## Complexity Science Doctoral Training Centre

### CO903 Complexity and Chaos in Dynamical Systems

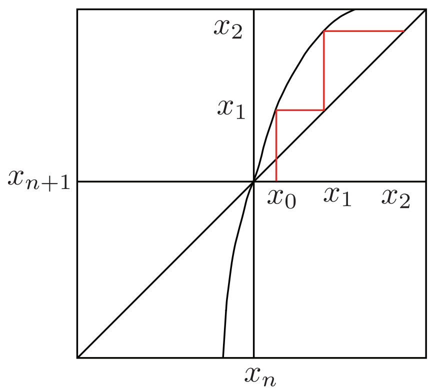
# 1 Introduction to Dynamical Models

## Difference equations

A difference equation (or an iterated map) can be written as

$$x_{n+1} = f(x_n),$$

where $x_n$ is the value of a variable $x$ at some time $n$, and $f$ is a function that describes the relationship between the value of $x$ at time $n$ and $n+1$. This equation can be analysed using a simple geometric interpretation, called the cobweb construction below. Given an initial condition $x_0$, draw a vertical line until it intersects the graph of $f$: that height is given by $x_1$. Then return to the horizontal axis and repeat the procedure to get $x_2$ from $x_1$.




## Differential equations

Differential equations describe the evolution of systems in continuous time (whereas iterated maps - in discrete time). A system of ordinary differential equations (ODEs):

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, ..., x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, ..., x_n). \end{aligned}$$

Here $\dot{x}_i \equiv \mathrm{d}x_i/\mathrm{d}t$.

A single ODE is a mathematical equation for an unknown function of one independent variable that relates the values of the function itself and of its derivative. In general, ODEs define the rates of change of the variables in terms of the current state. To solve a system of ODEs means to find continuous functions $x_1(t)$,..., $x_n(t)$ of the independent variable $t$ that, along with their derivatives, satisfies the system of equations.

**Autonomous systems**: $f_i(x_1, ..., x_n)$; **nonautonomous systems**: $f_i(x_1, ..., x_n, t)$.

## 1.1 First-order systems

$$\dot{x} = f(x)$$

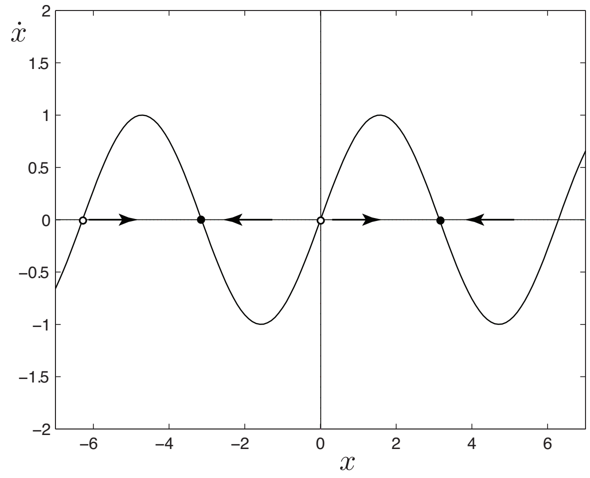
**Reminder:** Linear ODE

$$\dot{x} = -\frac{x}{\tau}, \qquad x(0) = x_0.$$

$$\text{Solution:} \qquad x(t) = x_0 e^{-t/\tau}$$

One of the most basic techniques of dynamics is to interpret a differential equation as a vector field.

**Example 1.** $\dot{x} = \sin x$




## Fixed points and stability

A solution $x(t)$ of differential equation starting from the initial condition $x_0$ is also called a *trajectory*. The fixed points (equilibrium solutions, steady-states) are defined by $f(x^*) = 0$.

- A fixed point is defined to be *stable* if all sufficiently small disturbances away from it damp out in time.
  - If all solutions of the dynamical system that start out near an equilibrium point $x^*$ stay near $x^*$ forever, then $x^*$ is *Lyapunov stable*.
  - More strongly, if all solutions that start out near $x^*$ converge to $x^*$, then $x^*$ is *asymptotically stable*.
  - The notion of *exponential stability* guarantees a minimal rate of convergence.
- If disturbances grow in time a fixed point is defined to be *unstable*.

**Example 2.** Find all fixed points and classify their stability

$$\dot{x} = x^2 - 1$$

.

**Example 3. Population Growth**: Classify the fixed points of the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

## Linear stability analysis

Let $x^*$ - a fixed point and $\eta(t) = x(t) - x^*$ - a small perturbation.

$$\dot{\eta} = f(x^* + \eta)$$

> **Reminder:** Taylor series of $f$ about the point $x^*$
>
> $$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x^*) \frac{(x - x^*)^n}{n!},$$
>
> where $f^k(x^*)$ is the $k$th derivative of the function $f$ evaluated at $x^*$.

Then $f(\eta + x^*) = f(x^*) + f'(x^*)\eta + O(\eta^2) \Rightarrow$

$$\dot{\eta} = f'(x^*)\eta$$

- $\eta(t)$ grows exponentially if $f'(x^*) > 0 \Rightarrow x^*$ - unstable
- $\eta(t)$ decays exponentially if $f'(x^*) < 0 \Rightarrow x^*$ - stable

## Local existence and uniqueness (in $\mathbb{R}$)

**Example 4.**

$$\dot{x} = x^{1/3} \qquad x(0) = 0$$

The point $x = 0$ is a fixed point, so one solution is $x(t) = 0$ for all $t$. Also

$$\int \frac{\mathrm{d}x}{x^{1/3}} = \int \mathrm{d}t \Rightarrow \frac{3}{2}x^{2/3} = t + C$$

Initial data ensures $C = 0$ so $x(t) = (2t/3)^{3/2}$. The co-existence of solutions (non-uniqueness) is due to the fact that $x^{1/3}$ is not continuously differentiable.

**Theorem**: Consider the initial value problem (IVP)

$$\dot{x} = f(x), \qquad x(0) = x_0$$

Suppose that $f$ is continuously differentiable on an open interval $I \subset \mathbb{R}$ with $x_0 \in I$. Then the IVP has a solution $x(t)$ on some interval $(-\tau, \tau)$ about $t = 0$ and the solution is unique.

**Example 5.**

$$\dot{x} = 1 + x^2, \qquad x(0) = 0$$

$$\int \frac{\mathrm{d}x}{1 + x^2} = \int \mathrm{d}t \Rightarrow \tan^{-1} x = t + C$$

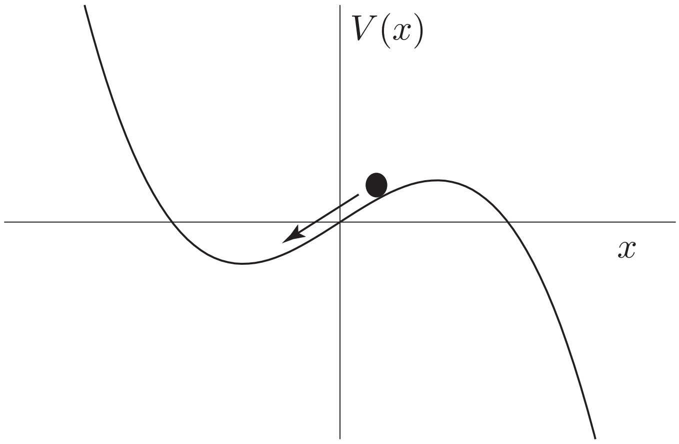
Initial data ensures $C = 0$ so $x = \tan t$ and only exists for $-\pi/2 < t < \pi/2$. This is an example of *blow-up* where $x(t)$ reaches $\infty$ in finite time.

As we have seen, all flows on a line either approach a fixed point or diverge to $\pm\infty$. In essence overshoot and damped oscillations can never occur in a first order system. It is never possible to have periodic motion for the first order system $\dot{x} = f(x)$, $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. (Motion on a circle may however lead to periodic motion).

## Potentials

For a dynamical system $\dot{x} = f(x)$, the potential $V(x)$ is defined by

$$f(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$




## Numerical simulation of dynamical systems

This is a vast subject in its own right. It is a powerful means of obtaining insight into the behaviour of nonlinear dynamical systems (thanks to cheap high speed computers) by obtaining approximate solutions to problems that are analytically intractable.

**Euler's method** Given a dynamical systems $\dot{x} = f(x)$ and a known solution at time $t$ we approximate the solution at time $t + \Delta t$ by $x(t + \Delta t)$ where

$$x(t + \Delta t) = x(t) + f(x(t))\Delta t$$

Introducing the notation $x_n = x(t_n)$ where $t_n = n\Delta t$ we have the following iterative scheme

$$x_{n+1} = x_n + f(x_n)\Delta t$$

**Improved Euler method** One problem with the Euler method is that it estimates the derivative $\dot{x}$ only at the left end of the time-interval between $t_n$ and $t_{n+1}$. A better approach is to use the average derivative across this interval. First construct the estimate $\widetilde{x}_{n+1} \approx x_n + f(x_n)\Delta t$ and then take the average of $f(x_n)$ and $f(\widetilde{x}_n)$ and use that to construct the next step in the iterative scheme. The improved Euler method is then given by

$$\begin{aligned} \widetilde{x}_n &= x_n + f(x_n)\Delta t & \text{the trial step} \\ x_{n+1} &= x_n + \frac{1}{2}[f(x_n) + f(\widetilde{x}_n)]\Delta t \end{aligned}$$

This scheme tends to a smaller *error* $E = |x(t_n) - x_n|$ for a given step-size $\Delta t$. For the (first order) Euler method $E \propto \Delta t$, whilst for the (second order) modified Euler $E \propto (\Delta t)^2$.

An accurate and commonly used scheme is the so-called fourth-order **Runge-Kutta scheme**:

$$x_{n+1} = x_n + \Delta t \left[\frac{1}{6}k_1(n) + \frac{1}{3}k_2(n) + \frac{1}{3}k_3(n) + \frac{1}{6}k_4(n)\right]$$

where

$$
\begin{aligned}
k_1(n) &= f(x_n) \\
k_2(n) &= f(x_n + \frac{1}{2}\Delta t k_1(n)) \\
k_3(n) &= f(x_n + \frac{1}{2}\Delta t k_2(n)) \\
k_4(n) &= f(x_n + \Delta t k_3(n))
\end{aligned}
$$

Rather than developing code from scratch it is useful to work with a tried and tested piece of software such as XPPAUT.

**Dynamical Systems Software**

1. AUTO 2000 is a software package for continuation and bifurcation problems in ordinary differential equations.

2. BOV-method Nonpublic software for the computation of normally hyperbolic invariant manifolds in discrete dynamical systems.

3. CANDYS QA Computer Analysis of Nonlinear Dynamical Systems Qualitative Analysis. A software package for numerical bifurcation analysis of dynamical systems.

4. CONTENT 1.5 is designed to perform simulation, continuation, and normal form analysis of dynamical systems. The current version supports bifurcation analysis of ODE's, iterated maps, and evolution PDE's in the unit interval.

5. DDE-BIFTOOL is a Matlab package for numerical bifurcation analysis of delay differential equations with several fixed discrete delays.

6. DsTool is a toolkit for exploring dynamical systems. It can do simulation of diffeomorphisms and ODE's, find equilibria and compute their one-dimensional stable and unstable manifolds.

7. Dynamics is designed for the exploration of two-dimensional maps, both diffeomorphisms and noninvertible maps.

8. Dynamics Solver is intended to solve initial and boundary-value problems for continuous and discrete dynamical systems. It is possible to draw phase-space portraits, Poincaré maps, Lyapunov exponents, cobweb diagrams, histograms and bifurcation diagrams.

9. GAIO is experimental software for the approximation of invariant sets and invariant measures in dynamical systems.

10. Geomview is an interactive program written at the Geometry Center for viewing and manipulating geometric objects in three dimensions, or in three three-dimensional projections of four dimensional space.

11. Global Manifolds 1D Software for globalizing one-dimensional stable and unstable manifolds for maps in $\mathbb{R}^2$

12. Global Manifolds 2D Nonpublic software for globalizing two-dimensional stable and unstable manifolds for maps in $\mathbb{R}^3$

13. Multifario is a set of subroutines and data structures for computing manifolds of dynamical systems.

14. PDECONT 1.01 This code implements some Newton-Picard single shooting algorithms for the continuation of periodic solutions of large-scale problems. The method tries to exploit the fact that many systems have only low-dimensional dynamics. Although the primary goal was the computation of periodic solutions of PDE's, the code can also be used for large ODE systems.

15. Phaser is a universial simulator for dynamical systems that provides a powerful, yet inviting, computing environment specifically crafted for the graphical and numerical simulations of differential and difference equations, from linear to chaotic.

16. Strong (Un)Stable Manifolds Software for computing one-dimensional strong stable and unstable manifolds for vector fields.

17. XPP/XPPAUT is a package for simulating and numerically solving dynamical systems. XPP can handle Differential equations, Delay equations, Volterra integral equations, Discrete dynamical systems, Markov processes, and Bifurcations.

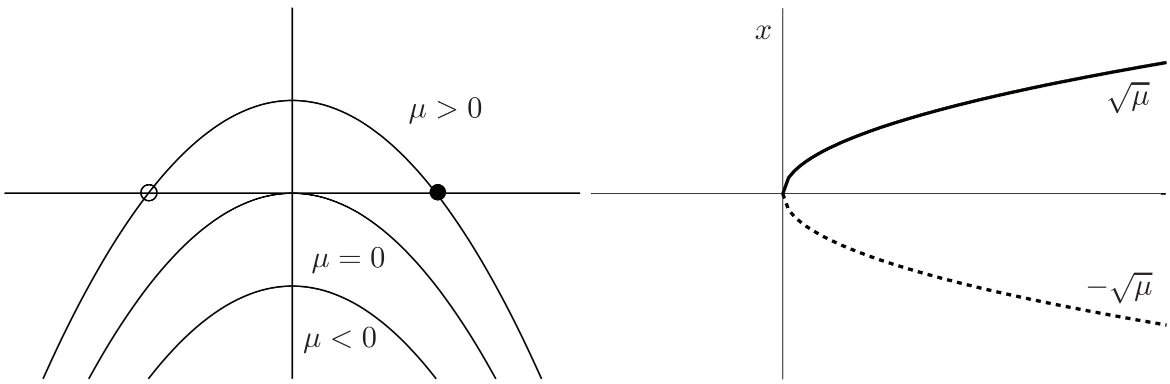
For an up-to-date list see the home-page of Hinke Osinga at http://www.enm.bris.ac.uk/staff/hinke/dss/

## Bifurcations

The qualitative structure of a flow can change as a parameter is varied. These qualitative changes are called *bifurcations* and the parameter values at which they occur are called *bifurcation* points.

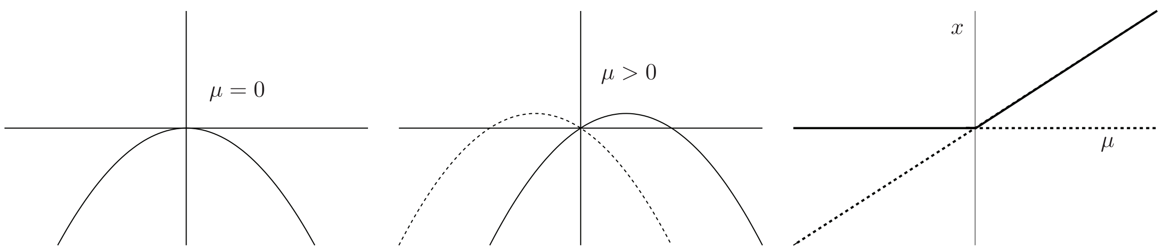
### Saddle-Node bifurcation

$$\dot{x} = \mu - x^2$$




### Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$



**Example 6.**

$$\dot{x} = r \ln x + x - 1$$

Fixed point at $x = 1$. Let $u = x - 1$, then

$$\begin{aligned} \dot{u} &= r \ln(1+u) + u \approx r\left(u - \frac{u^2}{2} + \ldots\right) + u \\ &= (r+1)u - \frac{1}{2} r u^2 \end{aligned}$$

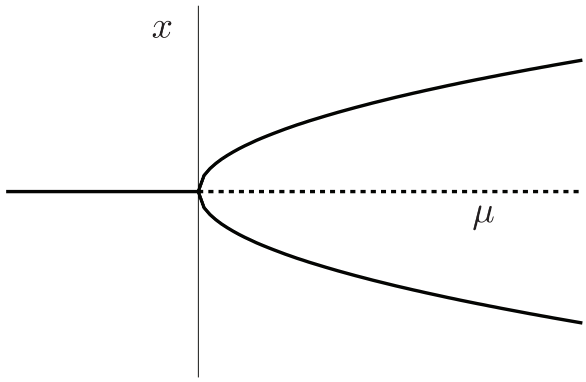
Rescale ($v = (r/2)u$):

$$\dot{v} = (r+1)v - v^2$$

By a near identity change of co-ords we have found the *normal form* for the dynamics (valid close to the bifurcation point).

**Pitchfork bifurcation: supercritical**

$$\dot{x} = \mu x - x^3$$




Shows critical slowing down at $\mu = 0$:

$$\int \frac{\mathrm{d}x}{x^3} = -\int \mathrm{d}t \Rightarrow x = \sqrt{\frac{1}{2(t+C)}}, \quad C = \frac{1}{2x_0^2}(x_0 \neq 0)$$

For large $t$, $x \sim t^{-1/2}$: power law decay rather than exponential $\mathrm{e}^{\mu t}$.

**Example 7. Overdamped harmonic oscillator**

Consider an anharmonic oscillator with force

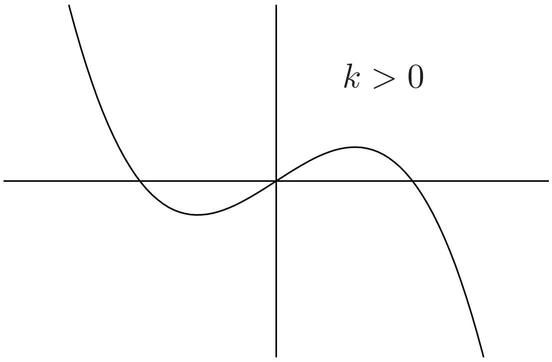
$$F(x) = kx - x^3$$

Newton's law of motion gives

$$m\ddot{x} + \gamma \dot{x} = F(x)$$

where $m$ is the mass and $\gamma$ the damping coefficient. Suppose that $m$ is small and $\gamma$ large. Then to a good approximation

$$\boxed{\gamma \dot{x} = kx - x^3}$$




$k > 0$

The equilibrium are determined by $\dot{x} = 0$.

1. $k < 0$: $F(\overline{x}) = 0$ when $\overline{x} = 0$
2. $k > 0$: $F(\overline{x}) = 0$ when $\overline{x} = 0, \pm\sqrt{k}$

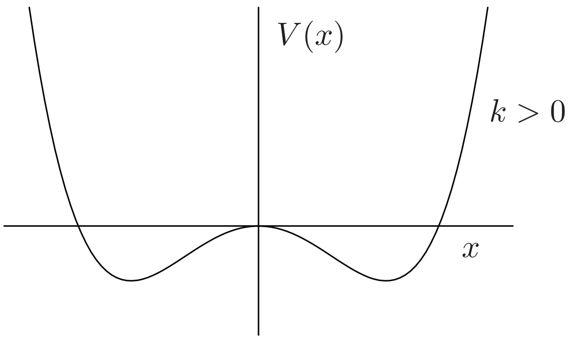
The potential function $V(x)$ is

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

where

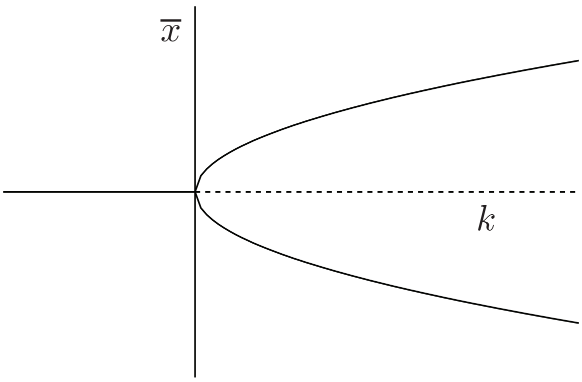
$$\boxed{V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}x^4}$$

Bistability:



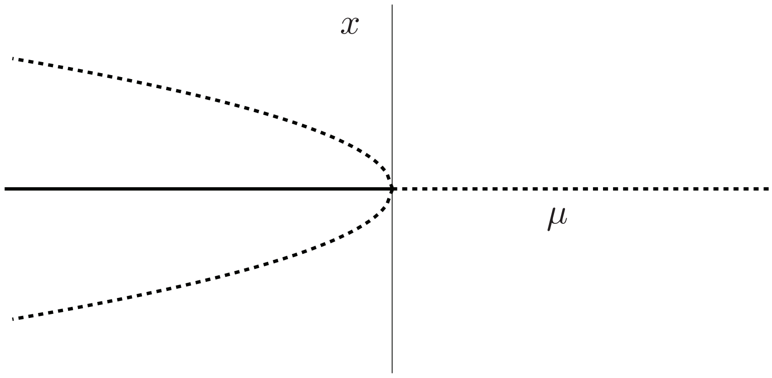

$k > 0$

It follows that $\overline{x} = 0$ is stable when $k < 0$, but for $k > 0$, $\overline{x} = 0$ becomes unstable and is replaced by two stable equilibria at $\overline{x} = \pm\sqrt{|k|}$.



**Pitchfork bifurcation: subcritical**

$$\dot{x} = \mu x + x^3$$




**Example 8.**

$$\dot{x} = \mu x + x^3 - x^5$$

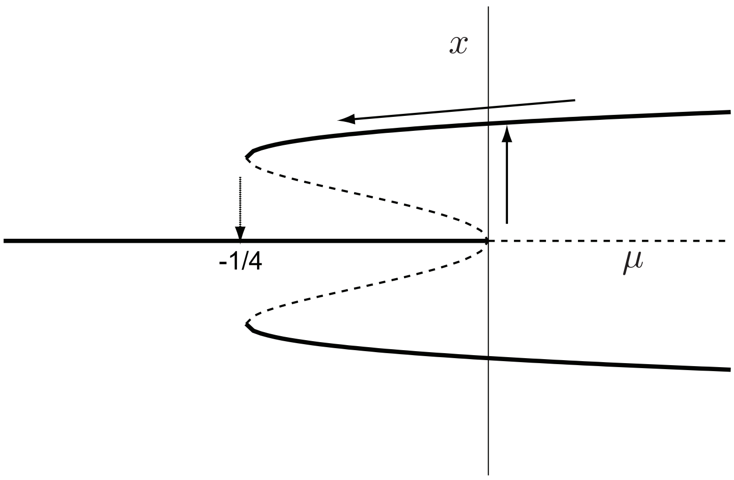
Fixed points

$$-(\mu + x^2) + x^4 = 0 \qquad \text{and} \qquad x = 0$$

roots:

$$x^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

If $\mu > 0$ then $x^2 = (1 + \sqrt{1+4\mu})/2$: total of three fixed points. If $-1/4 < \mu < 0$, $x^2 = (1 \pm \sqrt{1+4\mu})/2$: total of five fixed points. Define $\mu_c = -1/4$.

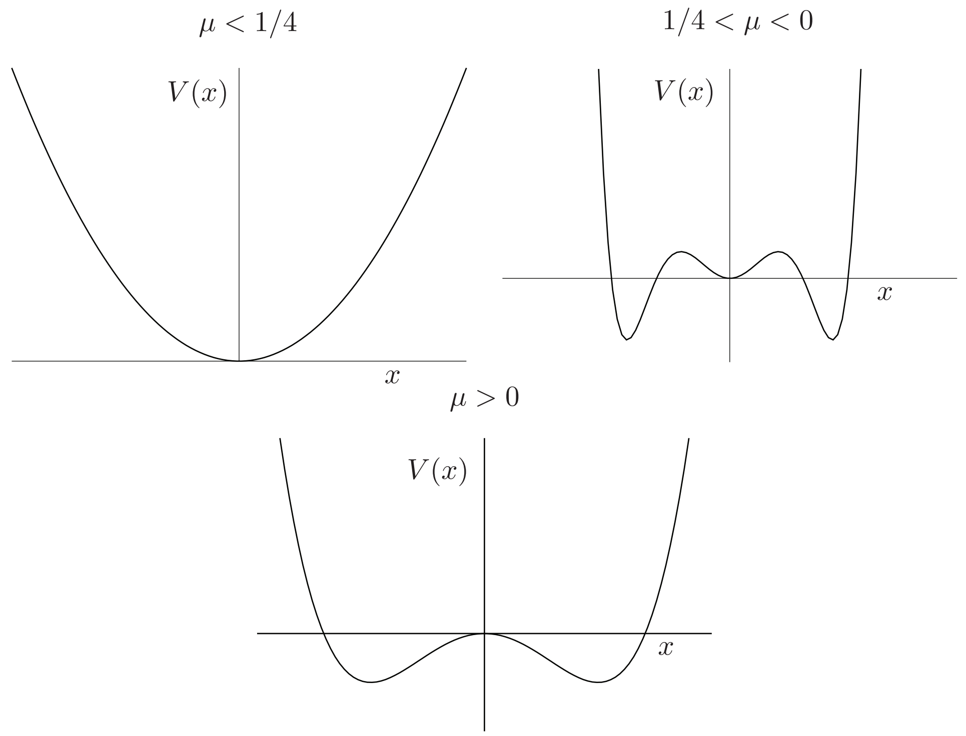



1. In range $\mu_c < \mu < 0$ there co-exist 3 stable fixed points (and 2 unstable). There is *multi-stability*. (Local not global stability). Initial conditions determine the final state.

2. Bifurcation at $\mu_c$ is a saddle-node bifurcation.

3. System exhibits hysteresis and jump phenomenon.

4. If $x^5$ term was absent then blow up could occur.




**Exercise.** Find the fixed points and determine their stability for the following examples. In each case sketch a bifurcation diagram and find the bifurcation values of the parameter.

(a) $\dot{x} = \mu - |x|$

(b) $\dot{x} = \mu - x^2 + 4x^4$

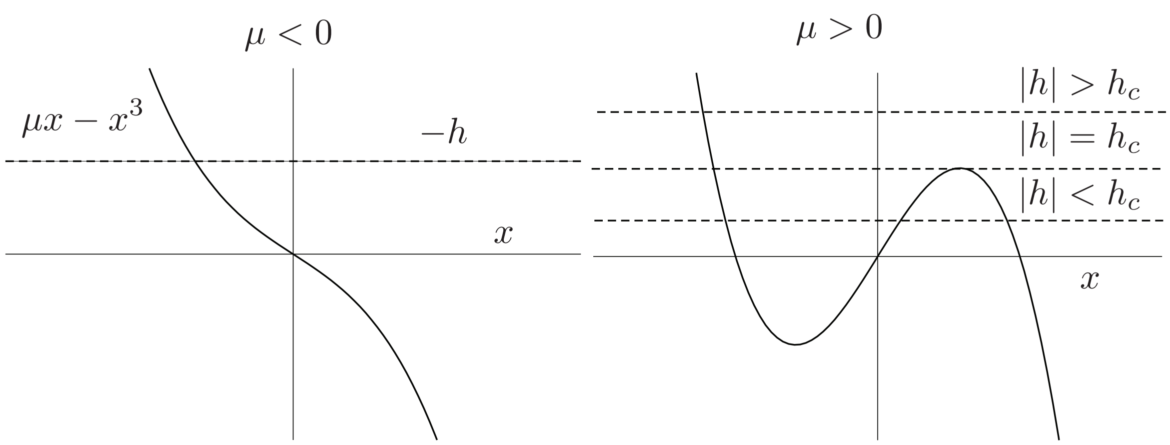
**Cusp singularity**

The pitchfork bifurcation is common in problems with reflection symmetry. Imperfections break this symmetry.

Q: Which way will a rod buckle?

$$\dot{x} = h + \mu x - x^3$$

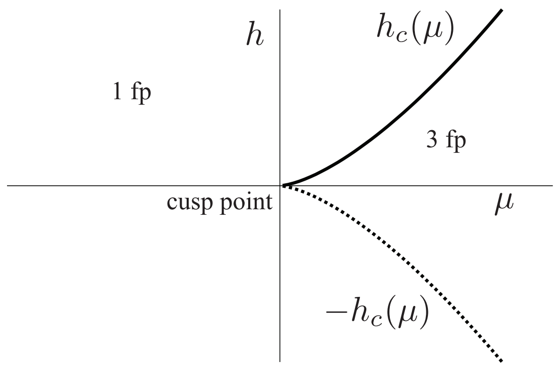
Co-dimension 2 rather than co-dimension 1.

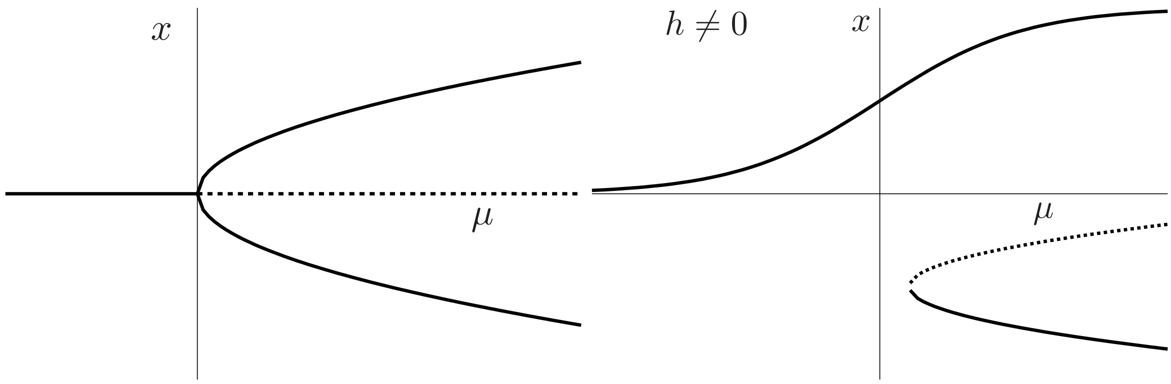


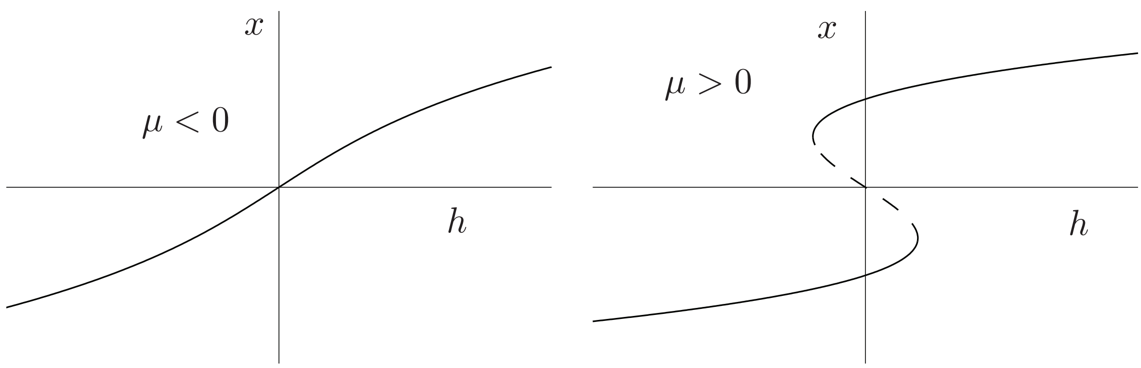
Critical case: horizontal line is tangent to min or max of $f(x) = \mu x - x^3$. Local max/min at $x = \pm\sqrt{\mu/3}$.

$$h_c(\mu) = \frac{2\mu}{3}\sqrt{\frac{\mu}{3}}$$

At $h = \pm h_c(\mu)$ there is a saddle-node bifurcation. There are two bifurcation curves $\pm h_c(\mu)$.




Jump phenomenon and catastrophe theory.

**Example 9.** Budworm population dynamics:

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}, \qquad A, B, R > 0$$

The budworm population $N(t)$ grows logistically (first term) in the absence of predators. The second term describes mortality due to predation (mainly by birds).

Non-dimensionalise: $x = N/A$.

$$\frac{A}{B}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{R}{B}Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2} \equiv f(x)$$

Introduce

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

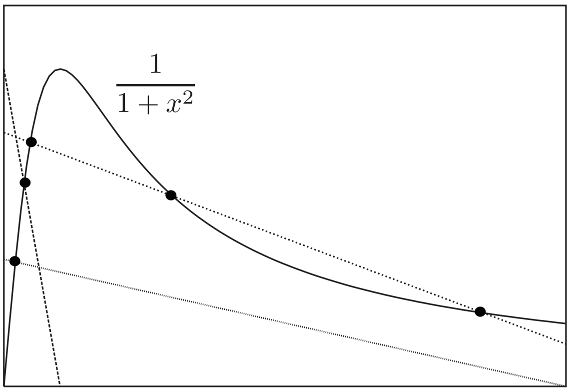
Fixed point at

$$\overline{x} = 0, \qquad r\left(1 - \frac{\overline{x}}{k}\right) - \frac{\overline{x}}{1+\overline{x}^2} = 0$$

Linearisation:

$$f'(x) = r - \frac{2rx}{k} - \frac{2x}{(1+x^2)^2}$$

so $f'(0) = r > 0$, so $\overline{x} = 0$ is unstable. Other roots may be found graphically by finding the intercepts of $x/(1+x^2)$ and $r(1-x/k)$:




Hence there can be either 1, 2 or 3 interceptions depending upon the choice of $(k, r)$. For example when there are three fixed points $c > b > a > 0$, then since $x = 0$ is unstable a is stable, b unstable and c stable. We compute the details of the bifurcation in the following manner:
Saddle-node occurs when $r(1-x/k)$ intersects $x/(1+x^2)$ tangentially. Thus we require $\overline{x}$ (given by $f(\overline{x}) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[r\left(1 - \frac{x}{k}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{1+x^2}\right]$$

or that

$$-\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}, \qquad x = \overline{x} \tag{1}$$

Substitution of $r/k$ into the fixed point equation gives

$$r = \frac{2\overline{x}^3}{(1+\overline{x}^2)^2}$$

Substitution into (1) gives

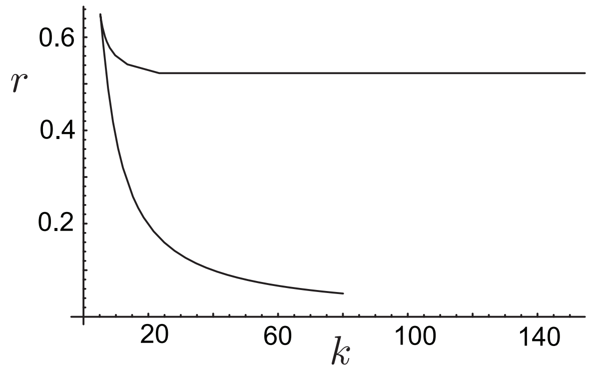
$$k = \frac{2\overline{x}^3}{\overline{x}^2 - 1}$$

Since $k > 0$, we require $x > 1$. The bifurcation curve is defined by $(k(\overline{x}), r(\overline{x}))$ Challenge: plot the bifurcation curve ($r = r(k)$). In MATLAB you could try

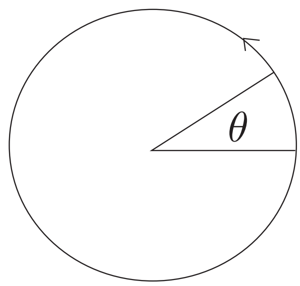
```
ezplot3('2*x.^3./(x.^2-1)','2*x.^3./(1+x.^2)^2','0',[1,15]);view(0,90);
```

In MATHEMATICA you could try

```
ParametricPlot[{2 x x x /(x x -1), 2 x x x/(1+x x)^2},{x,1,40}]
```




## Flows on the circle




Basic model of an oscillator:

$$\dot{\theta} = f(\theta), \qquad \theta \in [0, 2\pi)$$

where $f(\theta) = f(\theta + 2\pi)$.

**Uniform oscillator**

$$\dot{\theta} = \omega, \qquad \theta = \theta_0 + \omega t$$

Period $T = 2\pi/\omega$.

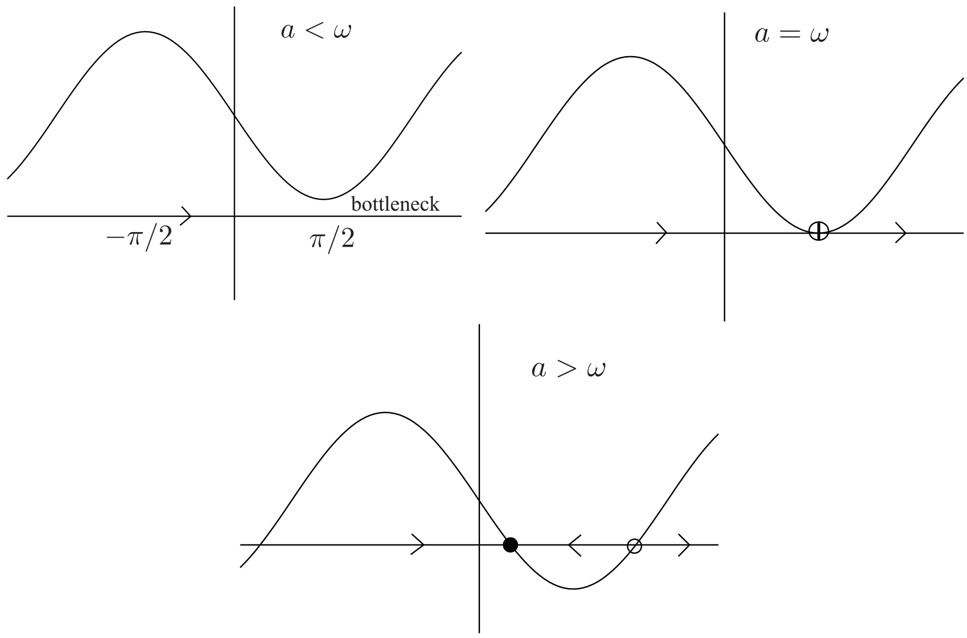
**Non-uniform oscillator**

$$\dot{\theta} = \omega - a \sin\theta$$

Consider here $\omega > 0$, $a \geq 0$ (similar results for negative $\omega$ and $a$).

$a < \omega$: Nonuniform flow which is fastest at $\theta = -\pi/2$ and slowest at $\pi/2$. When $a$ is only slightly less than $\omega$ the system takes a long time to pass through the bottleneck at $\theta = \pi/2$ after which it quickly traverses the rest of the circle.

$a > \omega$: There exists a stable-unstable pair of fixed points at $\sin^{-1}[\omega/a]$ born via a saddle-node bifurcation. Oscillations do not exist.




Period:

$$T = \int \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta} \mathrm{d}\theta = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\omega - a \sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

[Hint: use the substitution $u = \tan\theta/2$].

Close to $a = \omega$

$$T = \frac{2\pi}{\sqrt{\omega + a}} \frac{1}{\sqrt{\omega - a}} \approx \frac{\sqrt{2}\pi}{\sqrt{\omega}} \frac{1}{\sqrt{\omega - a}}$$

so that we have a square root scaling law.