Pulse-Coupled Oscillator Networks: Do Small Worlds Synchronize Quicker?

EPSRC

Max Planck Institute for Dynamics and Self-Organisation

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1. Introduction

Many complex real-world systems exhibit synchronization of their components. Examples include audiences applauding, cardiac pacemaker cells and fireflies flashing in synchrony. Mathematically they are often modelled as networks where the nodes represent the individual dynamical components and the edges represent interactions. An important question is how the topology of such networks affects synchronization. We focus on synchronization time: the time it takes for the system to re-synchronize after being perturbed from a fully synchronized state.

In particular, we look at pulse-coupled oscillators on directed networks with topology based on the Watts-Strogatz small-world network model [1]. A primary topological network property affecting synchronization is the average distance between nodes [2]. We numerically study networks with this quantity fixed and we anticipate that in the small-world regime synchronization is slower than for regular networks. This is surprising because studies so far have indicated that small-world networks have a better synchronizability [3].

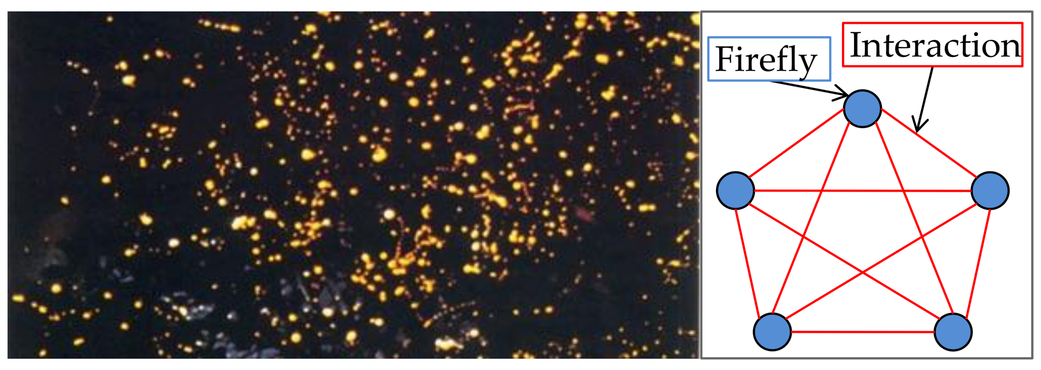


Figure: *Left: Fireflies flashing Right: Simple network model*

Left image from: www.dosomethingdifferent.com/experiences/kuala_selangor_fireflies_phenomena

2. Model Motivation

Our oscillator model (based on [4]) is motivated by neurons. Networks of neurons in the brain are known to show synchronized behaviour. It is thought this enables functions such as information transmission but it could also be related to diseases such as epilepsy.

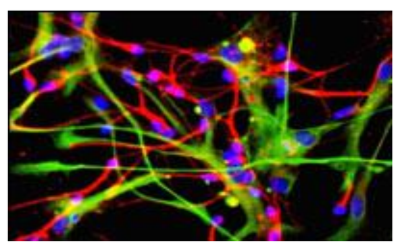


Figure: *Neural networks have a complex topology, possibly with small-world characteristics.*

Image from: news.bbc.co.uk/1/hi/programmes/click_online/4165420.stm

3. Neuron Oscillator Model

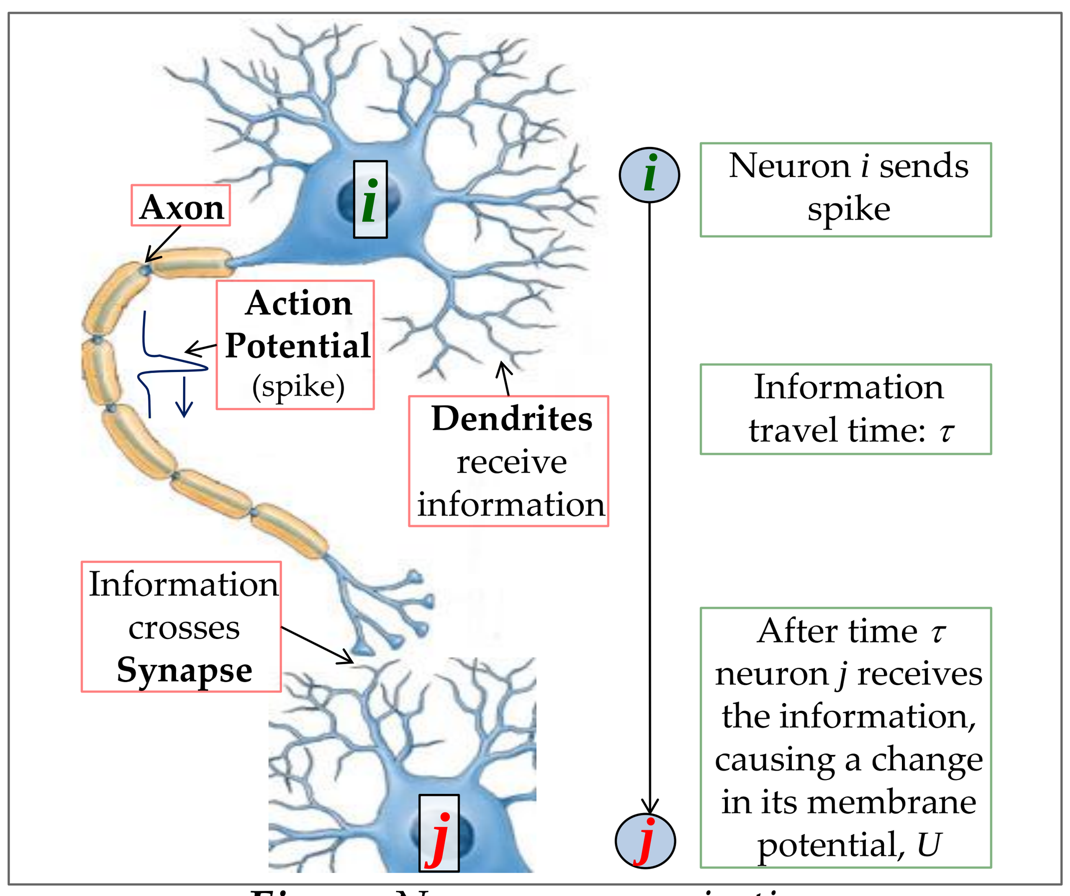
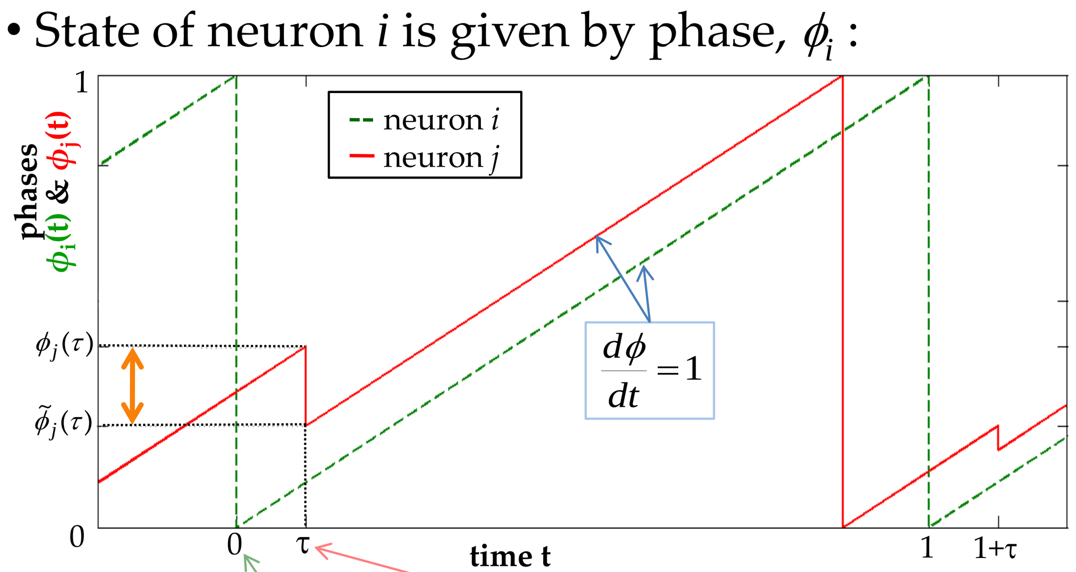


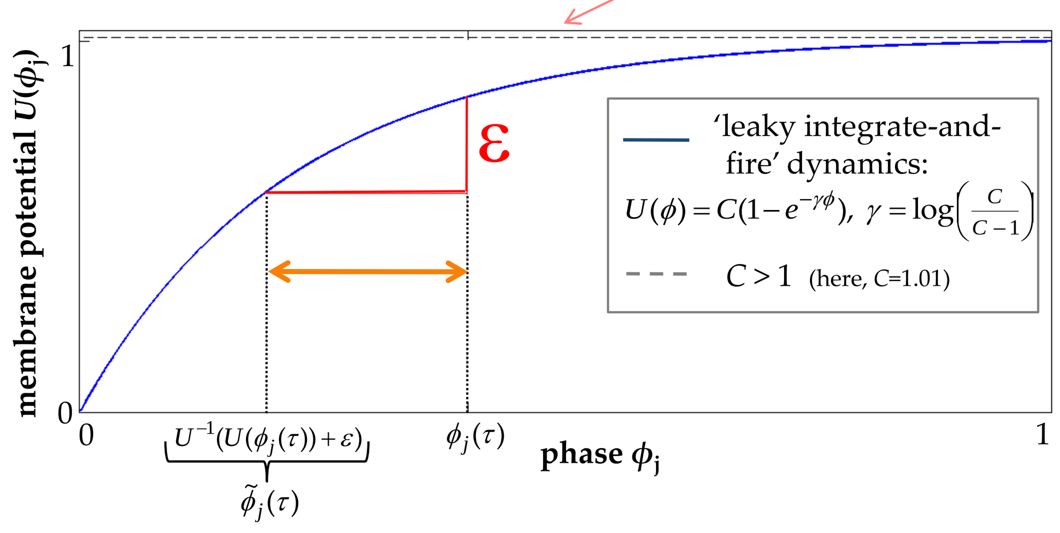
Figure: *Neuron communication*

Adapted from: cwx.prenhall.com/bookbind/pubbooks/morris5/chapter2/deluxe.html

- State of neuron i is given by phase, ϕ_i :



- At threshold ($\phi_i = U(\phi_i) = 1$) neuron i spikes and resets ($\phi_i = U(\phi_i) = 0$)
- After time τ neuron j's potential (U) changes by $\varepsilon < 0$ (inhibitory), which changes the phase



4. Network Topology

- We consider directed networks since neurons may communicate in one direction only.
- Four important properties of networks are:

1. **Number of nodes, N.**
2. **Average path length, L.**

   L_{ij} = shortest distance (number of edges) between nodes i and j.

$$L = \frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} L_{ij}$$

3. **Clustering coefficient, C.**

$$C_i = \frac{\text{number of triplets containing node } i}{\text{number of possible triplets containing node } i}$$

$$C = \frac{1}{N} \sum_{i=1}^{N} C_i$$

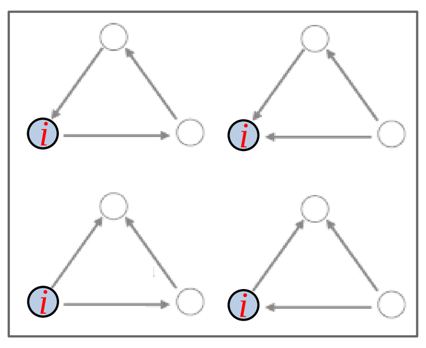


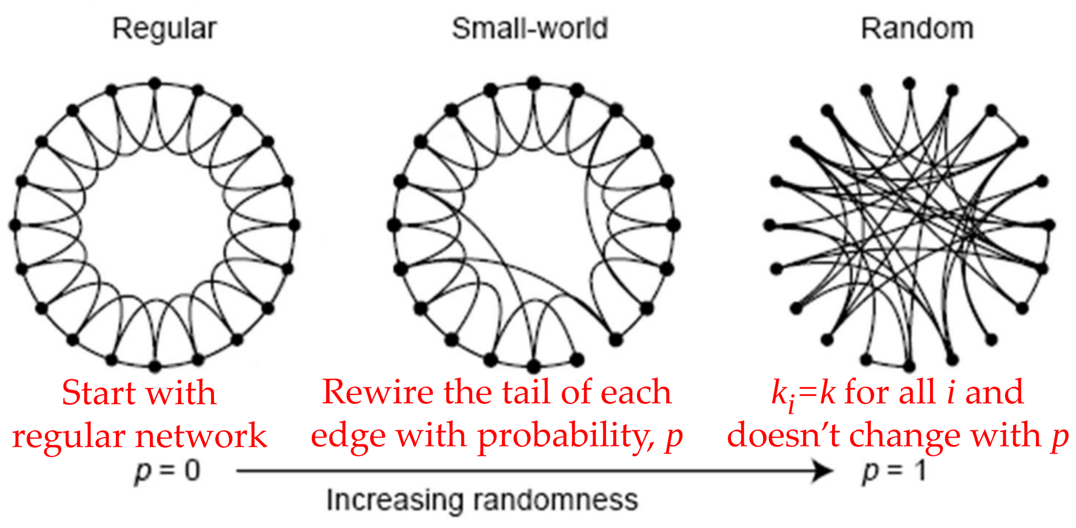
Figure: *Possible Triplets*

4. **Average in-degree, k.**

   k_i = number of edges entering node i.

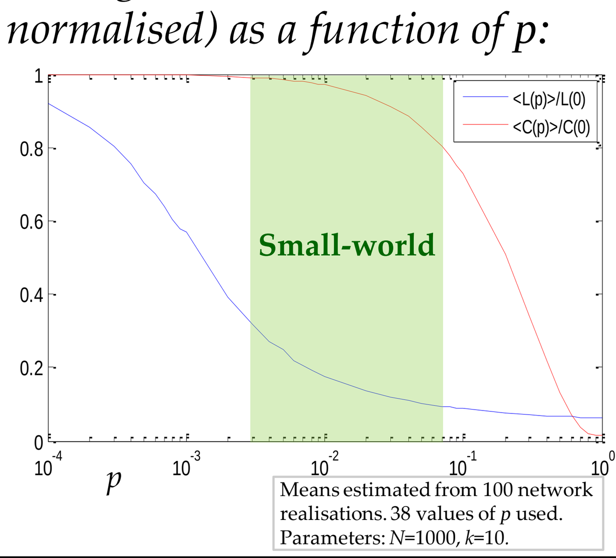
$$k = \frac{1}{N} \sum_{i=1}^{N} k_i$$

- Method used to produce networks of varying topology (based on Watts-Strogatz small-world model [1]):



Result: *Mean average path length <L> and mean clustering coefficient <C> (both normalised) as a function of p:*

Definition: Small-world networks have a large clustering coefficient and small average path length.



5. Synchronization

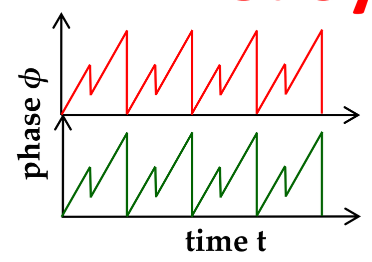


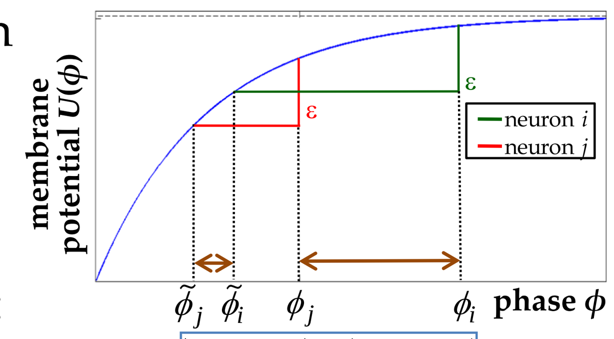
Figure: *Fully synchronized state.* $\phi_i(t) = \phi_j(t)$ *for all i,j and t.*

- **Simulation initial condition:**

  Oscillators are perturbed from the fully synchronized state: $\phi_i(0) = 0$ for all i.

  Perturbation ~ Uniform[$-\delta,\delta$] where $\delta < \tau/2$ (∗)

- Re-synchronization occurs due to:
  - (∗)
  - ε and k constant across nodes
  - concavity of $U(\phi)$: phases contract

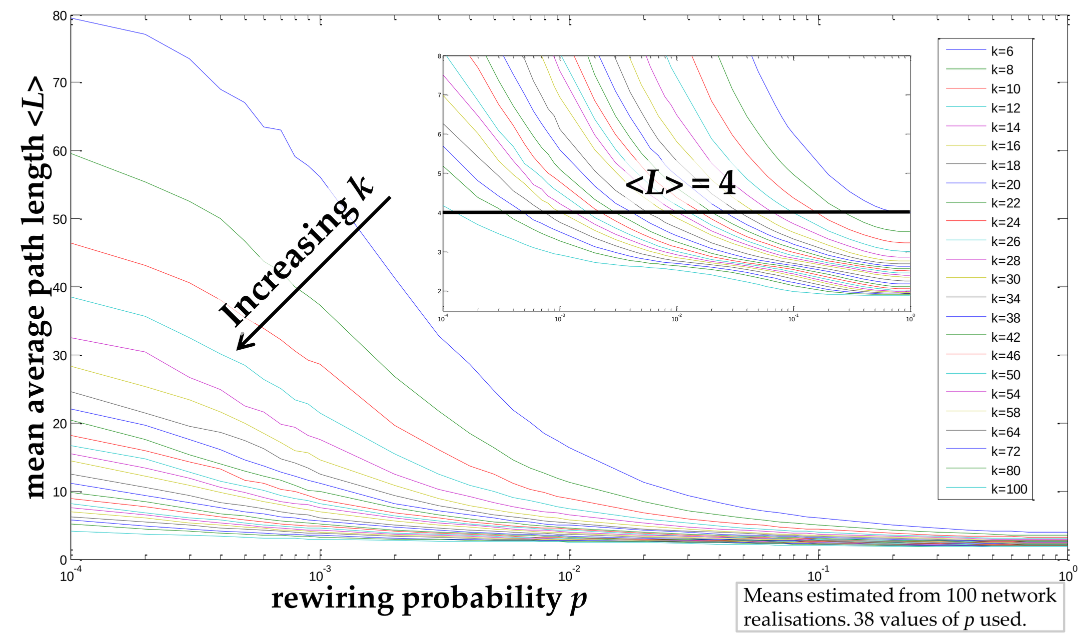


$$|\tilde{\phi}_i - \tilde{\phi}_j| < |\phi_i - \phi_j|$$

6. Results

- $N = 1000$

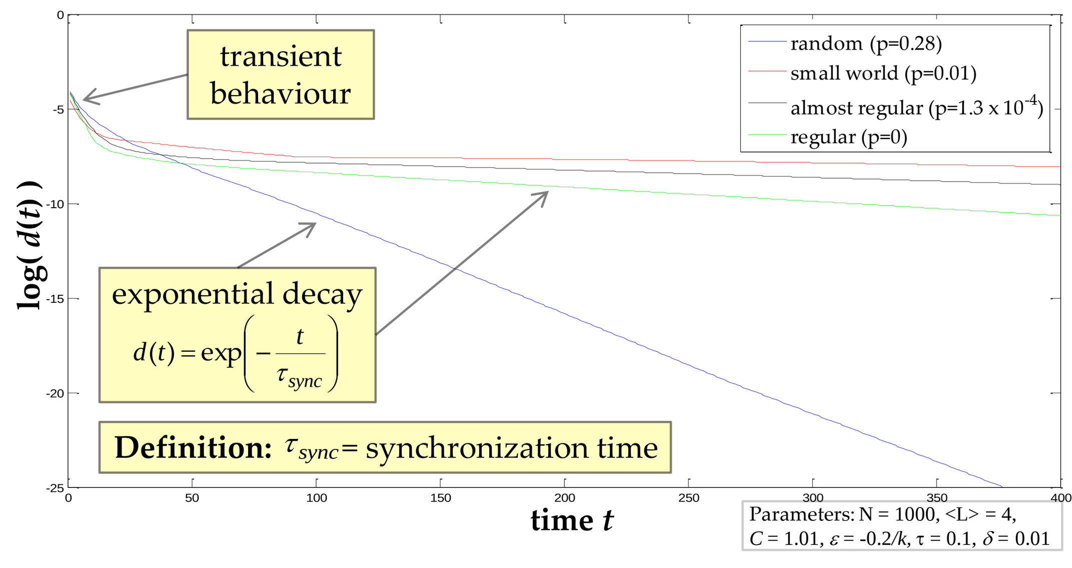
Result : *<L> as a function of p for varying in-degree k using Watts-Strogatz based method:*



- We numerically determine the average synchronization time as p varies for networks with **<L> = 4** (figure inset). We use the plot to choose appropriate k and p.
- $k\varepsilon$ **is fixed** as p varies so the total pulse received by an oscillator each period is fixed.
- We choose a reference oscillator i. When $\phi_i(t) = 0$, we measure the distance $d(t)$ from the fully synchronized state:

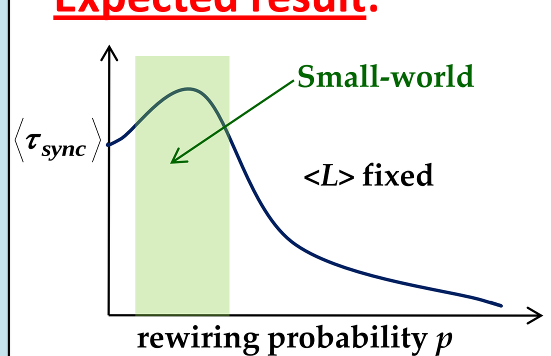
$$d(t) = \max_j(d_j(t)) \text{ where } d_j(t) = \min\{\phi_j(t), 1-\phi_j(t)\}$$

Preliminary Result : *One realization of d(t) for 4 values of p. Gradient gives synchronization time:*



- The preliminary results show that the synchronization is slower in the small-world regime than for a regular network but is faster for a random network. In particular, we see a non-monotonic dependence.

Expected result:



$\langle\tau_{sync}\rangle$ estimated by averaging over realisations of perturbations and networks for each p.

Further work:

- Compare with analytical results [5].
- Investigate reason for shape of curve.
- Investigate transient behaviour.

References

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[4] R.E. Mirollo and S.H. Strogatz. *SIAM J. Appl. Math.* **50:** 1645 (1990).
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