Ultrasonic transducers: From analytical modelling to design optimisation and validation

Álvaro Pérez Díaz, Centre for Complexity Science, University of Warwick
Supervisor: Dr Nishal Ramadas, Department of Physics, University of Warwick
Co-supervisor: Dr Gareth Alexander, Centre for Complexity Science, University of Warwick

Abstract—The most important part in an ultrasonic transducer is a piezoelectric substrate, which will accumulate electric charge in response to an applied mechanical stress. In the most typical configuration, one layer of such a piezoelectric material will be present altogether with some passive backing and matching layers, situated in the back and in the front of the transducer, respectively.

Complexity in ultrasonic transducers design has increased in the last decades: even with the most simple configuration there is a huge variety of possible designs: backing layers, matching layers, electrical load, piezoelectric elements, etc. All this makes intuitive design very difficult. When it comes to model this kind of device, there are three different choices: Finite Element (FE), equivalent circuit or analytical models. The latter provide physical insight into the processes taking place in the transducer, as well as being fast and suited for optimization purposes.

We shall distinguish two different kinds of materials, depending on whether they are piezoelectrically active or not:
• For non-piezoelectric materials: the two equations governing their behaviour are

\[
\begin{align*}
\Gamma &= Y \frac{\partial^2 \xi}{\partial x^2} \\
\rho \frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial \Gamma}{\partial x}
\end{align*}
\]

where (1a) is Hooke’s law between stress and strain, and (1b) is second Newton’s law for an infinitesimal volume element.

If we now combine both equations after taking partial derivative with respect to \(x\) in (1a):

\[
\frac{\partial \Gamma}{\partial x} = Y \frac{\partial^2 \xi}{\partial x^2} \Rightarrow \rho \frac{\partial^2 \xi}{\partial t^2} = Y \frac{\partial^2 \xi}{\partial x^2} \Rightarrow \frac{\partial^2 \xi}{\partial x^2} = v^2 \frac{\partial^2 \xi}{\partial x^2}
\]

where \(v^2 = \frac{Y}{\rho}\). This is the equation waves travelling in this non-piezoelectric material must satisfy.
• For piezoelectric materials: the equations are similar but in this case we have to include the piezoelectric behaviour of the material, represented by its piezoelectric constant \(e\), obtaining the set of equations:

\[
\begin{align*}
\Gamma &= Y \frac{\partial^2 \xi}{\partial x^2} - hD \\
E &= -h \frac{\partial \xi}{\partial x} + D/\varepsilon \\
\rho \frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial \Gamma}{\partial x}
\end{align*}
\]

II. Governing Equations

We will start by considering the physics behind piezoelectric and non-piezoelectric materials, having a look at their governing equations and including some assumptions that will let us derive our models. All the above altogether with some technical details that will constitute the framework for the first two models will shape section 2.

In the third and fourth sections we describe and analyze the models given in [2]-[5] respectively. The first one is the simplest model and will provide clear distinction between receiving and transmitting modes, as well as being a very good introduction to more complicated setups. The second one introduces arbitrary layer possibilities providing better behaviour.

Section 4 introduces a model based on [6] that uses a slightly different framework than the previous two ones, all the required tools will be detailed there. Originally it considered a different setup which is of no use for us, so we had to modify and redo the calculations to fit our purposes.

To finish, several results and experiments are described, providing evidence of the capabilities that this analytical models have when it comes to model real world ultrasonic transducers.
again, if we take partial derivative of (2a) with respect to 
\( x \) and combine it with (2c), we get
\[
\frac{\partial \xi}{\partial x} = Y \frac{\partial^2 \xi}{\partial x^2} - \frac{h D}{\partial x} \implies \frac{\partial \xi}{\partial x} = Y \frac{\partial^2 \xi}{\partial x^2} - \frac{h D}{\partial x}
\]
If we are considering a thickness mode, this is, we assume the transducer to vibrate predominantly in one dimension (thickness dimension), we can perform the following derivation:
\[
\frac{\partial D}{\partial y} = \frac{\partial D}{\partial x} = 0
\]
and using Gauss’ law together with the fact that there is no free charge inside the transducer (all the possible charge would be located at the electrodes) yields to
\[
\nabla \cdot D = \rho_{\text{inside}} = 0 \implies \frac{\partial D}{\partial x} = 0
\]
hence in our previous equation we can get rid of the last term:
\[
\frac{\partial^2 \xi}{\partial x^2} = Y \frac{\partial^2 \xi}{\partial x^2}, \quad \text{where} \quad v^2 = \frac{Y}{\rho}
\]
which is an identical wave equation to the one obtained for non-piezoelectric materials.

To solve the wave equation it is useful to use the Laplace transform, together with the assumption that all functions are zero at \( t = 0 \)
\[
\mathcal{L}[f(t)] = \overline{f}(t) = \int_0^\infty e^{-st} f(t) dt
\]
we shall apply it to our wave equation to obtain
\[
\frac{s^2}{v^2} \overline{\xi}(s, x) = \frac{\partial^2 \overline{\xi}(s, x)}{\partial x^2}
\]
and its general solution will be
\[
\overline{\xi}(s, x) = A e^{-sx/v} + B e^{sx/v}, \quad A, B \in \mathbb{R}
\]
From now on, for the rest of this chapter and in the two following ones, we shall use Laplace transformed functions, even if we omit the over line sign in some functions for the sake of clarity.

We can now relate the total force exerted over an area \( S \) normal to the \( x \) direction, \( F = \Gamma S \)
- For a non-piezoelectric material, using (1a)
\[
\Gamma = \frac{F}{S} = \frac{s^2}{v^2} \left[ -A e^{-sx/v} + B e^{sx/v} \right] \implies F = ZS \left[ -A e^{-sx/v} + B e^{sx/v} \right]
\]
where \( Z = \rho v S \) is the characteristic acoustic impedance of the material (note that in some texts the cross sectional area \( S \) is omitted).
- For a piezoelectric material, using (2a)
\[
\Gamma = Y \frac{s}{v} \left[ -A e^{-sx/v} + B e^{sx/v} \right] - hD
\]
and applying again Gauss’ law to the surface of the piezoelectric material (in our case, where the electrodes are located, this is front and rear faces)
\[
\int_S D \cdot ds = DS = \int_V \rho_{\text{charge}} = Q \implies Q \frac{S}{S} = D
\]
combining this results we obtain
\[
F + hQ = S Z_e \left[ -A e^{-sx/v} + B e^{sx/v} \right]
\]
We can also find the voltage across the piezoelectric transducer by integrating (2b) from \( x = 0 \) to \( x = L \) (all the thickness of the transducer):
\[
V = \int_0^L E dx = \int_0^L \left[ -h \xi + \frac{D}{\varepsilon} \right] dx = -h [\xi(L) - \xi(0)] + \frac{Q}{C_0}
\]
\( F \) where \( C_0 = S \varepsilon / L \) is the clamped or static capacitance of the transducer. We can rewrite our expression as follows
\[
V = -h [\xi(L) - \xi(0)] U, \quad \text{where} \quad U = \frac{s C_0 Z_e}{1 + s C_0 Z_e}
\]
being \( Z_e \) the impedance of an arbitrary electrical load connected across the transducer electrodes.

Note that if we had multiple piezoelectric layers, the overall voltage across the whole transducer would be the sum of the correspondent voltages for each of the layers
\[
V = \int_{a_1}^{b_1} E_1 dx + \ldots + \int_{a_n}^{b_n} E_n dx
\]

III. Linear Systems Model (LSM)

Fig. 1: Basic setup for a transducer with an infinite backing layer (at \( x = 0 \)) and in contact with a loading medium at its front face (\( x = L \)).

THE boundary conditions that apply for the wave equation solution are: continuity of particle displacements, \( \xi \) and continuity of forces across the interface, \( F_i \), at the front and rear face, resulting in four boundary equations:
\[
\begin{align*}
\xi_{1|x=0} &= \xi_{c|x=0} \\
\xi_{c|x=L} &= \xi_{2|x=L} \\
F_{1|x=0} &= F_{c|x=0} \\
F_{c|x=L} &= F_{2|x=L}
\end{align*}
\]
where each of the expressions has the following form:
\[
\begin{align*}
\xi_1(s) &= A_1 e^{-sx/v_1} + B_1 e^{sx/v_1} \\
\xi_c(s) &= A e^{-sx/v_c} + B e^{sx/v_c} \\
\xi_2(s) &= A_2 e^{-sx/v_2}
\end{align*}
\]
& \begin{align*}
F_1(s) &= sZ_1 \left( -A_1 e^{-sL/v_c} + B_1 e^{sL/v_1} \right) \\
F_2(s) &= sZ_2 \left( -A_2 e^{-sL/v_2} \right)
\end{align*}

hence the resulting equations are:

\begin{align*}
A_1 + B_1 &= A + B \\
Ae^{-sL/v_c} + Be^{sL/v_2} &= A_2 e^{-sL/v_2} \\
zs1 \left( -A_1 + B_1 \right) &= zs \left( -A + B \right) - hQ(s) \\
zs \left( -Ae^{-sL/v_c} + Be^{sL/v_2} \right) &= -zs2 \left( -A_2 e^{-sL/v_2} \right) - hQ(s)
\end{align*}

altogether with the expression for the voltage

\[ V = -h \left( A(e^{-sL/v_c} - 1) + B(e^{sL/v_2} - 1) \right) \mathcal{U}(s) \]

Now, combining (5a) and (5c) we obtain

\[ (1 - RF)A_1 = A - BRF + \frac{hQ(s)}{s(Z_c + Z_1)} \]

where \( RF = \frac{Z_c - Z_1}{Z_c + Z_1} \)

and similarly from (5b) and (5d)

\[ Ae^{-2sL/v_c}RB = -hQ(s)e^{sL/v_c} \frac{s(Z_2 + Z_c)}{s(Z_2 + Z_c)} \]

where \( RB = \frac{Z_c - Z_2}{Z_c + Z_2} \)

Both \( RF \) and \( RB \) are the reflection coefficients for the waves travelling into the piezoelectric medium.

Now combine equations (7) and (8) and solve for \( A \) and \( B \), using for example Cramer’s rule, to obtain:

\[ A = \frac{hQ(s)}{s(Z_c + Z_1)} + (RF - 1)A_1 - \frac{hQ(s)RF e^{-sL/v_c}}{s(Z_2 + Z_c)} \]

\[ B = \frac{-hQ(s)e^{-sL/v_c} - (1 - RF)A_1RB e^{-2sL/v_c} + \frac{hQ(s)RB e^{-2sL/v_c}}{s(Z_c + Z_1)}}{e^{-2sL/v_c} RB RF - 1} \]

We can now find the voltage across the transducer, plugging (9) and (10) into (6) to obtain

\[ V = h \left( RF \left( A_1(1 - RF) - \frac{hQ(s)}{s(Z_c + Z_1)} \right) \right) \]

\[ -K_B \frac{hQ(s)}{s(Z_c + Z_2)} \]

We shall now develop the receiver and transmitter models, using the framework just detailed. This has the advantage of showing very clearly both functioning modes, which are quite independent from each other.

Fig. 2: Basic setup for a transducer in receiver mode with an infinite backing layer (at \( x = 0 \)) and in contact with a loading medium at its front face (\( x = L \))

A. Receiver mode

In receiver mode, there is no emitterd left going wave, we will just consider the incident wave (\( A_1 \)).

Remember the equation for the force (4), together with the previous assumptions:

\[ F_1 = Z_1 s \left[ -A_1 e^{-sL/v_1} + B_1 e^{sL/v_1} \right] \]

\[ \Rightarrow F_1 = A_1 Z_1 s \Rightarrow A_1 = -\frac{F_1}{Z_1 s} \]

Also replace \( Q = -V/sZ_c \), which follows from Ohm’s law \( Z = V/I \), to obtain for the voltage

\[ V = \frac{hA_1KF(1 - RF)U(s)}{1 - h^2 \left[ \frac{KF}{Z_c + Z_1} + \frac{KB}{Z_c + Z_2} \right] U(s)} \]

and substituting \( A_1 = -\frac{F_1}{Z_1 s} \) yields to:

\[ V = \frac{-hKF RF U(s)/sZ_c}{1 - h^2 \left[ \frac{KF RF}{2} + \frac{KB RF}{2} \right] U(s)} \]

which is the transfer function relating the transforms of received voltage to incident force.

We can represent this transfer function using a block diagram with two positive feedback loops, see figure 3.

If we replace \( RB \) by \( RF \) and vice versa, we obtain a transfer function for the force applied in the rear face of the transducer. However, as we assume a infinitely long backing block and hence no returning wave, there will not be any rear face applied force.

B. Transmitter mode

This is an analogous situation, where now there is no incident wave (see figure 4). Setting \( A_1 = 0 \) into (11) yields to

\[ A = \frac{hQT_F - hQT_B R_F e^{-sL/v_c}}{2Z_e s e^{-2sL/v_c} RB RF - 1} \]

\[ -hQT_B e^{-sL/v_c} + hQT_F R_B e^{-2sL/v_c} \]

\[ B = \frac{2sZ_e}{e^{-2sL/v_c} RB RF - 1} \]
From this expression we can determine the operational impedance of the transducer, $Z_T$:

$$Z_T = \frac{V(s)}{sQ(s)} = \frac{1}{sC_o} \left[ 1 - \frac{h^2 C_o}{2 s Z_c} (T_F K_F + T_B K_B) \right]$$

The next step is to find the force at any point within the transducer. Recall the expression for the force in a piezoelectric layer

$$F(s) = hQ(s) \left\{ 1 + \frac{1}{2 s (e^{-2 s L/v_c} R_B R_F - 1)} \cdot \left[ (T_F - T_B R_F e^{-sL/v_c}) e^{-sL/v_c} + (T_B e^{-sL/v_c} - T_F R_B e^{-2sL/v_c}) e^{sL/v_c} \right] \right\}$$

We are now particularly interested in the front and back faces of the transducer. Thus, set $x = 0$ and $x = L$ respectively:

$$F_F(s) = F_{|x=0} = -hQ(s) K_F \frac{Z_1}{Z_c + Z_1} \quad (15)$$

$$F_B(s) = F_{|x=L} = -hQ(s) K_B \frac{Z_2}{Z_c + Z_2} \quad (16)$$

The final step will be, as we did for the receiver mode, to find a transfer function relating the force generated at the front face to the voltage across the transducer. Notice that we have obtained forces in both rear and front faces, but we are primarily interested in the front face.

Consider the following electrical setup:

$$\frac{I_T(s)}{e(s)} = \frac{Z_E}{Z_F(Z_o + Z_E) + Z_o Z_E} = \frac{Z_E/\left(Z_o + Z_E\right)}{Z_T + Z_o Z_E/\left(Z_o + Z_E\right)} = \frac{a(s)}{Z_T + b(s)} \quad \Rightarrow \quad I_T(s) = \frac{a(s) e(s)}{Z_T + b(s)} \\
\Rightarrow \quad Q = \frac{I_T(s)}{s} = \frac{a(s) e(s)}{Z_T + b(s)}$$

where

$$a(s) = \frac{Z_E}{Z_o + Z_E} \quad \& \quad b(s) = \frac{Z_o Z_E}{Z_o + Z_E}$$

We can now use the new expression for $Q(s)$ and (14) into (15) to obtain

$$\frac{F_F(s)}{e(s)} = -\frac{h a(s) Z_1/(Z_c + Z_1) K_F(s) Y(s)}{1 - h^2 Y(s)/(s Z_c) (T_F K_F/2 + T_B K_B/2)} \quad (17)$$

which is the desired transfer equation, with $Y(s) = (1 + s b(s) C_o)/C_o$.

In a completely similar fashion we can obtain the transfer function for the back face:

$$\frac{F_B(s)}{e(s)} = -\frac{h a(s) Z_2/(Z_c + Z_2) K_B(s) Y(s)}{1 - h^2 Y(s)/(s Z_c) (T_F K_F/2 + T_B K_B/2)} \quad (18)$$

This transfer function can be pictured again in a block diagram structure, see figure 6.

IV. LATTICE MODEL

Consider the setup outlined in figure 7, where medium 2, with a thickness $l_2$, is located between media 1 and 3, which are semi-infinite. We decompose forces into their backward $B_n$ and forward $F_n$ components.
the first and the second coming from (20a) and the last two ones from (20b). This can be written in matrix form:

\[
\begin{pmatrix}
(F_{10})
\begin{pmatrix}
B_{10}
\end{pmatrix}
& (1 + R_{12}) & -R_{12} \\
R_{12} & 1 & -R_{12}
\end{pmatrix}
\begin{pmatrix}
F_{20}
\end{pmatrix}
= F_{20}
\]

\[
\begin{pmatrix}
(F_{20})
\begin{pmatrix}
B_{20}
\end{pmatrix}
& (1 + R_{23}) & -R_{23} \\
R_{23} & 1 & -R_{23}
\end{pmatrix}
\begin{pmatrix}
F_{30}
\end{pmatrix}
= F_{30}
\]

where

\[
R_{12} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad R_{23} = \frac{Z_3 - Z_2}{Z_3 + Z_2}
\]

are the reflection coefficients for waves of force travelling into the piezo at the 1-2 and 2-3 interfaces respectively.

### A. Transducer model

Now consider a similar disposition, placing the piezoelectric layer between two passive layers.

\[
\begin{pmatrix}
V_t
\end{pmatrix} = -h \left[ (\xi_t)_{x_t = l_t} - (\xi_t)_{x_t = 0} \right] + \frac{Q_t}{C_o}
\]

where \(Q_t\) is the total electrical charge in the electrodes.

The mechanical boundary conditions are

\[
\begin{cases}
(\xi_{l_t})_{x_t = 0} = (\xi_t)_{x_t = 0} & (\xi_{l_t})_{x_t = l_t} = (\xi_t)_{x_t = l_t} \\
(\Gamma_{l_t})_{x_t = 0} = (\Gamma_t)_{x_t = 0} & (\Gamma_{l_t})_{x_t = l_t} = (\Gamma_t)_{x_t = l_t}
\end{cases}
\]

that is

\[
\begin{pmatrix}
(-F_{10} + B_{10})/Z_1 \\
(-F_{20} + B_{20})/Z_2
\end{pmatrix} = (-F_{10} + B_{10})/Z_1 \\
\begin{pmatrix}
(F_{10})
\begin{pmatrix}
B_{10}
\end{pmatrix}
& (1 + R_{12}) & -R_{12} \\
R_{12} & 1 & -R_{12}
\end{pmatrix}
\begin{pmatrix}
F_{20}
\end{pmatrix}
= \begin{pmatrix}
F_{20}
\end{pmatrix}
\]

\[
\begin{pmatrix}
(-F_{10} + B_{10})/Z_1 \\
(-F_{20} + B_{20})/Z_2
\end{pmatrix} = (-F_{10} + B_{10})/Z_1 \\
\begin{pmatrix}
(F_{10})
\begin{pmatrix}
B_{10}
\end{pmatrix}
& (1 + R_{12}) & -R_{12} \\
R_{12} & 1 & -R_{12}
\end{pmatrix}
\begin{pmatrix}
F_{20}
\end{pmatrix}
= \begin{pmatrix}
F_{20}
\end{pmatrix}
\]

and in matrix form we obtain

\[
\begin{pmatrix}
B_{10} \\
F_{10}
\end{pmatrix} = \begin{pmatrix}
R_{12} & 1 - R_{12} \\
1 + R_{12} & -R_{12}
\end{pmatrix} \begin{pmatrix}
F_{10} \\
B_{10}
\end{pmatrix} + \frac{h}{2} \begin{pmatrix}
R_{12} - 1 \\
R_{12} + 1
\end{pmatrix} Q_t
\]
The coefficients are found to be

\[ P_{11} = \{(R_1 - R_1 e^{-2sT}) + U(1 - e^{-sT})(1 - e^{-sT} - 1) \}
\cdot (1 + R_1 R_{-1} + 2(R_1 e^{-sT} - R_{-1})/[P_D)
\]
\[ P_{12} = \{(1 - R_{-1})(1 + R_1 e^{-sT} + U(1 - e^{-sT})(1 + e^{-sT})
\cdot (R_1 - 1)(R_{-1} + 1)/P_D
\]
\[ P_{13} = G(1 - e^{-sT})(R_{-1} - 1)(1 - R_{-1} e^{-sT})/P_D
\]
\[ P_{21} = \{(1 - R_{-1})(1 + R_1 e^{-sT} + U(1 - e^{-sT})
\cdot (1 + e^{-sT})(R_{-1} - 1)(R_{-1} + 1)/P_D
\]
\[ P_{22} = \{(R_1 - R_{-1}) e^{-2sT}) + U(1 - e^{-sT})(1 - e^{-sT}) - 1)
\cdot (1 + R_1 R_{-1} + 2(R_1 e^{-sT} - R_{-1})/[P_D)
\]
\[ P_{23} = G(1 - e^{-sT})(R_{-1} - 1)(1 - R_{-1} e^{-sT})/P_D
\]
\[ P_{31} = G(-sZ_E/Z_t)(1 + R_1)(1 - sT)(1 - R_{-1} e^{-sT})/P_D
\]
\[ P_{32} = G(-sZ_E/Z_t)(1 + R_{-1})(1 - sT)(1 - R_{-1} e^{-sT})/P_D
\]
\[ P_{33} = \{(1 + R_1)(1 - R_{-1} e^{-sT}) + (1 + R_{-1})(1 - R_1 e^{-sT})
\cdot [1 + (1 + R_1)(1 - R_{-1} e^{-sT}) + (1 + R_{-1})(1 - R_1 e^{-sT})]/P_D
\]

where

\[ U = \frac{h^2C_o}{2sZ_t(1 + sC_o Z_t)} \]
\[ P_D = \{(1 - R_1 R_{-1} e^{-2sT}) - U(1 - e^{-sT})(1 + R_1)
\cdot (1 - R_{-1} e^{-sT}) + (1 + R_{-1})(1 - R_{-1} e^{-sT})\}
\]

B. The multilayered acoustic lattice

Consider \( n+1 \) passive layers as pictured in the figure:

\[ \begin{array}{ccccccc}
\text{Layer 1} & \text{Layer 2} & \cdots & \text{Layer } i & \text{Layer } i+1 & \text{Layer } n & \text{Layer } n+1 \\
F_1 & F_2 & \cdots & F_i & F_{i+1} & F_n & F_{n+1} \\
B_{-1} & B_1 & \cdots & B_i & B_{i+1} & B_n & B_{n+1} \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{array} \]

Fig. 9: Passive multilayer setup, forces \( F \) & \( B \).

Consider the interface between layers \( i \) & \( i+1 \), the boundary conditions are

\[ \left\{ \begin{array}{l}
F_{i+1} / Z_i = (-F_{i+1} + B_{i+1} / Z_{i+1})/Z_i \\
F_{i+1} + B_{i+1} = -F_{i+1} + B_{i+1}
\end{array} \right. \]

hence

\[ \left\{ \begin{array}{l}
-F_{i+1} + B_{i+1} = Z_{i+1} / Z_i (-F_{i+1} + B_{i+1}) \\
F_{i+1} + B_{i+1} = F_{i+1} + B_{i+1}
\end{array} \right. \]

if we add and subtract both equations we obtain the following

\[ \left( \begin{array}{c}
F_{i+1,0} \\
B_{i+1,0}
\end{array} \right) = \left( \begin{array}{cc}
1 / 2 & 1 / 2 \\
1 / 2 & 1 / 2
\end{array} \right) \left( \begin{array}{c}
Z_{i+1} / Z_i \\
Z_{i+1} / Z_i
\end{array} \right) \left( \begin{array}{c}
F_{i+1,0} \\
B_{i+1,0}
\end{array} \right)
\]

with

\[ \left( \begin{array}{c}
F_{i+1,0} \\
B_{i+1,0}
\end{array} \right) = \left( \begin{array}{cc}
-e^{-sT} & 0 \\
0 & e^{sT}
\end{array} \right) \left( \begin{array}{c}
F_{i+1,0} \\
B_{i+1,0}
\end{array} \right)
\]
This allows us to rewrite the expression as
\[
\begin{pmatrix}
  F_{i+1,0} \\
  B_{i+1,0}
\end{pmatrix}
= \frac{1}{2} \begin{pmatrix}
  (1 + Z_{i+1}/Z_i) e^{-sT_i} & (1 - Z_{i+1}/Z_i) e^{sT_i} \\
  (1 - Z_{i+1}/Z_i) e^{-sT_i} & (1 + Z_{i+1}/Z_i) e^{sT_i}
\end{pmatrix}
\begin{pmatrix}
  F_{i,0} \\
  B_{i,0}
\end{pmatrix},
\]

where the coefficients are:
\[
W_{1i} = \{ U_{11}^B U_{22}^F - U_{12}^B U_{21}^F \}(P_{12} P_{21} - P_{11} P_{22}) + \\
+ P_{11} U_{12}^B U_{22}^F + U_{12}^B (1 - P_{22} U_{22}^B - P_{11} U_{21}^F) \}/W_D
\]
\[
W_{12} = U_{21}^B U_{22}^F P_{21}/W_D
\]
\[
W_{13} = U_{11}^F \{ P_{12} P_{23} - P_{13} P_{22} \} + P_{13} \}/W_D
\]
\[
W_{21} = U_{11}^B U_{22}^F P_{21}/W_D
\]
\[
W_{22} = \{ U_{22}^B U_{21}^F - U_{21}^B U_{22}^F \}(P_{21} P_{22} - P_{12} P_{21}) + \\
+ P_{22} U_{21}^B U_{22}^F + U_{21}^B (1 - P_{22} U_{22}^B - P_{12} U_{21}^F) \}/W_D
\]
\[
W_{23} = U_{21}^B \{ U_{21}^F P_{21} P_{13} - P_{12} P_{23} \} + P_{23} \}/W_D
\]
\[
W_{31} = \{ P_{31} U_{22}^B (1 - U_{21}^B P_{22}) + P_{31} U_{22}^B U_{21}^F P_{22} \}/W_D
\]
\[
W_{32} = \{ P_{32} U_{22}^B (1 - U_{21}^B P_{21}) + P_{32} U_{22}^B U_{21}^F P_{21} \}/W_D
\]
\[
W_{33} = \{ P_{33} + U_{21}^B (P_{31} P_{23} - P_{13} P_{21}) + U_{22}^B (P_{32} P_{23} - \\
- P_{22} P_{32}) + U_{21}^B U_{22}^B (P_{31} P_{23} - P_{13} P_{21}) + \\
+ P_{32} (P_{21} P_{13} - P_{23} P_{11}) P_{33} (P_{12} P_{22} - P_{12} P_{21})) \}/W_D
\]

with \( W_D = (1 - U_{21}^F P_{11})(1 - U_{22}^B P_{21}) - U_{11}^F U_{22}^B P_{12} P_{21} \).

We have then reduced all our multilayer system to a three inputs-outputs system. We can see that we recover the receiver and transmitter behaviour: for the receiver we will be interested in \( W_{31} \), as we receive the input force \( B_{(n+1),0} \) on the front face and all other inputs are set to zero. Similarly, for the transmitter we will look at \( W_{13} \), as we assume no input forces and we drive our device applying a voltage \( V_E \) across its electrodes. Similarly, \( W_{33} \) relates the applied voltage to the voltage found across the transducer.

V. HUANG’S MODEL OF MULTILAYERED ULTRASONIC TRANSDUCERS, MODIFIED WITHOUT THE INVERSION LAYER

<table>
<thead>
<tr>
<th>Backing layers</th>
<th>Piezoelectric</th>
<th>Matching layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>bN</td>
<td>...</td>
<td>a1</td>
</tr>
<tr>
<td>b1</td>
<td>a</td>
<td>...</td>
</tr>
<tr>
<td>zc</td>
<td>...</td>
<td>aM</td>
</tr>
<tr>
<td>Load medium</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 11: Setup for an arbitrary number of passive layers.

CONSIDER the multilayered setup detailed in figure 11. As explained in the introduction, this model uses a different framework, we no longer use Laplace transformed functions. Consider the interface between \( a \) and \( b1 \), keeping in mind that \( b1 \) is a passive layer and \( a \) is the active piezoelectric layer. The expressions for the particle velocities and pressures are:

At the left boundary of layer \( a \):
\[
\begin{cases}
  u_{aL,b} = A_a e^{-ik_a d_a} + B_{aLE} e^{-ik_a d_a} \\
  p_{aL,b} = z_a A_a e^{-ik_a d_a} - z_{aLE} B_{aLE} e^{-ik_a d_a} + hD
\end{cases}
\]

At the right boundary of layer \( b1 \):
\[
\begin{cases}
  u_{b1R,b} = A_{b1LE} e^{-ik_{b1} d_{b1}} + B_{b1LE} e^{-ik_{b1} d_{b1}} \\
  p_{b1R,b} = z_{b1} A_{b1LE} e^{-ik_{b1} d_{b1}} - z_{b1LE} B_{b1LE} e^{-ik_{b1} d_{b1}}
\end{cases}
\]
And the boundary conditions impose that both velocities and pressures must be the equal:

\[
\begin{align*}
A_a e^{i k_a d_a/2} + B_a e^{-i k_a d_a/2} &= = A_{b1} e^{-i k_{b1} d_{b1}/2} + B_{b1} e^{i k_{b1} d_{b1}/2} \\
\alpha_a &\left( A_a e^{i k_a d_a/2} - B_a e^{-i k_a d_a/2} \right) + hD = = \alpha_{b1} \left( A_{b1} e^{-i k_{b1} d_{b1}/2} - B_{b1} e^{i k_{b1} d_{b1}/2} \right)
\end{align*}
\] (28a)

(28b)

we will denote from now on \(\alpha_j = k_j d_j\).

Now, the voltage across the only active layer of the transducer looks like

\[
V = \int_{-d_a/2}^{d_a/2} E \, dx = -h \left( \xi_{d_a/2} - \xi_{-d_a/2} \right) + D/\varepsilon \, d_a = \frac{\hbar}{i \omega} \left( e^{i \omega a/2} - e^{-i \omega a/2} \right) (A_a - B_a) + D/\varepsilon \, d_a
\] (29)

and rearranging we can find the expression for \(D\):

\[
D = \frac{\varepsilon V}{d_a} - 2 \frac{h \varepsilon}{\omega d_a} \sin \left( \alpha_a/2 \right) (A_a - B_a)
\]

Thus, following Huang’s calculations, we can now represent equations (28a) and (28b) in the following matrix form

\[
\begin{pmatrix} 1 & 0 \\ 0 & z_{b1}' \end{pmatrix} \begin{pmatrix} \cos(-\alpha_{b1}/2) & i \sin(-\alpha_{b1}/2) \\ i \sin(-\alpha_{b1}/2) & \cos(-\alpha_{b1}/2) \end{pmatrix} \begin{pmatrix} H_{b1} \\ F_{b1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z_{a}' \end{pmatrix} \begin{pmatrix} \cos(-\alpha_a/2) & i \sin(-\alpha_a/2) \\ i \sin(-\alpha_a/2) & \cos(-\alpha_a/2) \end{pmatrix} \begin{pmatrix} H_a \\ F_a \end{pmatrix} + \begin{pmatrix} 0 \\ 1 - k_f^2 \mu_a F_a \end{pmatrix}
\] (30)

where \(k_f^2 = \frac{\hbar^2 \varepsilon}{Y} \), \(\mu_a = 2 / \alpha_a \sin(\alpha_a/2)\),

\[
H_a = z_0 d_a / h V (A_a + B_a) \quad \text{and} \quad F_a = z_0 d_a / h V (A_a - B_a)
\]

It is easy to check that we recover equations (28a) & (28b) if \(z_0 = Y / v_a\). Rewrite now equation (30) as

\[
M_{b1} G_{b1}^+ x_{b1} = M_a G_{a1}^+ x_{a1} + s
\]

and repeating this calculations for all the interfaces we obtain

\[
\begin{align*}
M_{bN} G_{bN}^- x_{bN} &= M_{b(N-1)} G_{b(N-1)}^+ x_{b(N-1)} \\
M_{b2} G_{b2} x_{b2} &= M_{b1} G_{b1} x_{b1} \\
M_{b1} G_{b1} x_{b1} &= M_a G_{a1}^+ x_a + s \\
M_{a} G_{a} x_{a} + s &= M_{a1} G_{a1}^+ x_{a1} \\
M_{a(M-1)} G_{a(M-1)}^- x_{a(M-1)} &= M_{aM} G_{aM}^+ x_{aM}
\end{align*}
\] (31a)

(31b)

(31c)

(31d)

(31e)

which are \(2(N+M)\) linear equations, and \(G_{bN}^− = G_{aM}^+ = I_d \quad \& \quad A_{bN} = B_{aM} = 0\).

In the very front and back layers there is no incoming and returning wave (respectively), this is \(A_{bN} = B_{aM} = 0\), resulting in:

\[
\begin{align*}
x_{aM} &= (H_{aM} F_{aM}^{-1} A_{aM} \frac{z_0 d}{\varepsilon h V}) = (A_{aM}' \frac{z_0 d}{\varepsilon h V}) \\
x_{bN} &= (H_{bN}^{-1} F_{bN} A_{bN} \frac{z_0 d}{\varepsilon h V}) = (B_{bN}' \frac{z_0 d}{\varepsilon h V})
\end{align*}
\]

Due to the form of the equations, we can now reduce them using transmission matrices, the procedure is detailed below. We shall start with the front layers: consider first equation (31d):

\[
M_a G_{a} x_{a} + s = M_{a1} G_{a1}^+ x_{a1}
\]

from the next equation we can write

\[
x_{a1} = (M_{a1} G_{a1}^+)^{-1} (M_{a2} G_{a2}^+) x_{a2}
\]

Similarly, from the following one

\[
x_{a2} = (M_{a2} G_{a2}^+)^{-1} (M_{a3} G_{a3}^+) x_{a3}
\]

and so on. Substituting the obtained expressions we get to:

\[
M_a G_{a} x_{a} + s = M_{a1} G_{a1}^+ [(M_{a1} G_{a1}^+)^{-1} (M_{a2} G_{a2}^+)] \cdots [(M_{a(N-1)} G_{a(N-1)}^+)^{-1} (M_{aN} G_{aN}^+) - I].
\]

\[
M_{aM} G_{aM}^+ x_{aM} = [(M_{a1} G_{a1}^+) (M_{a2} G_{a2}^+) \cdots (M_{a(N-1)} G_{a(N-1)}^+) (M_{aN} G_{aN}^+)] x_{aM} =
\]

\[
T_{a1}^+ \cdots T_{a(N-1)}^+ M_{aM} G_{aM}^+ x_{aM} = T_{aN}^+ M_{aN} G_{aN}^+ x_{aN} + s
\]

where \(T_{a}^+\) is the transmission matrix from the front layer to the transducer front face. Each of the single transmission matrices is as follows

\[
T_{am}^+ = M_{am} G_{am}^+ (M_{am} G_{am}^+)_{am}^{-1} =
\]

\[
\begin{pmatrix} 1 & 0 \\ 0 & z_{am}' \end{pmatrix} \begin{pmatrix} \cos(\alpha_m/2) & i \sin(\alpha_m/2) \\ i \sin(\alpha_m/2) & \cos(-\alpha_m/2) \end{pmatrix} =
\]

\[
\begin{pmatrix} \cos(\alpha_m/2) & i \sin(\alpha_m/2) \\ i \sin(\alpha_m/2) & \cos(-\alpha_m/2) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & z_{am}' \end{pmatrix} =
\]

\[
\left[ (\cos^2(\alpha_m/2) - \sin^2(\alpha_m/2)) 2i z_{am}' \cos(\alpha_m/2) \sin(\alpha_m/2) \right] =
\]

\[
\left[ (\cos^2(\alpha_m/2) - \sin^2(\alpha_m/2)) i z_{am}' \sin(\alpha_m/2) \cos(\alpha_m/2) \right]
\]

(32)

where we have used the following trigonometric identities

\[
cos^2 x - \sin^2 x = \cos(2x) \quad \text{and} \quad \sin x \cos x = \sin(2x)/2.
\]

We can perform the same calculations for the backing layers, starting with (31c)

\[
M_{b1} G_{b1} x_{b1} = M_{b} G_{b}^+ x_{b} + s
\]
from the previous equation we can write
\[ x_{b1} = (M_{b1} G_{b1}^+)^{-1} (M_{b2} G_{b2}^-) x_{b2} \]

Similarly, from the previous one
\[ x_{b2} = (M_{b2} G_{b2}^+)^{-1} (M_{b3} G_{b3}^-) x_{b3} \]
and so on. Substituting the obtained expressions we get to:
\[
M_a G_a^+ x_a + s = M_{b1} G_{b1}^i \left[ (M_{b1} G_{b1}^i)^{-1} (M_{b2} G_{b2}^-) \right] \\
\cdot \left[ (M_{b2} G_{b2}^+)^{-1} (M_{b3} G_{b3}^-) \right] \cdots \left[ (M_{b(N-1)} G_{b(N-1)}^-)^{-1} \right].
\]
\[
\cdot \left[ (M_{bN} G_{bN}^i)^{-1} \right]\cdot x_{bN} = [M_{b1} G_{b1}^i (M_{b1} G_{b1}^i)^{-1}].
\]
\[
\cdot [M_{b2} G_{b2}^i (M_{b2} G_{b2}^i)^{-1} \cdots (M_{b(N-1)} G_{b(N-1)}^-)^{-1}] = \left(T_{b1} \cdots T_{b(N-1)} \right) M_{bN} G_{bN} x_{bN} = \]
\[= T_{bN}^+ M_{bN} G_{bN} x_{bN} = M_a G_a^+ x_a + s.
\]

where \( T_{bN} \) is the transmission matrix from the back layer to the transducer rear face. Each of the single transmission matrices is as follows, in a completely analogous fashion as we did before:
\[
T_{bN} = (M_{bN} G_{bN} G_{bN}^-)^{-1} = \left( \begin{array}{c c}
\cos(-\alpha_a/2) & i/z_{\alpha_a}^i \sin(-\alpha_a/2) \\
iz_{\alpha_a}^i \sin(-\alpha_a/2) & \cos(-\alpha_a/2)
\end{array} \right) \]
(33)

Using (32) and (33) we can rewrite equations (31a)-(31e) in the more compact matrix form
\[
\begin{align*}
T_{bN} M_{bN} G_{bN} x_{bN} &= M_a G_a^+ x_a + s \\
M_a G_a^+ x_a + s &= T_{aM} M_{aM} G_{aM} x_{aM}
\end{align*}
\]
but \( M_a, G_{aM}^+ \) and \( G_{bN}^- \) can be set to the identity matrix, yielding to
\[
\begin{align*}
T_{bN} M_{bN} G_{bN} x_{bN} &= G_a^+ x_a + s \\
G_a^+ x_a + s &= T_{aM} M_{aM} x_{aM}
\end{align*}
\]
where
\[ x_{aM} = (A'_{aM} \ A_{aM}) \quad \text{and} \quad x_{bN} = (B'_{bN} \ B_{bN}) \]

As a last step, we can write (34) as a single linear system
\[ Ax = b. \]
From (34) we obtain:
\[
\begin{cases}
a_{32} B'_{bN} - [\cos(\alpha_a/2) H_a + i \sin(\alpha_a/2) F_a] = 0 \\
a_{42} B'_{bN} - [i \sin(\alpha_a/2) H_a + \cos(\alpha_a/2) F_a] - [1 - k_T^2 \mu_a F_a] = 0 \\
a_{11} A'_{aM} - [\cos(-\alpha_a/2) H_a + i \sin(-\alpha_a/2) F_a] = 0 \\
a_{21} A'_{aM} - [i \sin(-\alpha_a/2) H_a + \cos(-\alpha_a/2) F_a] - [1 - k_T^2 \mu_a F_a] = 0
\end{cases}
\]

where
\[ a_{11} = [T_{a}^+ (1, 1) + T_a^+ (2, 2) z_{aM}^-] \]
\[ a_{21} = [T_a^+ (2, 1) + T_a^+ (2, 2) z_{aM}^-] \]
\[ a_{32} = -[T_b^- (1, 1) + T_b^- (1, 2) z_{bN}^-] \]
\[ a_{42} = -[T_b^- (2, 1) + T_a^+ (2, 2) z_{bN}^-] \]

Thus we can finally write
\[
\begin{bmatrix}
a_{11} & 0 & -c_a^- & -i s_a^\mp \\
a_{21} & 0 & -i s_a^- & -c_a^+ + k_T^2 \mu_a \\
0 & a_{32} & c_a^+ & i s_a^\mp \\
0 & a_{42} & i s_a^\mp & c_a^- - k_T^2 \mu_a
\end{bmatrix} \cdot \begin{bmatrix} A_{aM} \\ B_{bN}^- \\ H_a \\ F_a \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

where \( c_a^\pm = \cos(\pm \alpha_a/2) \) and \( s_a^\pm = \sin(\pm \alpha_a/2) \).

We can easily solve this system using for example Cramer's rule.

To finish with this model, we can write the expression for the transducer impedance as follows: consider equation (29)
\[ V = \frac{h}{2 \varepsilon} \left( e^{+i \alpha_a/2} - e^{-i \alpha_a/2} \right) (A_a - B_a) + D/\varepsilon \Rightarrow \\
\Rightarrow \frac{D d_a}{\varepsilon} = V \left[ 1 - \frac{h^2 \varepsilon}{2 \alpha_a} \sin(\alpha_a/2) z_{\alpha_a} d_a \right] (A_a - B_a) \]
\[ \Rightarrow \frac{D d_a}{\varepsilon} = V \left[ 1 - k_T^2 \mu_a F_a \right] \Rightarrow D = \frac{\varepsilon V}{d_a} \left( 1 - k_T^2 \mu_a F_a \right) \]
thus the electrical current through the transducer is
\[ I = i_{\omega} DS = i_{\omega} \frac{\varepsilon V}{d_a} (1 - k_T^2 \mu_a F_a) = i_{\omega} C_a (1 - k_T^2 \mu_a F_a) V \]
and the impedance
\[
Z = \frac{V}{I} = \frac{1}{i_{\omega} C_a} \frac{1}{1 - k_T^2 \mu_a F_a}
\]

### VI. Analytical results

In the three previous sections we have introduced and defined three different analytical models that varied in capabilities and modelling techniques. For a considered transducer, all will provide analytical results for the transmitter and receiver force-voltage relations and the electrical impedance behaviour.

To test our models against FEM and real data from the laboratory, we have considered three test cases whose specifications can be found in the following table:

<table>
<thead>
<tr>
<th>Material</th>
<th>Test Case 1</th>
<th>Test Case 2</th>
<th>Test Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT</td>
<td>25x25mm</td>
<td>20x20mm</td>
<td>20x20mm</td>
</tr>
<tr>
<td>Thickness</td>
<td>0.5mm</td>
<td>1mm</td>
<td>2mm</td>
</tr>
</tbody>
</table>

The three transducers are made of Lead zirconate titanate (PZT), one of the most common piezoelectric materials. Notice the lateral dimensions-thickness ratio for the three of them, beginning with a reasonably high value and decreasing afterwards. As the considered analytical models are 1D, we will check how well this approximation holds for real world devices.
A. Transducer impedance

The results for the impedance modulus are shown in figures 12,13 & 14. The spike on the right hand side of the plot corresponds to the main resonance region, while the descending curve on the left hand side corresponds to the lateral resonance behaviour.

![Fig. 12: Electrical impedance modulus for test case 1](image1)

![Fig. 13: Electrical impedance modulus for test case 2](image2)

We can see good agreement with the real data for the three cases, although the FE model’s predictions are closer to reality. Due to the lack of lateral dimensions and any kind of attenuation or loss mechanism we can see our models predicting a narrower and earlier peak. Due to a laboratory problem, the data for the second test case is incomplete, but we can still appreciate its behaviour. All the plots are normalized to the unit due to some magnitude discordances among the models.

It is not shown in the pictures for the sake clarity, but we are also able to predict the impedance peak’s harmonics in higher frequencies, agreeing as well with the laboratory data.

Similarly, we show below the results for the impedance phase (figures 15, 16 & 17), we can see good agreement predicting the peak area.

![Fig. 14: Electrical impedance modulus for test case 3](image3)

![Fig. 15: Electrical impedance phase for test case 1](image4)

We have then tested some realistic devices and found that we can successfully predict their behaviour in a fast and clean way using analytical models. Let’s see now how to interpret the impedance results: when designing or analyzing a given transducer, we want to locate its impedance peaks, which will be the desired operating frequencies, as we will get the best behaviour there. This is then the first step in the design process.

B. Transducer bandwidth

We will define bandwidth as the range of frequencies for which the signal power has dropped to half of its maximum value (-3 dB). In the case of ultrasonic transducers, the signal will be the force output when transmitting and the voltage output when receiving (completely analogous results using...
Our results for the three transducers are presented below (figures 18, 19 & 20). The obtained fractional bandwidth is probably not very accurate as our analytical models predict narrower peaks, but as we will see later it still provides good qualitative behaviour.

A mismatch between transducer and loading medium acoustic impedances will result in poor bandwidth and energy transfer, being them increased as both impedances get closer. Hence, a preliminary test for our models will consist of varying the theoretical loading medium impedance and find the maximum in the fractional bandwidth curve, which is shown below (figure 21).

The black vertical line shows the transducer impedance,
ideally we should get the maximum exactly at that point, but due to the tendency of finding the maximum impedance peaks a bit shifted, we find a maximum point which is close to the theoretical result. The third test case, the one further from 1D approximation, presents some drift for higher impedances.

\[
\text{FBW optimization for a varying loading medium impedance}
\]

![Graph showing FBW optimization](image)

\[
\text{Fig. 21: Acoustical impedance optimization}
\]

C. Bandwidth and matching layers

When designing an application specific transducer with a given loading medium, the matching layers come into play: their purpose it to minimize the acoustic impedance mismatch between the transducer and the loading medium.

Literature gives the ideal impedance value for a single matching layer as the geometric mean of the transducer and loading medium impedances: \( Z = \sqrt{Z_t Z_l} \). The other important aspect of the matching layers is their thickness: optimal thickness for a single matching layer is a quarter of the operating wavelength (\( \lambda = c/f \), being \( c \) the speed of sound in the selected material and \( f \) the frequency). When the thickness of the piezoelectric layer is \( \lambda/2 \) this guarantees that waves that are reflected within the matching layer are in phase when exiting the layer, hence maximum energy transmission is achieved. [7]

We have tested this results analytically using lattice model, the results plotted below: we have considered a single matching layer with optimum impdance, and different thicknesses varying from zero to twice the thickness of the piezoelectric layer. The first dashed line corresponds to \( \lambda/4 \) thickness, and the gap between any two consecutive lines is \( \lambda/2 \), this will also keep reflected waves in phase. It is worth mentioning that our piezoelectric thicknesses are way larger than the optimum \( \lambda/2 \) explained before, so we may see discrepancies with the optimum matching thickness as well.

We can see very good agreement in the two first test cases: higher bandwidth is obtained as predicted theoretically. For

the first one, the model predicts as well high bandwidth peaks at the successive \( \lambda/2 \) jumps. The second one seems to have destructive wave interaction for early jumps, being maybe stabilized at a later stage. However it is difficult to judge because the dashed lines cut the bandwidth curve at very steep points. The third case is more difficult to interpret: we definitely see the spike behaviour located around the dashed lines, but it does not agree with the expected behaviour. This may be caused by numerical instabilities or by a too small lateral dimensions-thickness ratio, making the 1D model insufficient for this kind of analysis. In this case, further analysis using laboratory instrumentation or FEM should be used.

Summarizing, nice agreement with theory is found using the two first test cases, while the third one needs further testing. It seems to work fine for transducers thoroughly satisfying our assumptions. Deeper and broader analysis must be performed for other transducers to check the validity of this models.

\[
\text{Test Case 1 - Optimum matching impedance}
\]

![Graph showing Test Case 1](image)

\[
\text{Test Case 2 - Optimum matching impedance}
\]

![Graph showing Test Case 2](image)

D. Communication systems: pitch-catch

As a last setup for our models we will consider simulating a complete transducer system, that is typically seen in practical
applications: we place two ultrasonic transducers at both sides of an in-between medium, this can be a wall or other kind of object (see figure 18).

Air  Piezo  In-between medium  Piezo  Air

\[
\begin{array}{c|c|c|c|c}
Z_1.v_1 & Z_2.v_1 & Z_2.v_2 & Z_2.v_3 & Z_3.v_3 \\
\end{array}
\]

Fig. 25: Pitch-catch transmission system

One of them will act as a transmitter and the other one will receive the transmitted wave. This kind of systems have started being used in places where conventional wired communication is not possible, for example in submarines or in any kind of pressurized enclosure. This kind of ultrasonic communication problem is usually approached using an ad-hoc procedure, get the transducers in place and test them using a short pulse trying to obtain the frequency signature that would be used to model more complex waves.

Hence we can try to apply analytical modelling to this problem: we shall use the linear systems model for both transmitter and receiver. To use this model we have to assume that the in-between medium’s thickness is much larger that the transducer, this shoud not be a problem though. We will hit the transmitter with an ideal pulse and obtain the force output in frequency domain, finally we will plug it into the receiver, obtaining the final voltage output. We could use an inverse Fourier transform to get the time domain representation and then use convolution to obtain the final output.

The analytical model gives the following: applying a voltage \( e \) to the transmitter, the transfer function for the output force is, using equation (17)

\[
F_{F1} = -\frac{e \cdot h \cdot a(s)Z_2/(Z_{c1} + Z_2) K_{F1}(s)Y(s)}{1 - h^2Y(s)/(sZ_c) (T_{F1}K_{F1}/2 + T_{B1}K_{B1}/2)}
\]

where the coefficients are detailed in the third section, considering the new index notation specified in the figure.

Then, for the receiver, using (12)

\[
V = \frac{-F \cdot h \cdot K_{F2}T_{F2}U(s)/sZ_{c2}}{1 - h^2 \left(\frac{K_{F2}T_{F2}}{2} + \frac{K_{B2}T_{B2}}{2}\right) U(s)/s^2Z_cZ_{c2}}
\]

\[
= \frac{h \cdot K_{F2}T_{F2}U(s)/sZ_{c2}}{1 - h^2 \left(\frac{K_{F2}T_{F2}}{2} + \frac{K_{B2}T_{B2}}{2}\right) U(s)/s^2Z_cZ_{c2}}
\]

\[
e \cdot h \cdot a(s)Z_2/(Z_{c1} + Z_2) K_{F1}(s)Y(s)
\]

\[
1 - h^2Y(s)/(sZ_c) (T_{F1}K_{F1}/2 + T_{B1}K_{B1}/2)
\]

To test the results we have used the first test case transducer as transmitter and receiver. The in-between medium is chosen to be steel (\( Z_2 = 45 \text{MRayl} \)) with large enough thickness. The results are presented below.

The thick curve is the generated force in the transmitter, and the thin one is the generated voltage in the receiver. We can see that we receive a similar but attenuated signal. Similar results are obtained using the other two test cases. The system can be improved introducing matching layers and chosing a better backing than air. It would also be interesting to obtain an analytical model comprising both transmitter and receiver in a compact way, i.e. solve the wave equations for the present boundary conditions (see figure 10).

VII. CONCLUSIONS

We have explained and tested three different analytical models for ultrasonic transducers in a variety of experiments to check their behaviour, obtaining typically good results, although when the transducer gets a bit far from the models’ assumptions, predictions break and one has to be more careful.

Our models allow us to find the optimal operating frequencies for each considered transducer. Once this is
achieved, our interest will be maximizing the bandwidth and the energy transmission. This will be done using matching layers, whose optimum impedance and thickness can be obtained analytically.

Thus analytical models are a good tool for design and optimization purposes, providing an insight into the underlying physics. This kind of models are also very fast and can be easily coded, in contraposition with FE models which are much slower and computationally expensive, and usually come in the form of commercial software packages.

**VIII. ACKNOWLEDGEMENT**

I would like to thank Nishal Ramadas for all the help and training provided.

**REFERENCES**


