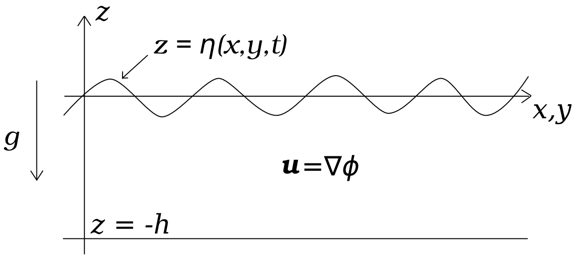

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  $\Longrightarrow \mathbf{u} = \nabla\phi$.

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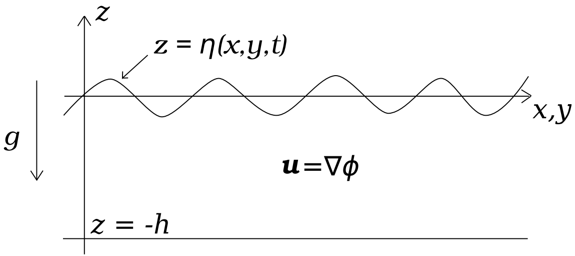

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  $\implies \mathbf{u} = \nabla \phi$.
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  $\nabla \cdot \mathbf{u} = 0$
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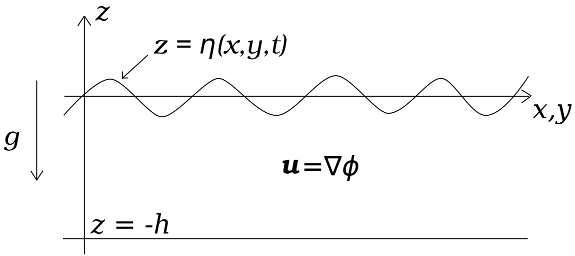

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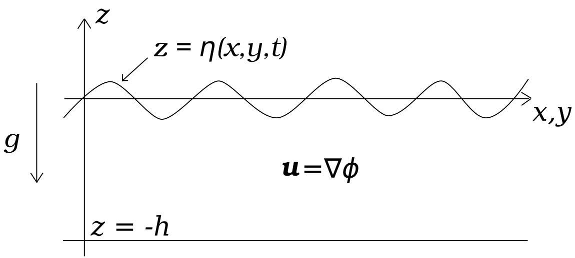
Definition

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, \eta(\mathbf{x}, t), t).$$

Boundary conditions



Dynamic

$$\frac{\partial\psi}{\partial t}+\frac{1}{2}(\nabla\phi)^2|_{z=\eta}+g\eta=0$$

Kinematic

$$\frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial z}\quad \text{on } z=\eta(\mathbf{x},t)$$

$$\frac{\partial\phi}{\partial z}=0\quad \text{on } z=-h.$$

Together with Laplace equation fully specify system.

Wave solution

Fourier transform

$$f(\mathbf{k}) = \frac{1}{2\pi} \int f(\mathbf{x}) e^{-i(\mathbf{k}\cdot\mathbf{x})} \,\mathrm{d}\mathbf{x}$$

$$f(\mathbf{x}) = \frac{1}{2\pi} \int f(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x})} \,\mathrm{d}\mathbf{k}$$

- Wave steepness; $\alpha = k\eta(\mathbf{k})$, assumed small.
- $\eta(\mathbf{k}, t) = \eta_0(\mathbf{k}) e^{i\omega(\mathbf{k})t}$ and $\psi(\mathbf{k}, t) = \psi_0(\mathbf{k}) e^{i\omega(\mathbf{k})t}$

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Dispersion relation

$$\omega(\mathbf{k}) = \sqrt{gk\tanh kh}$$

$\omega(\mathbf{k}) \to \sqrt{gk}$ as $h \to \infty$.

Hamiltonian Equations of motion

- $\eta(\mathbf{k}, t)$ and $\psi(\mathbf{k}, t)$ are canonical variables satisfying Hamiltonian equations [Broer 74, Miles 77].
- Assume random phase and amplitude and take infinite region then small nonlinearity limit.

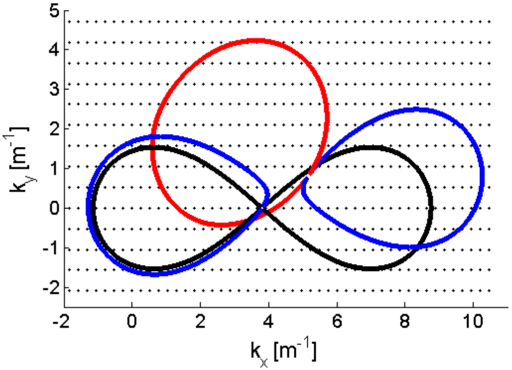
$\Rightarrow$ Derive a kinetic equation describing evolution of the spectrum in terms of wave-action (average amplitude).

$$\frac{\partial n_k}{\partial t} = \int W_{\mathbf{k_1},\mathbf{k_2}}^{\mathbf{k_3},\mathbf{k}} \left(n_{k_1} n_{k_2} (n_{k_3} + n_k) - n_{k_3} n_k (n_{k_1} + n_{k_2}) \right) \delta(k_1 + k_2 - k_3 - k)\delta(\omega_1 + \omega_2 - \omega_3 - \omega)\, \mathrm{d}k_{1,2,3}$$

Resonant interactions

Resonant manifold

$$\omega(\mathbf{k_1}) + \omega(\mathbf{k_2}) - \omega(\mathbf{k_3}) - \omega(\mathbf{k_4}) = 0, \quad \mathbf{k_1} + \mathbf{k_2} - \mathbf{k_3} - \mathbf{k_4} = \mathbf{0},$$



$$\mathbf{k}_{d1} = (3.810, 0)$$
$$\mathbf{k}_{d2} = (5.166, 0.818)$$

$$2\mathbf{k}_{d1} = \mathbf{k}_3 + \mathbf{k}_4.$$
$$\mathbf{k}_{d1} + \mathbf{k}_{d2} = \mathbf{k}_3 + \mathbf{k}_4.$$
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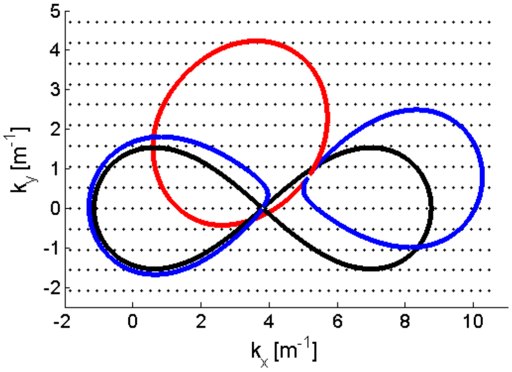
Finite region

1. Initially at rest then a finite number of modes are driven.



Finite region

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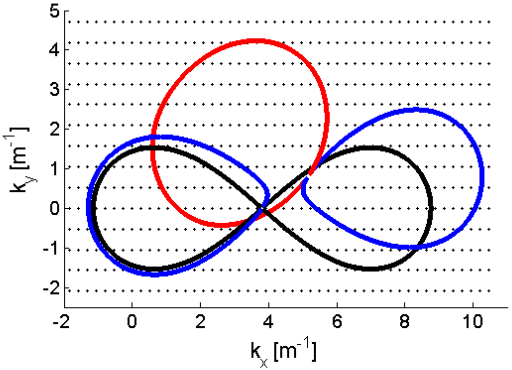
2. Cascade arrest due to k-space discreteness leads to accumulation of energy near forcing scales.



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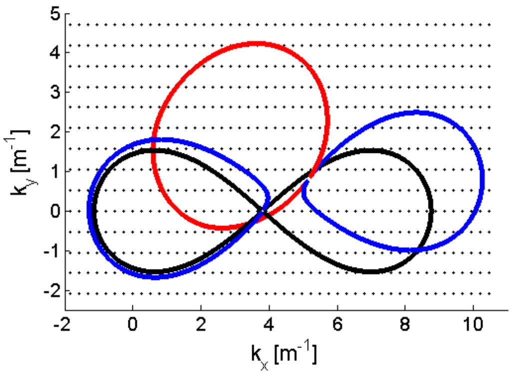
3. This leads to widening of the nonlinear resonance.



Finite region

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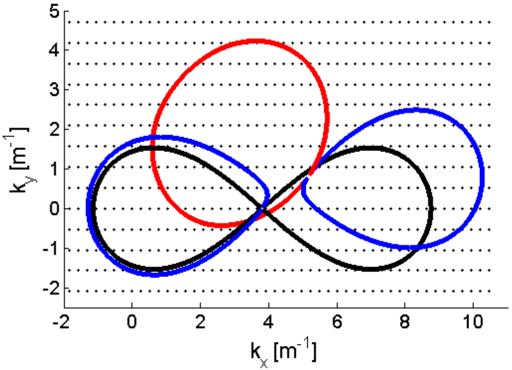
3. This leads to widening of the nonlinear resonance.
4. Sufficient widening triggers a cascade.



Turbulence cascade

Wave energy spectrum

$$E_f = \int e^{2\pi f t' i} \langle \eta(\mathbf{x}, t)\eta(\mathbf{x}, t + t') \rangle dt'$$

- Zakharov Filonenko (1967) power law solution to wave kinetic equation.

$$E_f \sim f^{-4}.$$

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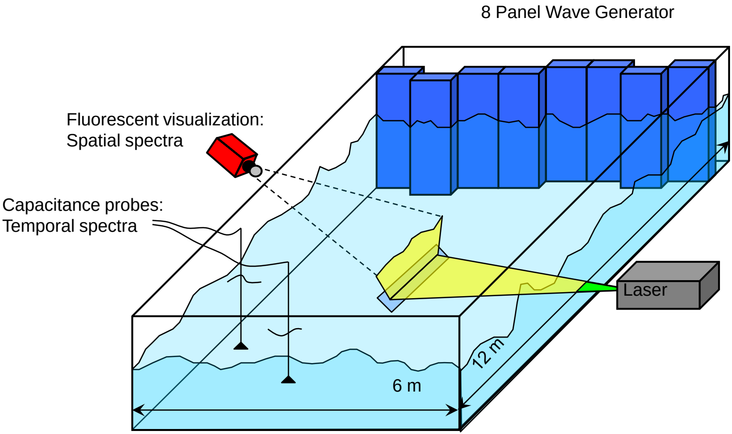
Other power law power spectra

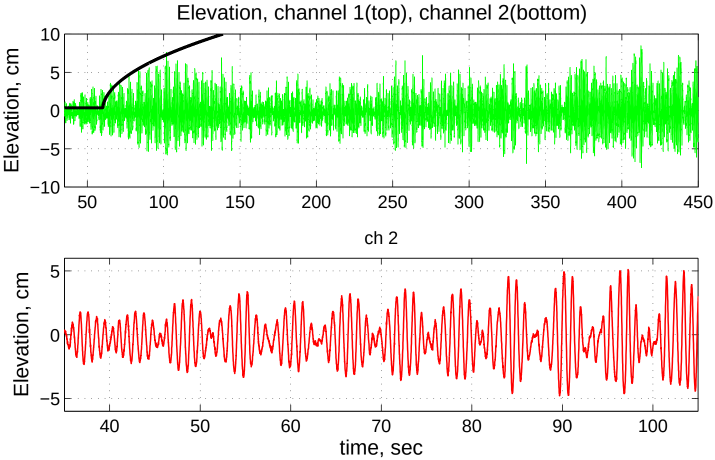
- Phillips (58), $E_f \sim f^{-5}$
- Kuznetsov (04), $E_f \sim f^{-(3+D)}$
- Nazarenko (06), $E_f \sim f^{-6}$

Experimental Setup



[Total Environment Simulator at “The Deep” Geography Department, University of Hull]

Data

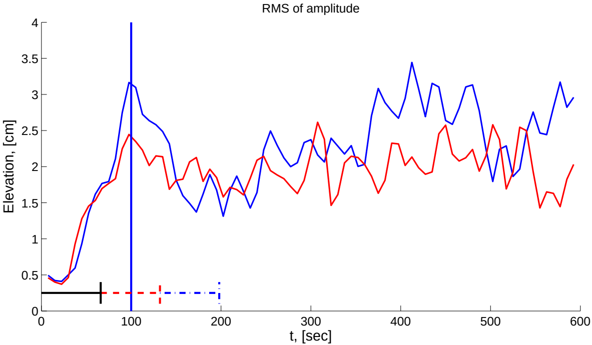

Elevation, channel 1(top), channel 2(bottom)
Elevation, cm
10
5
0
−5
−10
50
100
150
200
250
300
350
400
450
ch 2
Elevation, cm
5
0
−5
40
50
60
70
80
90
100
time, sec

Initial maximum

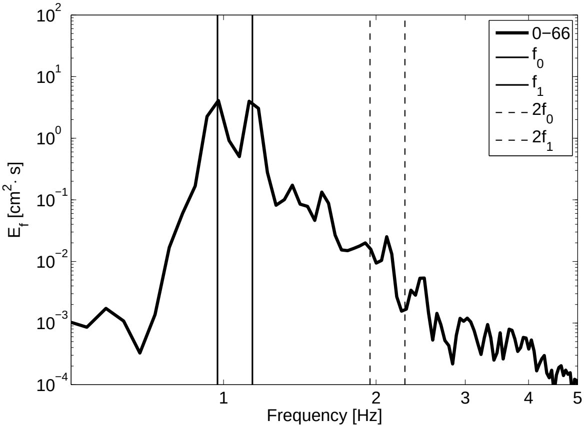
Root mean squared surface elevation

Averaged of 10 second windows

$$A = \sqrt{\langle(\eta - \bar{\eta})^2\rangle}.$$

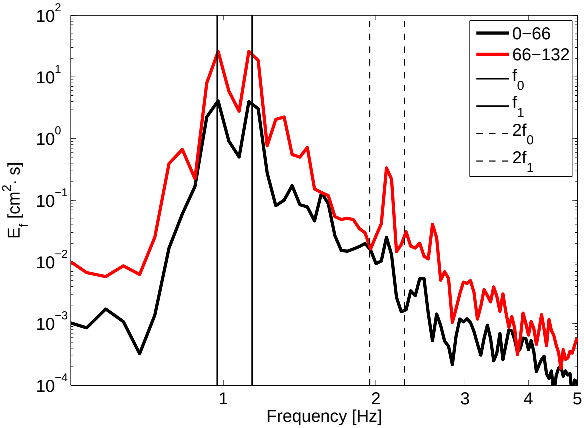


Spectra



Welch algorithm with Hann windows of length 20.48s averaged over minimum of 5 spectra.

Spectra



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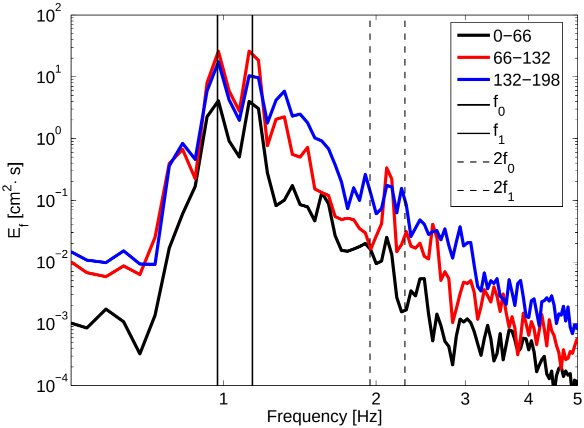

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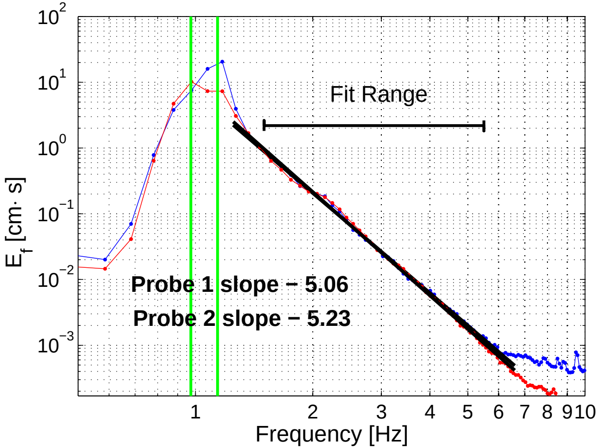
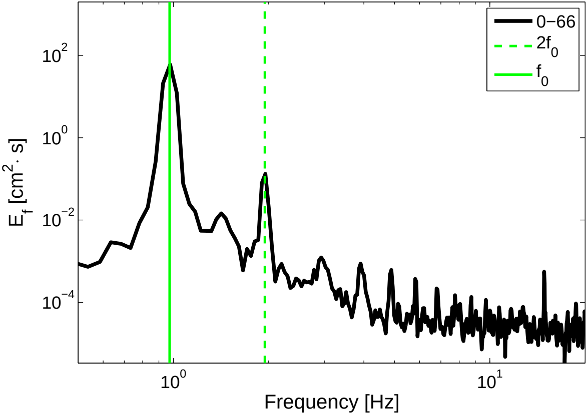


Welch algorithm with Hann windows of length 10.24s averaged over minimum of 500 spectra.

Spectra



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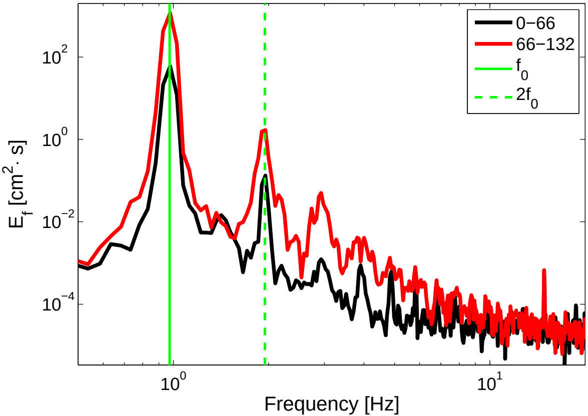
Spectra



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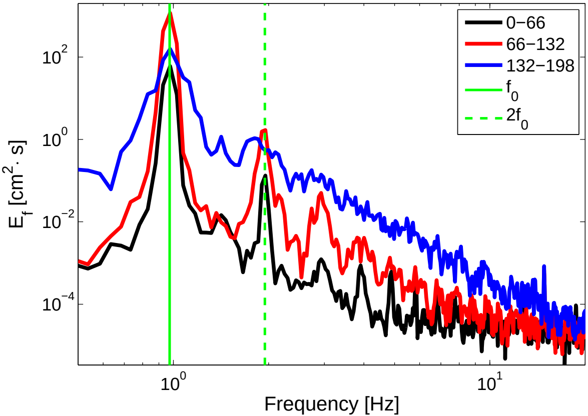
Spectra



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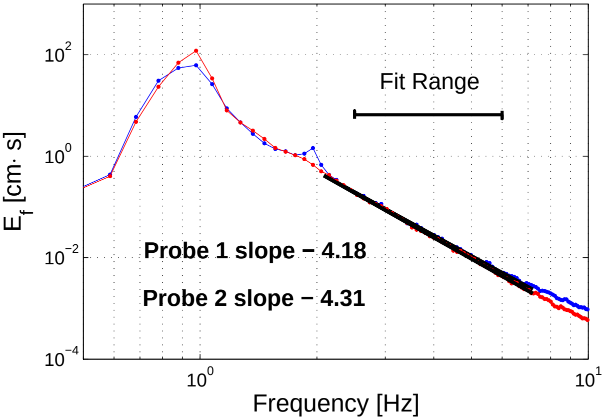

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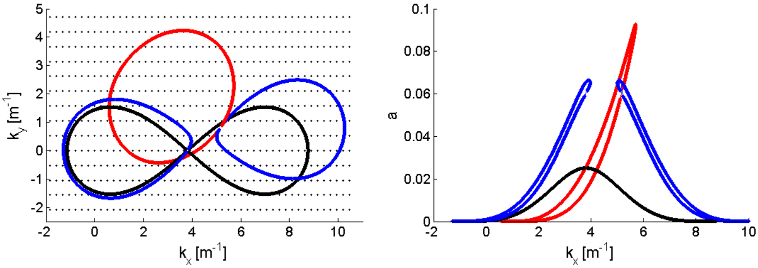
4-wave resonant interactions



(a) Resonant manifold (b) Interaction coefficient

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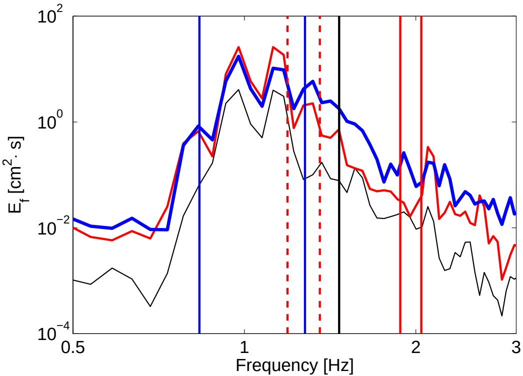
Presence of quasi-resonance in spectra



$$|2\omega(\mathbf{k}_{d1}) - \omega(2\mathbf{k}_{d1} - \mathbf{k}_4) - \omega(\mathbf{k}_4)| = 0.0003 \quad \text{black}$$

$$|\omega(\mathbf{k}_{d1}) + \omega(\mathbf{k}_{d2}) - \omega(\mathbf{k}_{d1} + \mathbf{k}_{d2} - \mathbf{k}_4) - \omega(\mathbf{k}_4)| = 0.0052 \quad \text{blue}$$

$$|\omega(\mathbf{k}_{d1}) - \omega(\mathbf{k}_{d2}) + \omega(\mathbf{k}_{d2} - \mathbf{k}_{d1} + \mathbf{k}_4) - \omega(\mathbf{k}_4)| = 0.0051 \quad \text{red.}$$

Conclusion and further work

Conclusions:

1. Accumulation of energy at driving frequency before the onset of nonlinear interactions supports resent theoretical work.
2. Greater accumulation for single driving frequency (or well spaced driving frequencies).
   Largest interaction coefficient with two close driving modes.
3. Predict energy transfer based on closest quasi-resonance with eigenmodes of the Harbour.

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Further work:

1. More experiments for various driving rates and modes.
2. Supported by numerical simulations.
3. More accurate estimates of interaction coefficient.

Acknowledgements

Thanks to

- Petr Denissenko for supervision and
- Sergey Nazarenko for co-supervision.
- Sergyei Lukaschuk for the experimental results.
- Colm Connaughton for many useful discussions.

- Warwick Complexity Science DTC
- Funding from the EPSRC

- All data supplied by Denissenko and Lukaschuk from the total environment simulator at "The Deep", University of Hull Geography Department.

- $\eta(\mathbf{k}, t)$ and $\psi(\mathbf{k}, t)$ are canonical variables satisfying Hamiltonian equations [Broer 74, Miles 77].

$$\frac{\partial \eta(\mathbf{x}, t)}{\partial t} = \frac{\delta H}{\delta \psi(\mathbf{x}, t)}, \quad \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{\delta H}{\delta \eta(\mathbf{x}, t)}$$

Where $H = K + \Pi$,

$$K = \frac{1}{2} \int \int_{-h}^{\eta} (\nabla \psi)^2 \, \mathrm{d}z \, \mathrm{d}\mathbf{x}$$

$$\Pi = \frac{1}{2} g \int \eta^2 \, \mathrm{d}\mathbf{x}$$

- Assume small nonlinearity (wave steepness)

⇒ Expand the Hamiltonian as an integral power series in conjugate variables.

- Canonical transformations reduce the Hamiltonian.
- Assume random phase and amplitude and take infinite region then small nonlinearity limit.

⇒ Derive a kinetic equation describing evolution of the spectrum.

Resonant manifold

$$\omega(\mathbf{k_1}) \pm \omega(\mathbf{k_3}) \pm \omega(\mathbf{k_3}) = 0, \quad \mathbf{k_1} \pm \mathbf{k_2} \pm \mathbf{k_3} = \mathbf{0},$$
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