CO904 Assignment II

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1 The Vicsek Model

1.1 Introduction

The Vicsek model [1] is a continuous space, discrete time model of self-propelled particles which act according to a universal and simple rule to determine their behaviour: each particle moves with a constant absolute velocity, but at each time step its orientation is updated to the average of the particles in its interaction radius r with an additional random contribution. We will be concerned with particle motion in two dimensions, however the concept is easily extendible to d dimensions.

We can draw a direct analogy with ferromagnetic type models (for example, the XY model since we are looking at $d = 2$), with the rule for the tendency of neighbouring sites to align spin corresponding to the tendancy of particles in the Vicsek model to align their neighbours direction of motion. One key difference between these models is that particles in the Vicsek model are not constrained to lattice sites, as such the model is inherently dynamic, allowing local density fluctuations. Note that in the limit of zero particle velocity we can essentially retrieve the XY model. The inclusion of a noise contribution corresponds to imperfect update of individual orientation, specifically with the assumption that understanding of neighbour directions is exact, however the alignment of that individual is subject to some error [2]. These random perturbations are analogous to temperature in ferromagnetic type models.

The Vicsek model is interesting particularly for application to biological systems as a minimal model for social clustering, for example with birds or fish, who form flocks and schools and display a natural collective behaviour, and even bacteria who may act cooperatively to survive [1]. It can also be extended in a variety of ways, such as the inclusion of particle repulsion, different noise types, bipolar particles, and others [2]. It may surprising then that a simple model based on a single local rule can produce interesting global dynamics, as the Vicsek model does.

1.2 Simulation

Simulations were performed on a two dimensional square cell of size $L \times L$ with periodic boundary conditions. We looked to reproduce results similar to Vicsek, therefore chose certain fixed parameters to facilitate comparison. From Figure 2 of his 1995 paper on this model [1] we decided to use a fixed global density ρ of 4.0. This inferred values for cell size L

for varied number of of particles N via $\rho = N/L^2$. The ratio of speed v to interaction range r is set at 0.1, and we choose a fixed interaction radius r of 1.0, as Vicsek had, inferring a fixed speed of 0.1. The inverse speed actually acts like a "thermalisation parameter" which describes how often neighbours check each others orientation whilst within interaction range [2]. With a value of 0.1 we lie in he high-thermalisation regime, which actually guarantees isotropic diffusion. Within this regime ($v \leq 0.1$) the choice of speed does not actually affect the nature of the phase transition [2], so this is a sound choice of parameter. Note that since this model drives the particles with a constant speed, net momentum of interacting particles is not conserved (which actually belies the existence of a kinetic phase transition) [1]. We look to vary the noise parameter in the range $[0.5, 5.0]$ in steps of 0.05 to gain good resolution of the variation in order parameters.

We are interested in how the system of particles orients itself as it evolves, therefore we require an order parameter. If we consider the bulk (average) velocity of the particle flow, it is intuitive that is there is no preferred direction and all particles are moving randomly, then individual contributions will cancel to leave close to zero, however if there is a well defined order and direction of motion, this will be close to unitary (if normalised). We therefore define the average normalised velocity to use as our order parameter:

$$v_{ave} = \frac{1}{vN} \left| \sum_{i=1}^{N} \mathbf{v}_i \right|$$

Now that our parameters are known and set, we can simulate this model as follows. We start by defining our initial conditions. We used a random initial configuration of particle positions $\mathbf{x}$ and orientations θ. At each time step we update the positions of each particle based on its velocity, and update the orientation of each particle based on the mean orientation of its neighbours within its interaction range. Note that the simple arithmetic mean is not sufficient, instead a typical mean of angles (via arctan of velocity components) is required. This orientation was then augmented by a random number distributed with uniform probability on the interval $[-\eta/2, \eta/2]$, where η is the magnitude of the noise as selected for this run. This then defined new velocity values for the particles. Note that update of these value is performed simultaneously each time step. The order parameter was also calculated at each time step, based on the updated particle values.

As a slight aside, identification of neighbours is typically a limiting factor in scaling to large N (as a naive implementation scales quadratically). Some experimentation was done on implementing a "cell lists" algorithm to achieve linear scaling, although sample code provided does not include this, to keep things clear. Frankly, in MATLAB there appeared to be a drawback to this algorithm in that the read/write time to a dynamic cell array was not negligible, although it was marginally faster for around $N > 4000$. It would be expected that an implementation in C/C++ (or more time spent to implement smoothly in MATLAB) with non-dynamic lists would be significantly faster, although very little time was not spent in this area. As mentioned, some MATLAB code is attached to this assignment, which although intentionally not comprehensive, should provide some idea as to how this model was simulated specifically.

1.3 Phase Transition

Figure 1 shows the variation of the order parameter, average velocity v_{ave}, as noise η is varied between $[0.5, 5.0]$ in steps of 0.05. It is evident that the system goes from a state of high order when noise is low, that is that there is a defined global direction, to a state of low order, with no defined global direction. This transition in our non-equilibrium system occurs as the magnitude of the noise of the system is varied, and noting again its analog with temperature in ferromagnetic type systems, we notice how the behaviour of it is very similar to critical equilibrium systems [1]. What is even more of an indication that this sort of a phase transition is occurring is the scaling of the so-called critical region with system size - the transition becomes more abrupt as N increases. Comparing this with typical critical phenomena like percolation or the Ising model, where the transition look similar to the data in Figure 1 suggests that this phase transition could be continuous (second-order, where correlation lengths diverge).

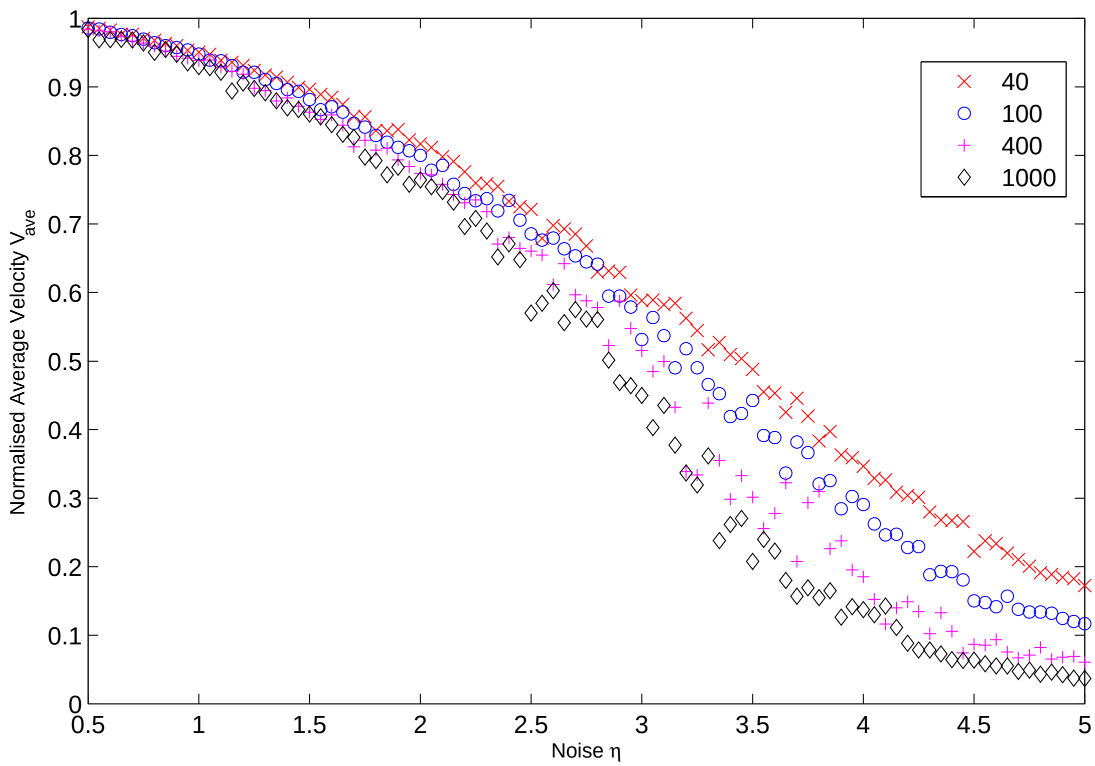


Figure 1: The order parameter, average velocity v_{ave}, versus the noise η for a selection of particle numbers N, for fixed density $\rho = 4.0$, and hence varied cell size L. Red crosses: N=40, L=3.16(3sf); Blue circles: N=100, L=5.0; Magenta pluses: N=400, L=10; Black diamonds: N=1000,L=15.8(3sf).

Although there has been some debate about the nature of this phase transition, with some claiming it is discontinuous, Baglietto, et al. [2] find relationships (usually for equilibrium systems) between critical exponents that could characterise this transition do hold, which indicates that despite this system being non-equilibrium, this transition is indeed continuous. What is the reason behind this change of behaviour? One possible physical explanation might be that, as the particles diffuse, there is mixing in the system, which results in long range correlations, and an effective interaction radius far greater than that for an individual particle.

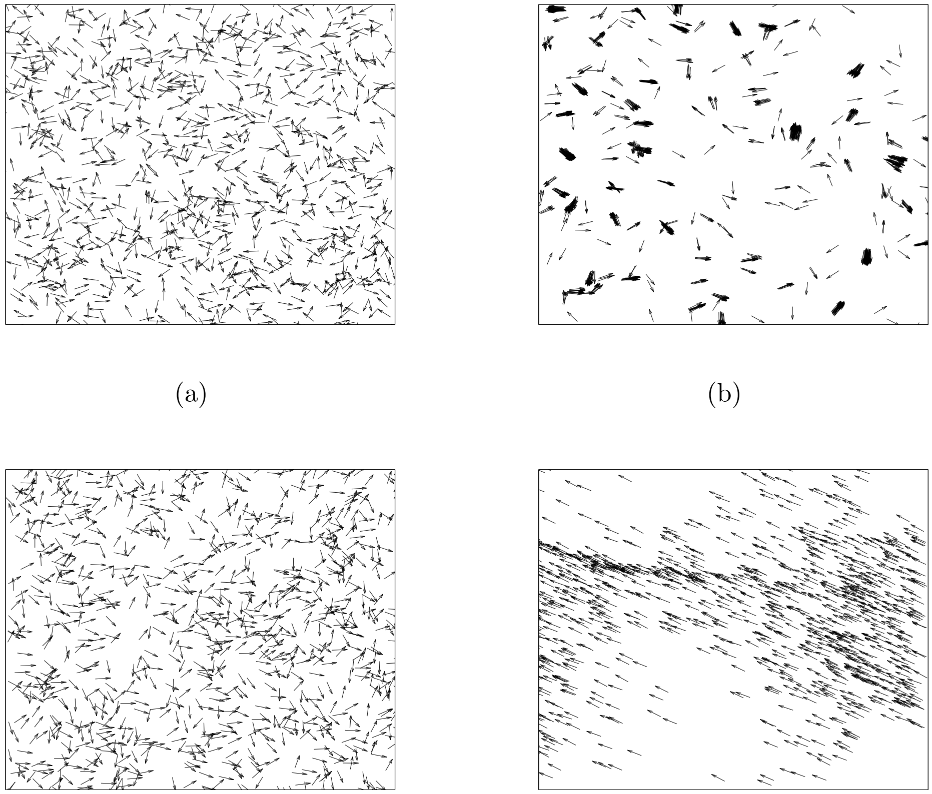
We would like to obtain a value for this critical point η_c. We do not however have a N value at infinity. To remedy this we look how the estimated critical point changes for each value of N and what it appears to converge to. To estimate the critical point for each value of N we use the rather crude method of windowing the data deemed to be to the left of the critical region by eye and fitting a straight line to it. We then look to see where this straight line intersects the noise axis, and how this value moves closer to zero as N increases. This proves to be terribly difficult with just four data points (need maybe an order of magnitude more to reach around Viscek's estimate), so I will state my best estimate for η_c as 3.92 (3sf), as the noise intersect for my largest reliable value for N (1000).

Note that this is a little poor for four values of N, and given longer I would like to optimise the simulation to scale for higher N and allow presumable a better estimation of the critical point. Of course comparing our value with Vicsek, et al., we are not quite close. In Figure 2 of his paper [1], we can see he has looked at $N = 4000, 10000$ and the difference from the $N = 400$ curve is profound. One area we would also like to look at in the future would be to calculate the critical exponents (which require a good estimate of η_c for a decent value) for the power law scaling of v_{ave} with noise and density [1]. I would also like to extend this by doing the most crucial thing first - a thorough error analysis so some proper confidence intervals can be placed on stated values(!). It should be noted that in order to counteract their absence, I have taken multiple repeats over each measurement (in an effort to scale down its standard error by $1/sqrt(n)$, of course assuming it is sensible to begin with). Larger values of N have been repeated more so, 7 times for $N = 1000$, since Vicsek mentioned that larger N has greater uncertainty [1].

1.4 Typical Configurations

When we graphically plot the velocity field of the particles on the two dimensional interaction area, we can see a macroscopic configurations which are qualitatively different from one another. This is an example of global behaviour variation due to differences in the local interaction rule.

Figure 2(a-d) show a selection of configurational snapshots in order to see the phase of the system. The velocity of each particle is represented by an arrow representing orientation information mainly. (a) At $t = 0$ we have a random distribution of positions and velocities for the particles. This corresponds to the high noise and low density regime. (b) For low noise, but also low density, particles slowly form clusters who all locally move in the same direction, however the global distribution of cluster orientations is random. (c) In the high density, high noise regime, there is some correlation between the particles, due to the sheer proximity of the particles, however it is largely drowned out by random fluctuations. (d) The density is large and the noise is low, allowing the motion to become macroscopically ordered, with particles all moving in the same direction. Since the initial condition is always a random distribution of positions and velocities, this symmetry is spontaneously broken as the particles select a direction randomly. This direction selection is just like what happens



(a) (b) (c) (d)

Figure 2: Configurational snapshots of velocity field. For all, $N = 1000$, so density parameterised by system size L. (a) Disordered phase; t=0, $\eta = 5.2$, $L = 250$. (b) Low density $L = 250$, low noise $\eta = 0.2$; local order, with clusters of locally aligned particles moving in globally random directions. (c) High density $L = 5$, high noise $\eta = 3.2$; group formation disrupted, however short gaps between particles force some correlation. (d) High density $L = 5$, low noise $\eta = 0.2$; Aligned phase, macroscopic order with all particles moving in one spontaneously chosen direction.

if you approach the phase transition from above, so reducing the noise, until the particles can suddenly interact with one another, allowing a defined direction to emerge and hence a non-zero normalised average velocity.

1.5 References

[1] Vicsek, et al., "*Novel Type of Phase Transition in a System of Self Driven Particles*, Phys. Rev. Lett. 75:1226, 1995.
[2] Baglietto, et al., "*Criticality and the Onset of Ordering in the Standard Vicsek Model*, Interface Focus 2, 708, 2012.
[3] A.-L. Barabási and H. E. Stanley, "*Fractal Concepts in Surface Growth*, Cambridge University Press, 1995.