Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

3 Introduction to chaos: Lorenz equations

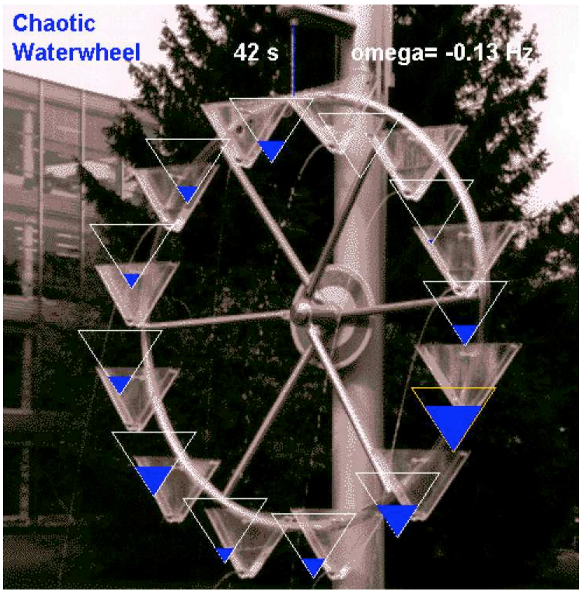
Consider the Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz\end{aligned}$$

Here $\sigma,\ r,\ b > 0$ are parameters. Ed Lorenz derived this three-dimensional system from a simplified model of convection rolls in the atmosphere (E. Lorenz (1963) Deterministic nonperiodic flow, *Journal of Atmospheric Sciences*, Vol. 20).

A chaotic waterwheel

A mechanical model of the Lorenz equations was invented by W. Malkus and L. Howard at MIT in the 1970s.



From the website of Fritz Gassmann
http://people.web.psi.ch/gassmann/waterwheel/WaterwheelLab.html

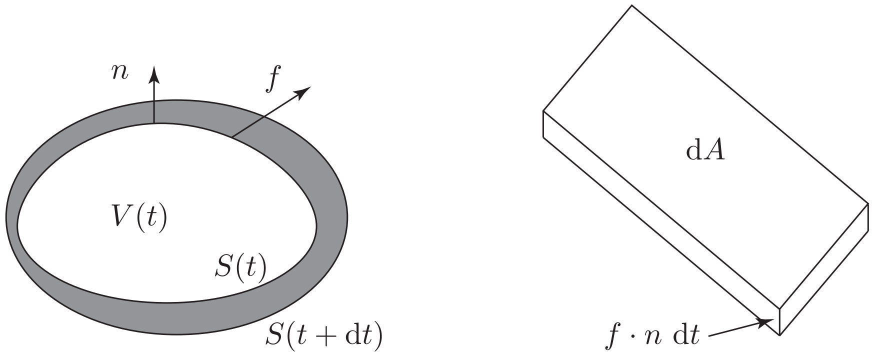
3.1 Properties of the Lorenz equations

- **Nonlinearity**
  The Lorenz equations have two nonlinearities.

- **Symmetry**
  If we replace $(x, y) \to (-x, -y)$, the equations stay the same. Hence if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$.

- **Volume contraction**
  The Lorenz system is **dissipative**: volumes in phase-space contract under the flow.

  Consider any three-dimensional system $\dot{x} = f(x), \qquad x \in \mathbb{R}^3$. Consider an arbitrary closed surface $S(t)$ of volume $V(t)$ in phase space. After dt time, S evolves into a new surface $S(t + \mathrm{d}t)$. What is its volume $V(t + \mathrm{d}t)$?



  Let n denote the outward normal on S. Then $f \cdot n$ is the outward normal component of velocity (since f is the instantaneous velocity of the points). Therefore in time dt a patch of area dA sweeps out a volume $(f \cdot n\ \mathrm{d}t)\mathrm{d}A$. Hence

$$V(t + \mathrm{d}t) = V(t) + \int_S (f \cdot n\ \mathrm{d}t)\mathrm{d}A$$
$$\Rightarrow \dot{V} = \frac{V(t + \mathrm{d}t) - V(t)}{\mathrm{d}t} = \int_S f \cdot n\ \mathrm{d}A.$$

  By the divergence theorem we have

$$\dot{V} = \int_V \nabla \cdot f\ \mathrm{d}V.$$

  For the Lorenz system

$$\nabla \cdot f = \frac{\partial}{\partial x}[\sigma(y - x)] + \frac{\partial}{\partial y}[rx - y - xz] + \frac{\partial}{\partial z}(xy - bz) = -\sigma - 1 - b < 0.$$

  Since the divergence is constant we have $\dot{V} = -(\sigma + 1 + b)V$ with solution $V(t) = V(0)\mathrm{e}^{-(\sigma+1+b)t}$. Thus volumes in phase space shrink exponentially fast.

- The Lorenz equations cannot have repelling fixed points or repelling closed orbits, since repellers are incompatible with volume contraction. Thus all fixed points must be sinks or saddles and closed orbits (if they exist).

- **Fixed points**
  Two types of fixed points: $(x^*, y^*, z^*) = (0, 0, 0)$ for all values of the parameters and a symmetric pair of fixed points

$$C^+ \equiv (x^* = y^* = \sqrt{b(r-1)},\ z^* = r - 1)$$
$$C^- \equiv (x^* = y^* = -\sqrt{b(r-1)},\ z^* = r - 1)$$

  when $r > 1$. At $r = 1$ 2 fixed pints coalesce with $(0, 0, 0)$.

- **Linear stability of the origin**
  The linearisation at the origin gives

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y$$
$$\dot{z} = -bz$$

The equation for $z(t)$ is decoupled and show that $z(t) \to 0$ exponentially fast.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv A \begin{pmatrix} x \\ y \end{pmatrix}$$

with $tr(A) = -\sigma - 1 < 0$ and $det(A) = \sigma(1 - r)$.

$r > 1$ $\Rightarrow$ the origin is a saddle point because $det(A) < 0$ (one outgoing and two incoming directions)

$r < 1$ $\Rightarrow$ all directions are incoming, $tr(A)^2 - 4det(A) = (\sigma - 1)^2 + 4\sigma r > 0$ and $(0, 0, 0)$ is a stable node

- **Global stability of the origin**
  We can show for $r < 1$ that the origin is globally stable (no limit cycles or chaos) by constructing a Lyapunov function. Consider $L(x, y, z) = x^2/\sigma + y^2 + z^2$. We have to show that if $r < 1$ and $(x, y, z) \neq (0, 0, 0)$, then $\dot{L} < 0$.

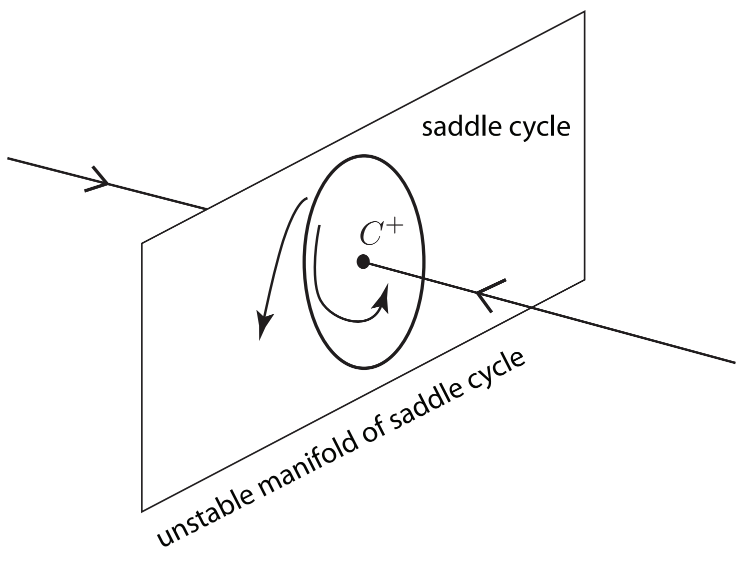
$$\frac{1}{2}\dot{L} = x\dot{x}/\sigma + y\dot{y} + z\dot{z} = -\left[x - \frac{r+1}{2}\right]^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right] y^2 - bz^2$$

We can show that $\dot{L} = 0$ only at $(0, 0, 0)$, otherwise $\dot{V} < 0$. Therefore the origin is globally stable.

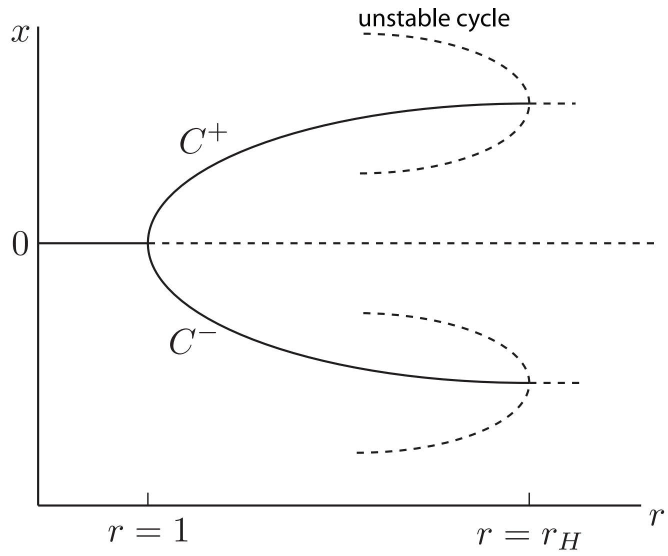
- **Stability of C^+ and C^-**
  Assume $r > 1$ so that C^+ and C^- exist. We can find that they are linearly stable for

$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

(assuming $\sigma - b - 1 > 0$). At $r = r_H$ - subcritical Hopf bifurcation
When $r < r_H$

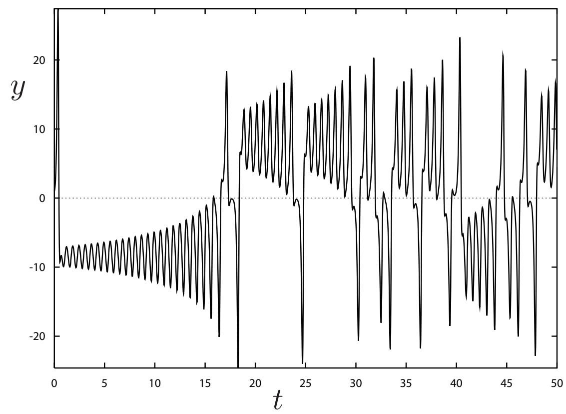


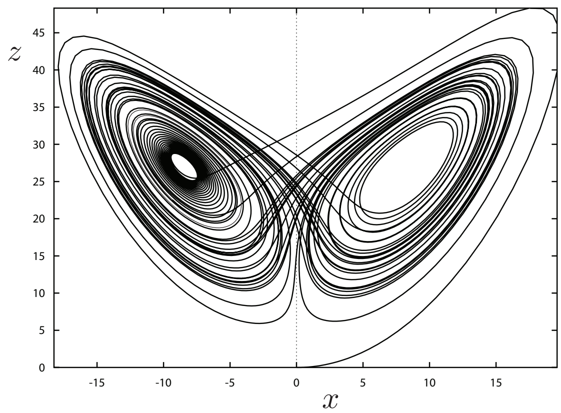
A saddle cycle - a new type of unstable limit cycle that is possible in phase space of three or more dimensions.



Chaos on a strange attractor

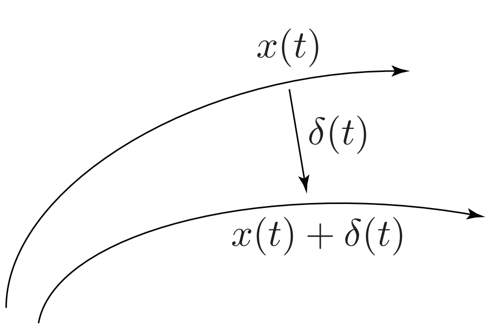
Lorenz used numerical integration to see what the trajectories would do in the long run. Parameters $\sigma = 10$, $b = 8/3$, $r = 28$. The motion is **aperiodic**.





Exponential divergence

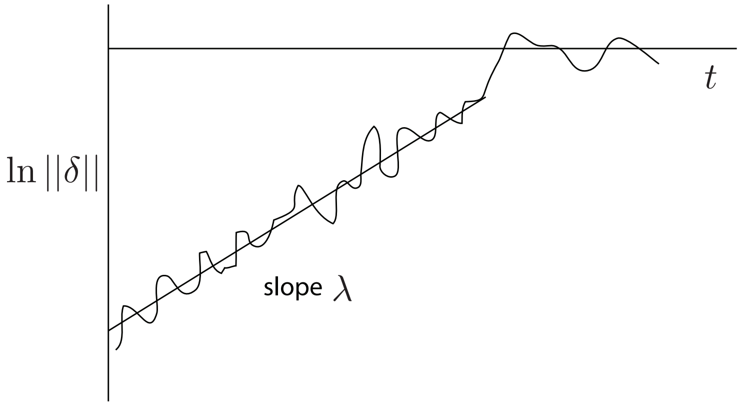
Consider $x(t)$ and $x(t) + \delta(t)$ with a vector of initial length $||\delta_0|| = 10^{-15}$.



It can be found numerically that

$$||\delta(t)|| \propto ||\delta_0|| e^{\lambda t}$$

where $\lambda = 0.9$.



In general an n-dimensional system has n different Lyapunov exponents

$$\delta_k(t) \propto \delta_k(0) e^{\lambda_k t}$$

Our λ is the largest Lyapunov exponent.

If a measures our tolerance, then our prediction becomes intolerable (when $||\delta(t)|| \geq a$) after a time

$$t \approx \frac{1}{\lambda} \ln \frac{a}{||\delta_0||}$$

3.2 Defining chaos

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

1. *Aperiodic long-term behaviour* means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as $t \to \infty$.

2. *Deterministic* means no noise.

3. *Sensitive dependence on initial conditions* means that nearby trajectories separate exponentially fast (the Lyapunov exponent $\lambda > 0$).

3.3 Defining attractor and strange attractor

An attractor Λ is a closed set A with the following properties

1. A is an invariant set: any trajectory $x(t)$ that starts in A stays in A for all time.

2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.

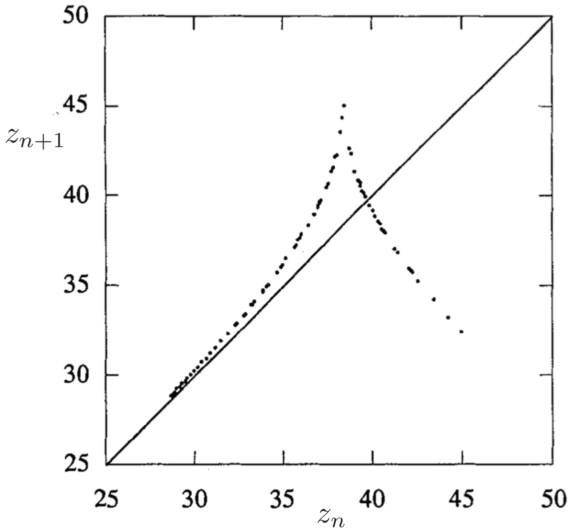
3. A *is minimal*: there is no proper subset of A that satisfies conditions 1 and 2.

Attractors with positive Lyapunov exponents are called strange attractors, and trajectories are called chaotic if at least one Lyapunov exponent is positive (i.e. there is sensitive dependence upon initial conditions). In a strange chaotic attractor the positive Lyapunov exponent indicates exponential spreading within the attractor in the direction transverse to the flow and the negative exponent indicates exponential contraction onto the attractor.

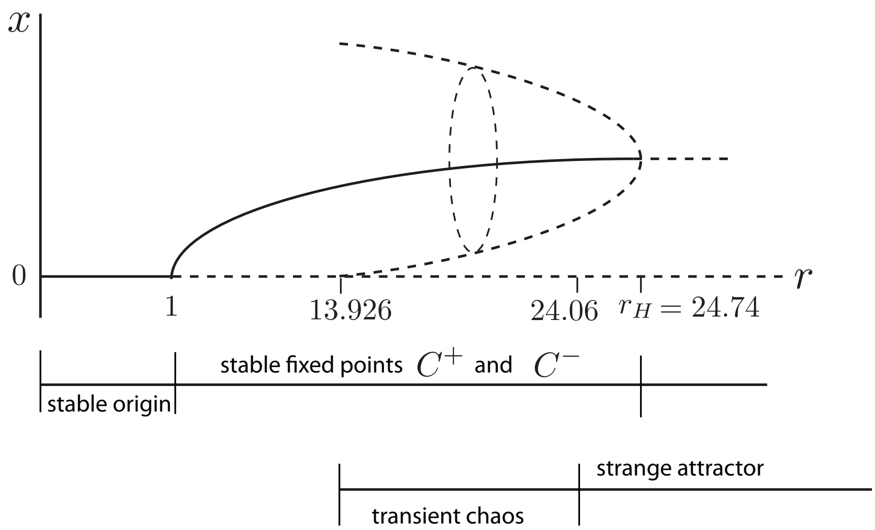
In nonlinear systems it is possible for more than one attractor to exist. To which attractor a trajectory ends up in depends upon initial conditions. The closure of the set of initial conditions which approach a given attractor is called a basin of attraction. In many nonlinear systems the boundary between basins is **not** smooth and has a **fractal** structure.

Lorenz map

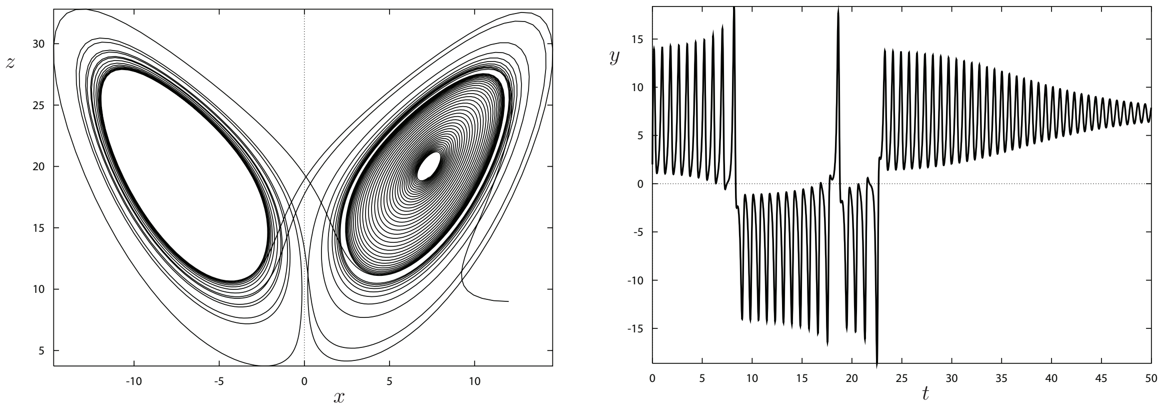
The main idea behind it is to show that z_n should predict z_{n+1}. The Lorenz equations can be integrated for a long time, then the local maxima of $z(t)$ are measured and plotted z_{n+1} vs. z_n to get the Lorenz map $z_{n+1} = f(z_n)$.



Exploring parameter space



Parameters: $\sigma = 10$, $b = 8/3$, $r = 21$:



Parameters: $\sigma = 10$, $b = 8/3$, $r = 350$: