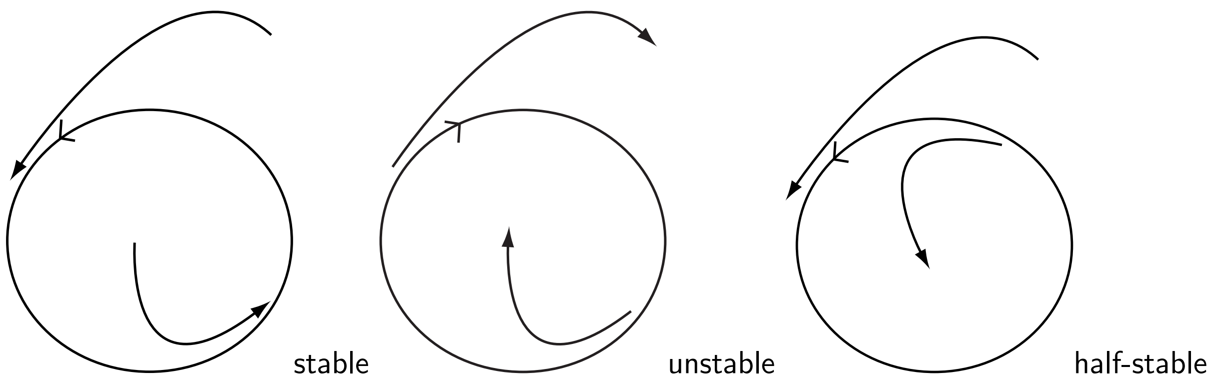
Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

2 Nonlinear oscillations

2.1 Limit cycles in $\mathbb{R}^2$

A limit cycle is an isolated closed trajectory.



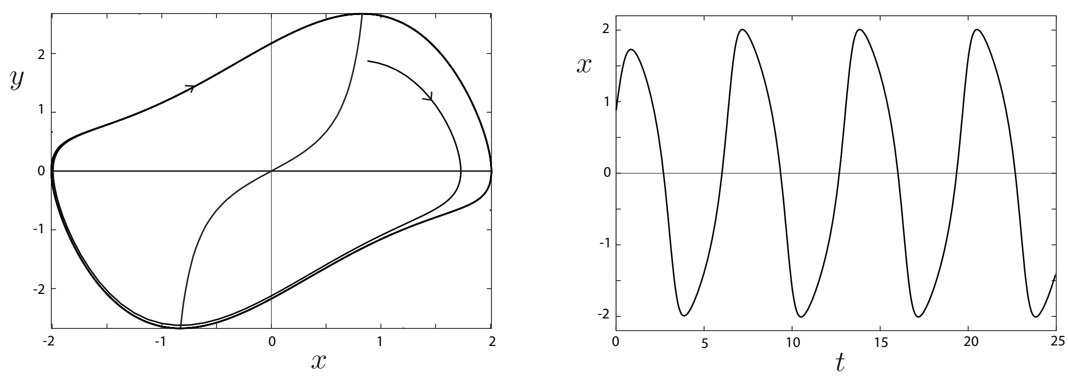
Limit cycles are often found in models that exhibit self-sustained oscillations. There are many example throughout the applied sciences: beating of a heart, chemical reactions, daily (circadian) rhythms in human body temperature and hormone secretion, dangerous self-excited oscillations in bridges (Takoma Narrows) and airplane wings.

Limit cycles are inherently a nonlinear phenomenon — a linear system can have closed orbits but they are not isolated (instead they foliate the phase-plane).

Example 1. Van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios. This equation looks like a simple harmonic oscillator, but with a nonlinear damping term $\mu(x^2 - 1)\dot{x}$. This term acts like ordinary positive damping for $|x| > 1$, but like negative damping for $|x| < 1 \Rightarrow$ it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.

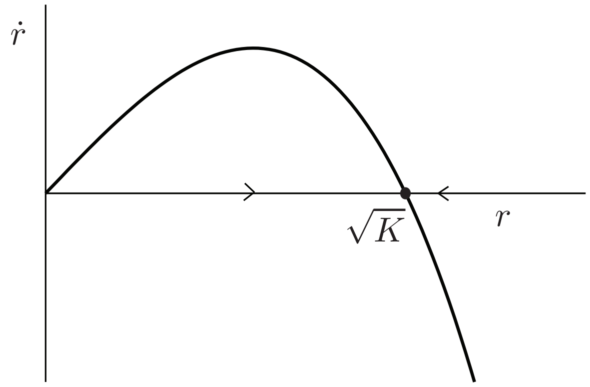


The van der Pol equation has a unique, stable limit cycle for each $\mu > 0$.

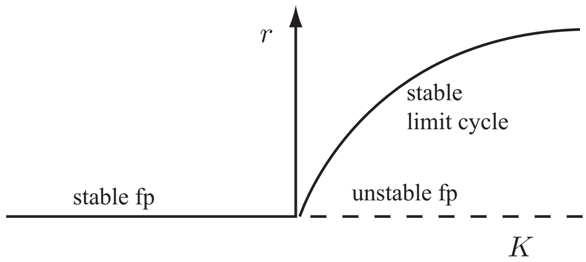
Example 2. Consider the following 2D system in polar coordinates

$$\boxed{\dot{r} = r(K - r^2), \qquad \dot{\theta} = 1}$$

For $K > 0$ there exists an unstable fixed point at the origin and a stable limit cycle at $r = \sqrt{K}$.



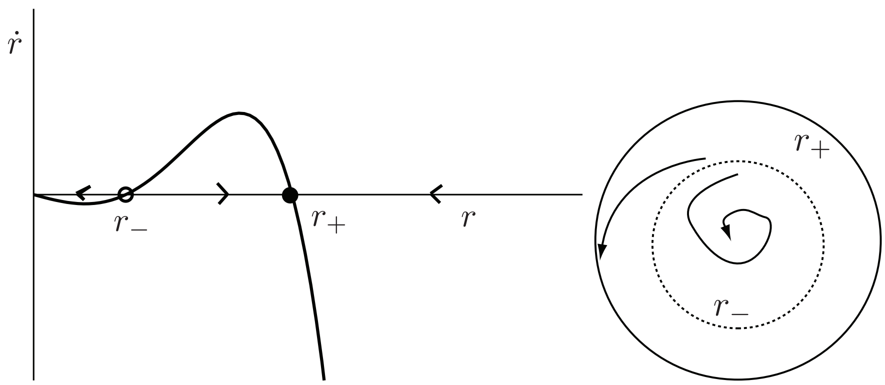
For $K < 0$ there is no limit cycle but only a stable fixed point at the origin.
We can represent qualitative change in the dynamics as the parameter K varies by a **bifurcation diagram**.



This is an example of a **super-critical Hopf bifurcation**. Since an arbitrarily small perturbation of the origin will produce a self-sustained oscillation the system is said to exhibit a **soft excitation**.

Example 3. Now consider the following example

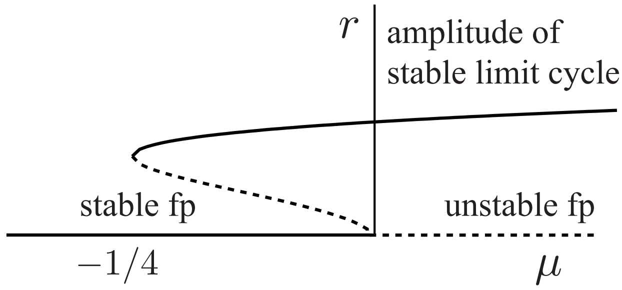
$$\boxed{\dot{r} = \mu r + r^3 - r^5, \qquad \dot{\theta} = 1}$$



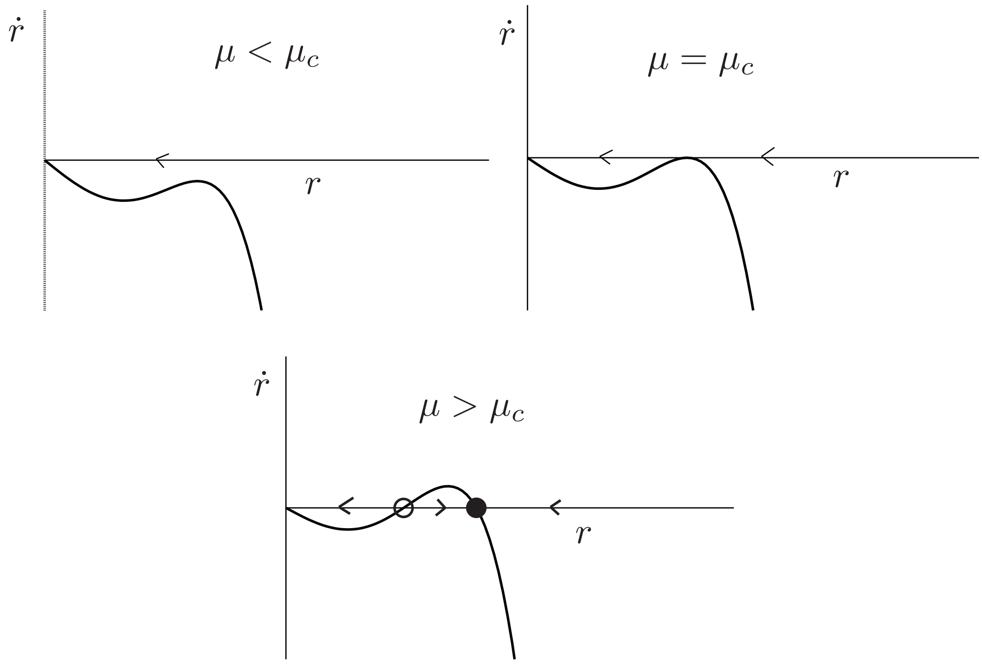
For $\mu < 0$ there exists a stable fixed point at the origin, an unstable limit cycle at $r = r_-$ and a stable limit cycle at $r = r_+$ where

$$r_{\pm}^2 = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\mu}\right]$$

For $\mu > 0$ there exists an unstable fixed point at the origin and a stable limit cycle at $r = r_+$.



At $\mu = 0$ there is a **sub-critical Hopf bifurcation**. The system also exhibits hysteresis; once large amplitude oscillations have begun they cannot be turned off by bringing μ back to zero. In fact they persist until $\mu = -1/4$ where the stable and unstable cycles collide and annihilate in a **saddle-node bifurcation** of limit cycles.



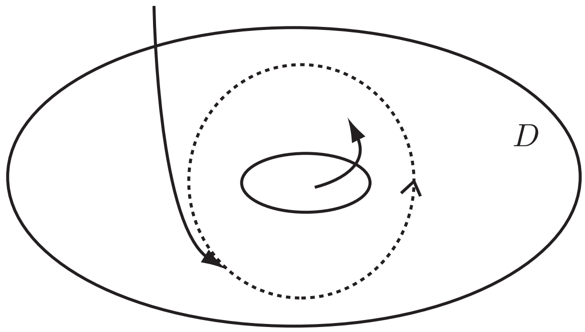
When the system passes through the sub-critical Hopf bifurcation it jumps from the fixed point to a large amplitude oscillation — termed a **hard excitation**. This is potentially dangerous in engineering applications.

Poincaré-Bendixson Theorem

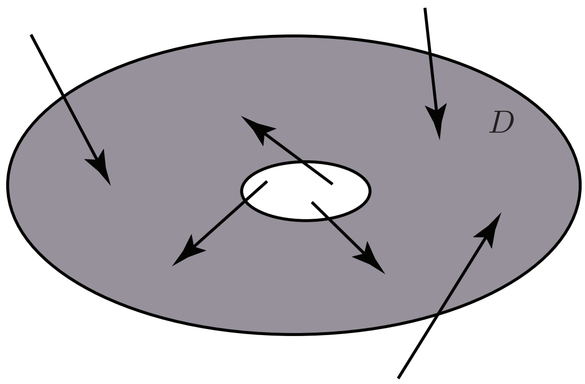
It is generally difficult to establish the existence of a limit cycle. In 2-D one has the following useful theorem:

Theorem: Suppose that there exists a bounded region D of phase-space such that any trajectory entering D cannot leave D. If there are no fixed points in D then there exists at least one periodic orbit in D.

Typically, D will be an annular region with an unstable focus or node in the hole in the middle (so trajectories enter the inner boundary) and all trajectories cross the outer boundary inwards.



The standard trick to apply the Poincaré-Bendixson theorem is to construct a *trapping region* D, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of D.



The Poincaré-Bendixson theorem tells us that the dynamics of planar systems is severely limited — if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. There is no CHAOS for planar systems!

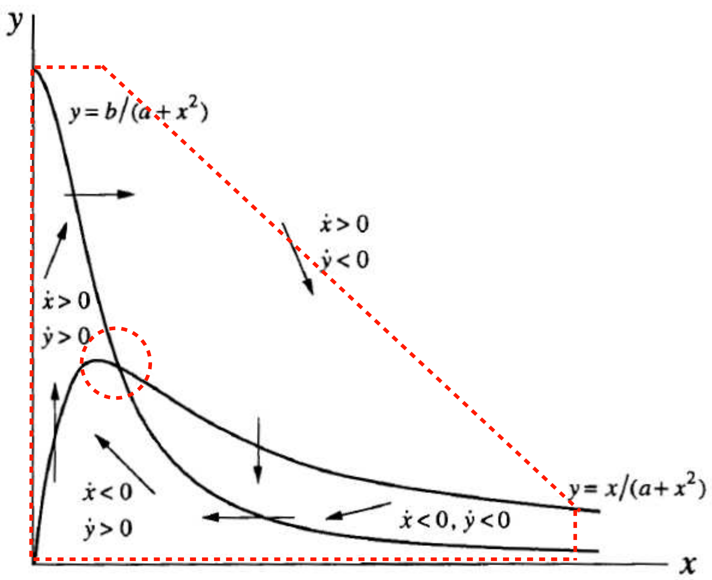
In higher-dimensional systems (in $\mathbb{R}^n,\ n \geq 3$) the Poincaré-Bendixson theorem no longer applies and trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*.

Example 4. In a fundamental biochemical process called glycolysis, living cells obtain energy by breaking down sugar. In yeast cells, for example, glycolysis can proceed in an oscillatory fashion, with concentrations of intermediate products varying periodically. A model of this process is given by

$$\dot{x} = -x + ay + x^2 y \equiv f(x, y)$$
$$\dot{y} = b - ay - x^2 y \equiv g(x, y)$$

where x and y are concentrations of ADP (adenosine phosphate) and F(6)P (Fructose-6 phosphate) and $a, b > 0$ are kinetic parameters. Construct a trapping region for this system.

Solution: First find the nullclines ($f(x, y) = 0 = g(x, y)$)



and then show that all trajectories are *inwards* in some region. To construct the bounding region consider large x and y. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\mathrm{d}y/\mathrm{d}x \approx -1$ along trajectories. Hence, the vector field at large x is parallel to the diagonal, which suggests comparing the sizes of $\dot{x}$ and $-\dot{y}$. So, consider

$$\dot{x} - (-\dot{y}) = -x + ay + x^2 y + (b - ay - x^2 y) = b - x$$

Hence $-\dot{y} > \dot{x}$ if $x > b$. This implies that the vector field points inward on the diagonal line (of the above figure) because $\mathrm{d}y/\mathrm{d}x$ is more negative than -1 and therefore the vectors are steeper than the diagonal — we have a trapping region! We must now find under those conditions which make the fixed point unstable (so as to repel orbits). Linearisation:

$$A = \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{bmatrix}$$

Fixed point

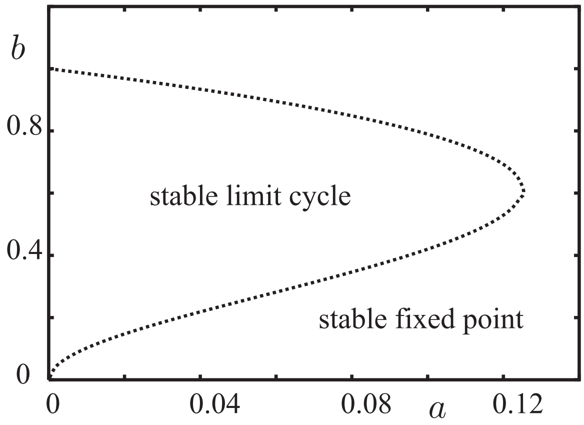
$$\overline{x} = b, \qquad \overline{y} = \frac{b}{a + b^2}$$

Determinant $\det A = a + b^2 > 0$ and

$$\mathrm{Tr}\, A = -\frac{b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2}$$

Hence the fixed point is unstable for $\mathrm{Tr}\, A > 0$ and stable for $\mathrm{Tr}\, A < 0$. The border of stability $\mathrm{Tr}\, A = 0$ occurs when

$$b^2 = \frac{1}{2}\left[1 - 2a \pm \sqrt{1 - 8a}\right]$$



Numerical integration shows that there is one stable limit cycle in the parameter regime which guarantees an unstable fixed point.

2.2 Relaxation oscillators

Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for the special case that $\mu \gg 1$ (strongly nonlinear limit). Using

$$\ddot{x} + \mu(x^2 - 1)\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + \mu(x^3/3 - x)\right]$$

and introducing

$$F(x) = \frac{x^3}{3} - x, \qquad w = \dot{x} + \mu F(x)$$

we may write

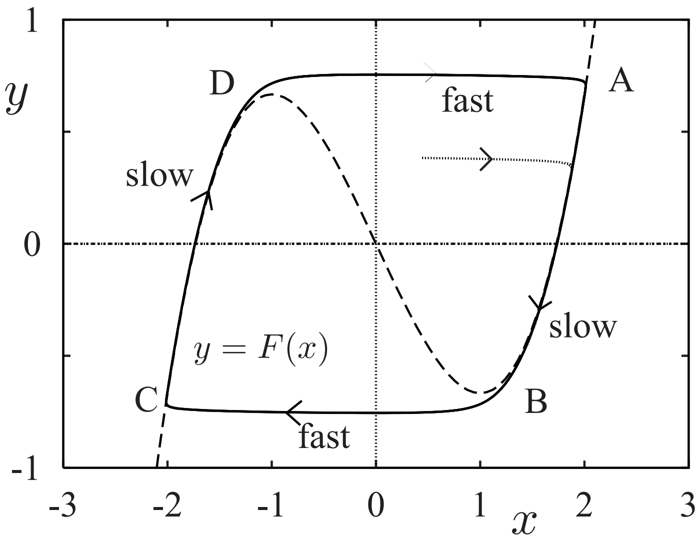
$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x$$

Hence, the van der Pol system has a planar representation:

$$\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x \end{aligned}$$

With the re-scaling $y = w/\mu$ we have

$$\begin{aligned} \dot{x} &= \mu[y - F(x)] \\ \dot{y} &= -\frac{1}{\mu}x \end{aligned}$$



Suppose that the initial condition is not too close to the cubic nullcline, i.e. $y - F(x) \sim O(1)$. Then $|\dot{x}| \sim O(\mu) \gg 1$ and $|\dot{y}| \sim O(1/\mu) \ll 1$; hence the velocity is large in the horizontal direction and small in the vertical direction, so trajectories move horizontally. If the initial condition is above the cubic nullcline then $y - F(x) > 0$ so $\dot{x} > 0$; the trajectory moves sideways towards the right-hand branch of the nullcline. Once the trajectory gets so close that $y - F(x) \sim O(1/\mu^2)$, then $\dot{x}$ and $\dot{y}$ become comparable (both being $O(1/\mu)$). The trajectory crosses the nullcline vertically (see the figure) and then moves slowly down the branch with a velocity $O(1/\mu)$, until it reaches the knee and can jump sideways.

The system has two widely separated time scales. The jumps take a time $O(1/\mu)$ and the crawls a time $O(\mu)$. The period of oscillation can be approximated by the time spent on the slow branches:

$$T \approx \int_{t_A}^{t_B} \mathrm{d}t + \int_{t_C}^{t_D} \mathrm{d}t = 2\int_{t_A}^{t_B} \mathrm{d}t \quad \text{by symmetry}$$

On the slow branch $y = F(x)$ so

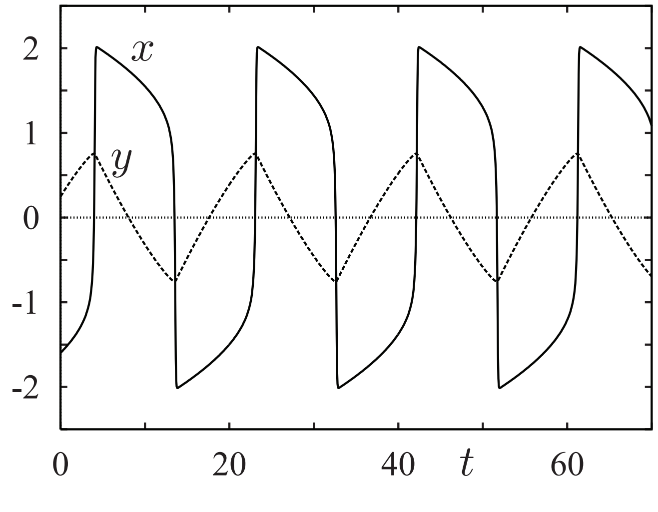
$$\dot{y} \approx \frac{\mathrm{d}y}{\mathrm{d}x}\dot{x} = F'(x)\dot{x} = (x^2 - 1)\dot{x}$$

Using $\dot{y} = -x/\mu$ we have that $\dot{x} = -x/[\mu(x^2 - 1)]$, so

$$\mathrm{d}t \approx -\frac{\mu(x^2 - 1)}{x}\mathrm{d}x$$

Now $x_A = 2$ and $x_B = 1$ (check this for yourselves) so

$$T \approx 2\int_2^1 -\frac{\mu}{x}(x^2 - 1)\mathrm{d}x = \mu[3 - 2\ln 2]$$



2
1
0
-1
-2
x
y
0
20
40
60
t