Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

2.3 Perturbation methods

In this section we learn how to deal with weakly nonlinear oscillators of the form

$$\boxed{\ddot{x} + x + \epsilon h(x, \dot{x}) = 0, \qquad \epsilon \ll 1}$$

Two fundamental examples are the Van der Pol oscillator

$$\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$$

and the Duffing equation

$$\ddot{x} + x + \epsilon x^3 = 0$$

Regular perturbation theory and its failures

As a first approach we seek solutions to the equations of a weakly nonlinear oscillator in the form of a power series in ϵ. If $x(t, \epsilon)$ is a solution then we write

$$\boxed{x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots}$$

Consider as a simple example the weakly damped linear oscillator (exactly soluble)

$$\ddot{x} + 2\epsilon\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1$$

The exact solution is

$$x(t, \epsilon) = (1 - \epsilon^2)^{-1/2} e^{-\epsilon t} \sin[(1 - \epsilon^2)^{1/2} t]$$

Using perturbation theory:

$$\left\{ \frac{d^2}{dt^2} + 2\epsilon \frac{d}{dt} + 1 \right\} (x_0 + \epsilon x_1 + \dots) = 0$$

Equating powers of ϵ gives

$$[\ddot{x}_0 + x_0] + \epsilon[\ddot{x}_1 + 2\dot{x}_0 + x_1] + O(\epsilon^2) = 0$$

Since this holds for all sufficiently small ϵ the coefficients of each power of ϵ must vanish separately

$$\begin{array}{lll} O(1) & \ddot{x}_0 + x_0 = 0 & x_0(0) = 0, \quad \dot{x}_0(0) = 1 \\ O(\epsilon) & \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0 & x_1(0) = 0, \quad \dot{x}_1(0) = 0 \end{array}$$

Solve the initial value problems in sequence. First

$$x_0(t) = \sin t$$

Substituting into the equation for x_1 gives

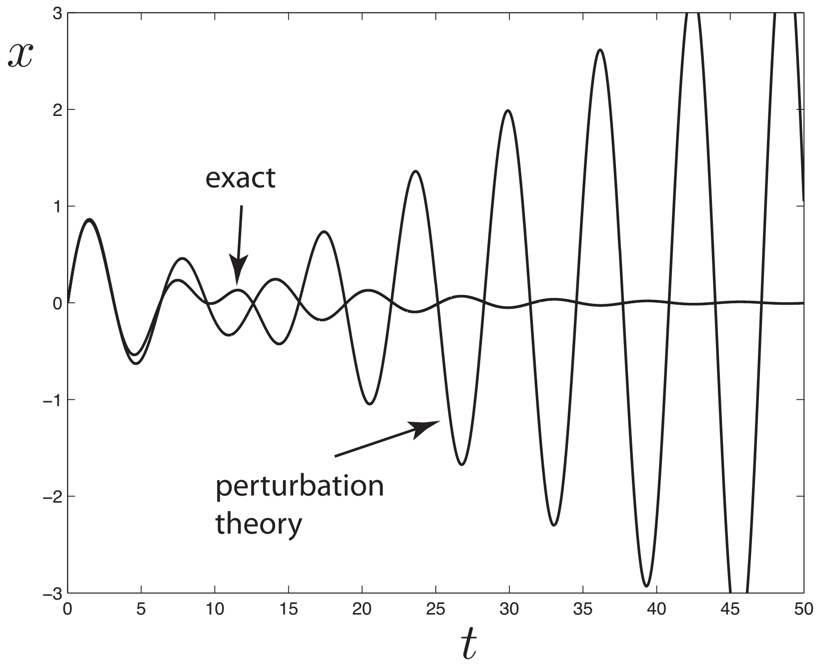
$$\ddot{x}_1 + x_1 = -2\cos t$$

which has the solution $x_1(t) = -t\sin t$. This is called a *secular* term since it grows without bound as $t \to \infty$. Thus according to perturbation theory

$$x(t, \epsilon) = \sin t - \epsilon t \sin t + O(\epsilon^2)$$

For fixed t this provides a convergent series expansion of the exact solution if ϵ is small enough ($\epsilon t \ll 1$). However, we are usually interested in the behaviour for fixed ϵ not fixed t. In that case we can only expect the perturbation approximation to work for $t \ll O(1/\epsilon)$. There are two major problems:

- The true solution exhibits two time-scales: a fast time $t \sim O(1)$ for the sinusoidal oscillation and a slow time $t \sim 1/\epsilon$ over which the amplitude decays. The perturbation approximation completely misrepresents the slow time-scale behaviour ($e^{-\epsilon t} \approx 1 - \epsilon t + O(\epsilon^2 t^2)$).
- The frequency of oscillations of the exact solution is $\omega = (1-\epsilon^2)^{1/2} \approx 1 - \epsilon^2/2$, which is shifted slightly from the frequency $\omega = 1$ of the perturbation approximation. After a very long time $t \sim O(1/\epsilon^2)$ this frequency error will have a cumulative effect (super slow time-scale).



We would like an approximation that captures the behaviour for all t, or at least for large t.

Method of multiple scales (two-timing)

The previous example suggests that we should consider the series expansion

$$x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$$

where $\tau = t$ denotes the fast $O(1)$ time and $T = \epsilon t$ denotes the slow time. We treat τ and T as independent variables. Then the chain rule implies that

$$\dot{x} = \partial_\tau x + \epsilon \partial_T x$$

Substituting the series solution and collecting powers of ϵ, we find

$$\dot{x} = \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2)$$
$$\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\epsilon^2)$$

Example 1. Weakly damped linear oscillator:

$$\ddot{x} + 2\epsilon\dot{x} + x = 0$$

So assuming the two-time expansion we have

$$\partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} + 2\partial_{T\tau} x_0) + 2\epsilon\partial_\tau x_0 + x_0 + \epsilon x_1 + O(\epsilon^2) = 0$$

Equating powers of ϵ yields

$$\begin{aligned} O(1) \quad & \partial_{\tau\tau} x_0 + x_0 = 0 \\ O(\epsilon) \quad & \partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + 2\partial_\tau x_0 + x_1 = 0 \end{aligned}$$

General solution for $x_0(\tau, T)$ is

$$x_0(\tau, T) = A(T)\sin\tau + B(T)\cos\tau$$

Substituting into expression for x_1 gives

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 &= -2(\partial_{T\tau} x_0 + \partial_\tau x_0) \\ &= -2(A' + A)\cos\tau + 2(B' + B)\sin\tau \end{aligned}$$

where $' \equiv d/dT$. Since we want to have an approximation free of secular terms we set the coefficients of the resonant terms to zero

$$A' + A = 0, \qquad B' + B = 0$$

so that

$$A(T) = A(0)e^{-T}, \qquad B(T) = B(0)e^{-T}$$

Initial conditions determine $A(0)$ and $B(0)$ ($x(0) = 0$ and $\dot{x}(0) = 1$):

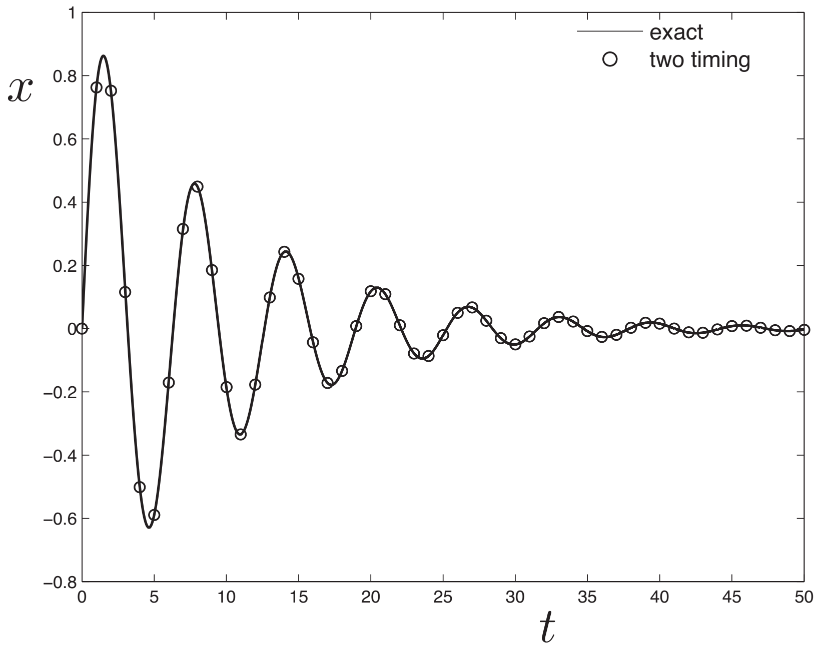
$$x(0) = x_0(0,0) + \epsilon x_1(0,0) + O(\epsilon^2) = 0$$
$$\dot{x}(0) = \partial_\tau x_0(0,0) + \epsilon(\partial_T x_0(0,0) + \partial_\tau x_1(0,0)) + O(\epsilon^2) = 1$$

i.e.

$$\begin{aligned} x_0(0,0) &= 0, \qquad & x_1(0,0) &= 0 \\ \partial_\tau x_0(0,0) &= 1, \qquad & \partial_\tau x_1(0,0) + \partial_T x_0(0,0) &= 0 \end{aligned}$$

Therefore $A(0) = 1$ and $B(0) = 0$ so

$$x(t, \epsilon) = e^{-T}\sin\tau + O(\epsilon), \qquad x_0(t) = e^{-\epsilon t}\sin t + O(\epsilon)$$



Example 2. Use the method of multiple scales to show that the Van der Pol oscillator has a stable limit cycle that is nearly circular with a radius $2 + O(\epsilon)$ and a frequency $\omega = 1 + O(\epsilon^2)$.

$$\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$$

Solution:

$$x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$$

Substitution into the Van der Pol equation and collecting powers of ϵ gives

$$\begin{aligned} &O(1) \qquad \partial_{\tau\tau} x_0 + x_0 = 0 \\ &O(\epsilon) \qquad \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - (x_0^2 - 1)\partial_\tau x_0 \end{aligned}$$

Writing the solution for x_0 in the form

$$x_0(\tau, T) = r(T)\cos[\tau + \phi(T)]$$

where r and ϕ are slowly varying amplitude and phase. Substitution into the expression for x_1 we find

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 = {} & 2(r' \sin(\tau + \phi) + r\phi' \cos(\tau + \phi)) \\ & + r\sin(\tau + \phi)[r^2 \cos^2(\tau + \phi) - 1] \end{aligned}$$

Now

$$\sin(\tau + \phi)\cos^2(\tau + \phi) = \frac{1}{4}[\sin(\tau + \phi) + \sin 3(\tau + \phi)]$$

so

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 = {} & [2r' - r + r^3/4]\sin(\tau + \phi) \\ & + 2r\phi' \cos(\tau + \phi) + r^3/4 \sin 3(\tau + \phi) \end{aligned}$$

To avoid secular terms we require

$$2r' - r + r^3/4 = 0, \qquad 2r\phi' = 0$$

i.e. $r' = r(4 - r^2)/8$ with solution $r(T) = 2/(1 + 3e^{-T})^{1/2}$ so that $r^* = 0$ is an unstable fixed point and $r^* = 2$ is a stable fixed point. Hence $r(T) \to 2$ as $T \to \infty$. Since $\phi' = 0$ when $\phi(T) = \phi_0$ we have that

$$x(t) \to 2\cos(t + \phi_0) + O(\epsilon) \qquad \text{as } t \to \infty$$

The angular frequency (to $O(\epsilon^2)$) is

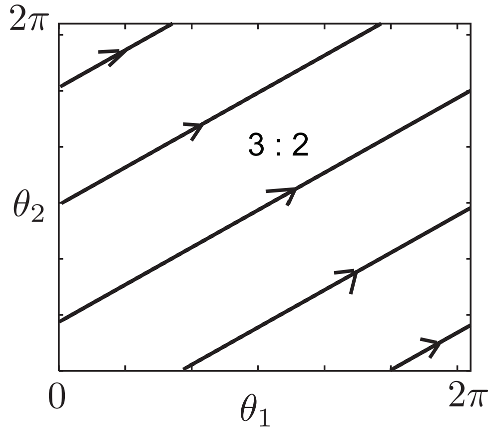
$$\omega = \frac{d\theta}{dt} = \left[\frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T}\right][\tau + \phi(T)] = 1 + \epsilon \frac{d\phi}{dT} = 1$$

2.4 Coupled oscillators

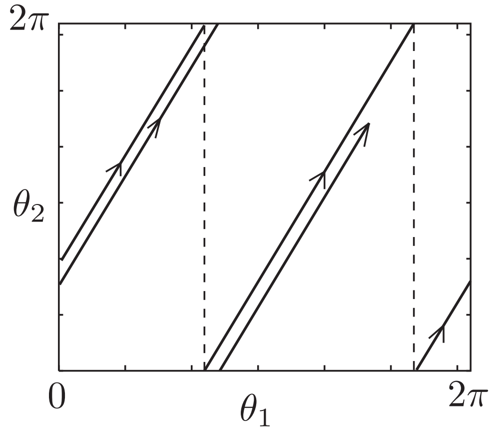
Consider the model

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

In the uncoupled state ($K_1 = K_2 = 0$) we have $\theta_1(t) = \theta_1(0) + \omega_1 t$ and $\theta_2(t) = \theta_2(0) + \omega_2 t$ such that $d\theta_2/d\theta_1 = \omega_2/\omega_1$. If the slope is rational, $\omega_2/\omega_1 = p/q$, $p, q \in \mathbb{Z}$, then all trajectories lie on closed orbits of the torus (with coords (θ_1, θ_2)).



For irrational slopes the flow is said to be **quasiperiodic**. Each trajectory is dense on the torus (i.e. comes arbitrarily close to any given point).



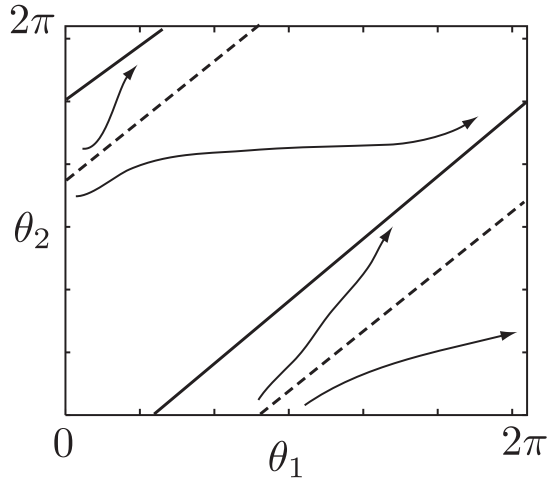
Introducing $\phi = \theta_1 - \theta_2$ the coupled system takes the form

$$\dot{\phi} = \omega_1 - \omega_2 - (K_1 + K_2)\sin\phi$$

There are two fixed points if $|\omega_1 - \omega_2| < K_1 + K_2$, defined by $\sin\phi^* = (\omega_1 - \omega_2)/(K_1 + K_2)$, and a saddle-node (tangent) bifurcation occurs when $|\omega_1 - \omega_2| = K_1 + K_2$. In this case $\dot{\phi} = 0$ so that $\dot{\theta}_1 = \dot{\theta}_2 = \text{constant} = \omega^*$, where

$$\omega^* = \omega_2 + K_2 \sin\phi^* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

We may regard ω^* as a co-operative frequency that is an emergent property of the coupled system. When no-cooperative frequency can be established the two oscillators cannot phase-lock (although they may still frequency lock).

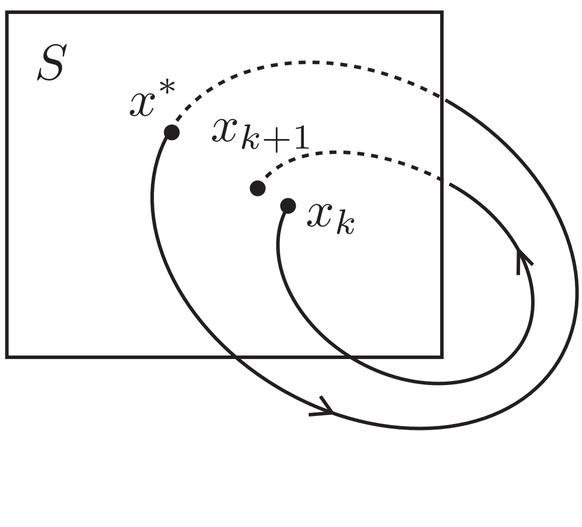


2.5 Poincaré maps

Poincaré maps are useful for studying the flows near a periodic orbit. Consider an n-dimensional system

$$\dot{x} = f(x).$$

Let S is an $n - 1$ dimensional surface of section. S is required to be transverse to the flow, i.e. all trajectories starting on S flow through it (not parallel to it).



The **Poincaré map** is a mapping from S to itself, obtained by following trajectories from one intersection with S to the next. If $x_k \in S$ denotes the kth intersection, then the Poincaré map is defined by

$$x_{k+1} = P(x_k).$$

Suppose that x^* is a fixed point of P, i.e. $P(x^*) = x^*$. Then a trajectory starting at x^* returns to x^* after some time T, and is therefore a closed orbit for the original system $\dot{x} = f(x)$.

Linear stability of limit cycle

Consider a system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,$$

with a closed orbit. To ask whether the orbit is stable or not, we ask whether the corresponding fixed point x^* of the Poincaré map is stable. Consider $x^* + v_0$ in S, where v_0 is a perturbation. Then after the first return to S

$$x^* + v_1 = P(x^* + v_0) = P(x^*) + [DP(x^*)]v_0 + \text{small terms},$$

where $DP(x^*)$ is an $(n-1) \times (n-1)$ matrix called the **linearised Poincaré map** at x^*. Since $x^* = P(x^*)$, we have

$$v_1 = DP(x^*)v_0.$$

> The stability criterion is expressed in terms of the eigenvalues λ_j of $DP(x^*)$: The closed orbit is linearly stable if and only if $|\lambda_j| < 1$ for all $j = 1, ..., n-1$.

(From the expression $v_k = \sum_{j=1}^{n-1} v_j(\lambda_j)^k e_j$, where e_j - eigenvectors and v_j - some scalars).
λ_j are called the **characteristic** or **Floquet multipliers** of the periodic orbit. In general, the characteristic multipliers can only be found by numerical integration.

Example 3.

$$\dot{r} = r(1-r^2), \qquad \dot{\theta} = 1$$

Let S be the positive x-axis. Compute the Poincaré map and show that the system has a unique periodic orbit and determine its stability. Find the characteristic multipliers for the limit cycle.

Let r_0 be the initial condition on S. Since $\dot{\theta} = 1$, the first return to S occurs after a period $T = 2\pi$. Then $r_1 = P(r_0)$ where

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi$$

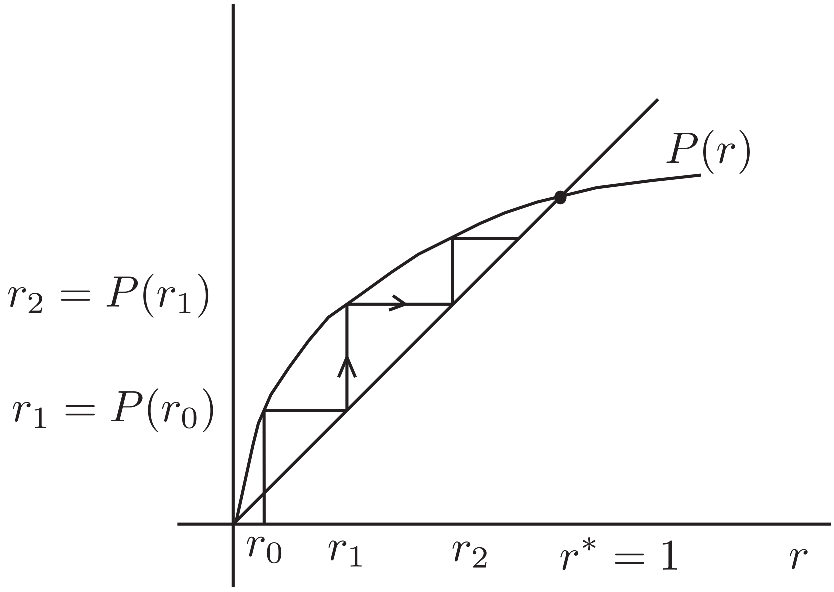
so

$$r_1 = \left[1 + e^{-4\pi}\left(\frac{1}{r_0^2} - 1\right)\right]^{-1/2}$$

Therefore

$$P(r) = \left[1 + e^{-4\pi}\left(\frac{1}{r^2} - 1\right)\right]^{-1/2}$$

We can show graphically that P has a unique stable fixed point at $r^* = 1$.



$$\frac{dP(r)}{dr} = e^{-4\pi} r^{-3} [1 + e^{-4\pi}(r^{-2} - 1)]^{-3/2} \qquad \Rightarrow$$

$$\left.\frac{dP(r)}{dr}\right|_{r^*=1} = e^{-4\pi} \qquad - \text{ Floquet multiplier}$$

$|e^{-4\pi}| < 1 \qquad \Rightarrow \qquad$ the closed orbit is linearly stable.