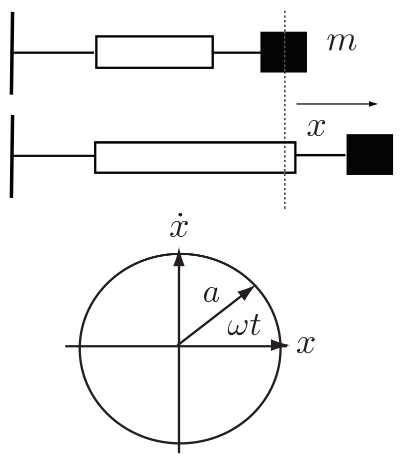
1.2 Second (and higher) order systems

We shall consider equations of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2, \qquad (x \in \mathbb{R}^n)$$

Harmonic oscillator



According to classical theory a simple harmonic oscillator is a particle of mass m moving under the action of a force $F = -kx$ (Hooke's law). Newton's laws of motion take the form

$$m\ddot{x} = -kx \qquad \text{or} \qquad \ddot{x} + \omega^2 x = 0, \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

The general solution to this differential equation is of the form

$$x(t) = A\cos\omega t + B\sin\omega t$$

which represents an oscillatory motion of angular frequency ω. The constants of integration A and B are determined by the initial conditions for x and $\dot{x}$, where

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

so that $x(0) = A$ and $\dot{x}(0) = B\omega$. An easy way to imagine the geometry of simple harmonic motion is to write the equations of motion as a second-order (linear!) system. Introduce $v = \dot{x}$, then

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

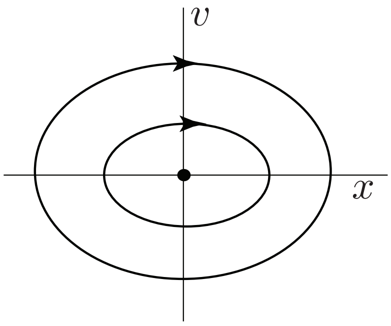
There is a fixed point at $(x, v) = (0, 0)$. Combining the above we have

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 \frac{x}{v}$$

After integrating this separable ODE we have

$$v^2 + \omega^2 x^2 = \text{constant}$$

as before (trajectories in phase space are elliptical).



Reminder - matrix and vector manipulation

The matrix $\boldsymbol{A}$ multiplying the vector $\boldsymbol{x}$ acts as a linear operator that produces a new vector $\boldsymbol{z}$:

$$\boldsymbol{z} = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

- Identity matrix

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Addition

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}, \qquad \boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

- Multiplication

$$cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}, \quad c = \text{constant}$$

- Differentiation

$$\mathrm{d}\boldsymbol{x}/\mathrm{dt} = \begin{pmatrix} \mathrm{dx}_1/\mathrm{dt} \\ \mathrm{dx}_2/\mathrm{dt} \end{pmatrix}$$

- The *trace* and *determinant* of the matrix $\boldsymbol{A}$

$$\begin{aligned} \mathrm{tr}(\boldsymbol{A}) &= a_{11} + a_{22} \\ \det(\boldsymbol{A}) &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

- Singularity: the matrix $\boldsymbol{A}$ is singular if $\det(\boldsymbol{A}) = 0$

Example 1. Consider the system

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}.$$

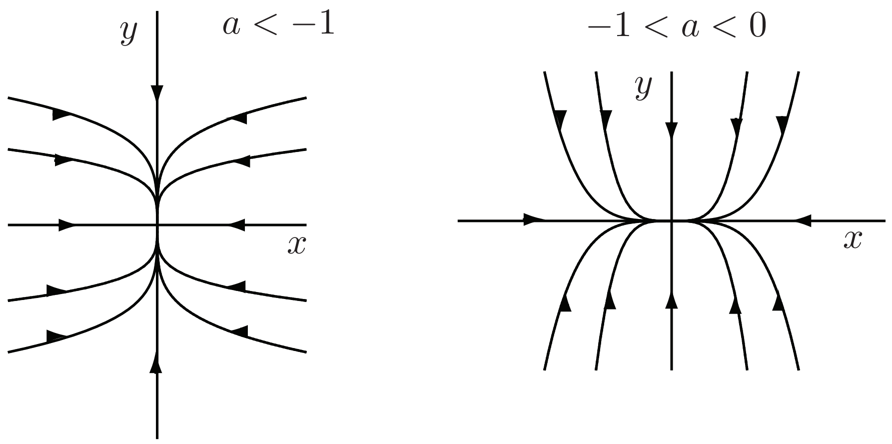
Matrix multiplication yields

$$\dot{x} = ax,$$
$$\dot{y} = -y.$$

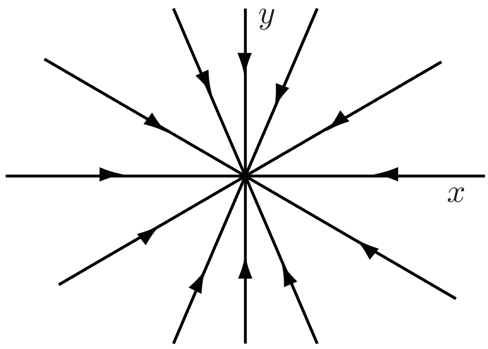
Since these two equations are uncoupled they can be solved separately

$$x(t) = x_0 \mathrm{e}^{at},$$
$$y(t) = y_0 \mathrm{e}^{-t}.$$

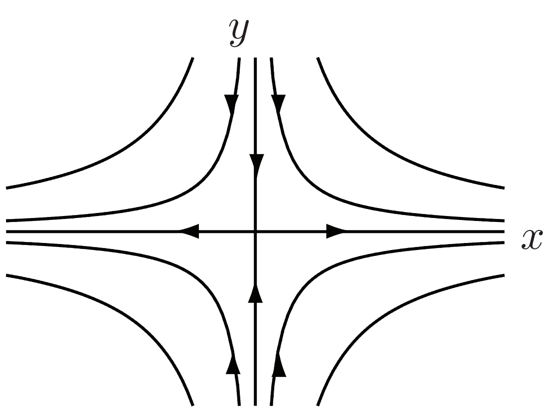
- **Stable nodes**: i) $a < -1$ and ii) $-1 < a < 0$



- **Star**: $a = -1$

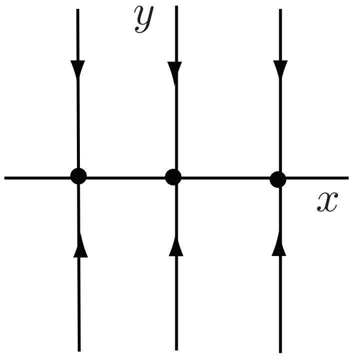


- **Saddle point**: $a > 0$



The y-axis is called the *stable manifold* of the saddle point x^*: the set of initial conditions x_0 such that $x(t) \to x^*$ as $t \to \infty$. The x-axis is called the *unstable manifold* of the saddle point x^*: the set of initial conditions x_0 such that $x(t) \to x^*$ as $t \to -\infty$.

- **Line of fixed points**: $a = 0$



1.3 Linear systems in $\mathbb{R}^2$

$$\dot{x}_1 = ax_1 + bx_2$$
$$\dot{x}_2 = cx_1 + dx_2$$

Introducing the vector $x = (x_1, x_2)^T$ we have

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Try a solution of the form

$$x = \mathrm{e}^{\lambda t} v$$

This leads to the linear homogeneous equation

$$Av = \lambda v.$$

v is an *eigenvector* of A with corresponding *eigenvalue* λ. For the system above to have a non-trivial solution we require that

$$\boxed{\det(A - \lambda I) = 0}$$

which is called the *characteristic equation*. Here I is the 2×2 identity matrix. Substituting the components of A into the characteristic equation gives

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

or

$$\lambda^2 - \mathrm{Tr}\, A\ \lambda + \det A = 0$$

so that

$$\boxed{\lambda_\pm = \frac{1}{2}\left[\mathrm{Tr}\, A \pm \sqrt{(\mathrm{Tr}\, A)^2 - 4 \det A}\right]}$$

The general solution for x(t):

$$x(t) = c_1 \mathrm{e}^{\lambda_1 t} v_1 + c_2 \mathrm{e}^{\lambda_2 t} v_2.$$

Exercise. Solve the initial value problem

$$\dot{x} = x + y, \qquad \dot{y} = 4x - 2y, \qquad (x_0, y_0) = (2, -3)$$

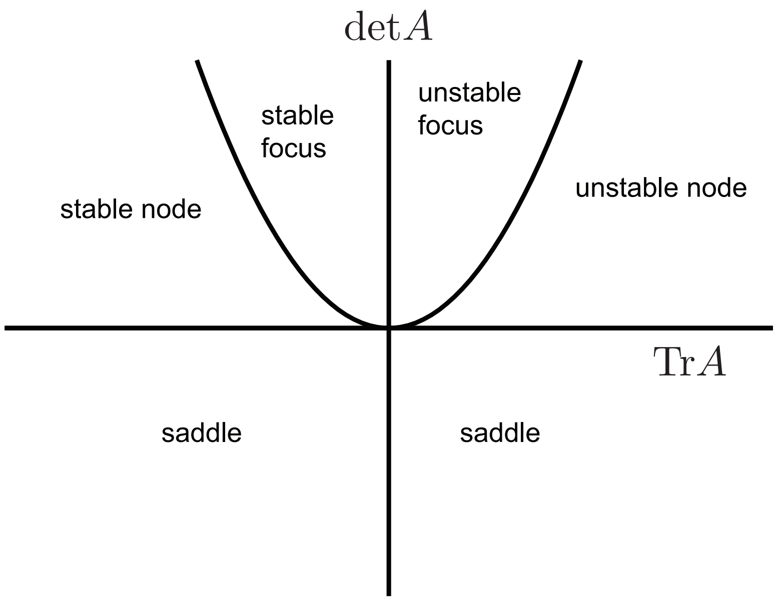
If $\lambda_{1,2}$ are complex ($\lambda_{1,2} = \alpha \pm i\omega$), the fixed point is either a *centre* or a *spiral*. Since $x(t)$ involves linear combinations of $e^{\alpha \pm i\omega}$, $x(t)$ is a combination of terms involving $e^{\alpha t}\cos(\omega t)$ and $e^{\alpha t}\sin(\omega t)$ (by Euler's formula $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$).

- If $\alpha < 0 \Rightarrow$ stable focus (or stable spiral)
- If $\alpha > 0 \Rightarrow$ unstable focus (or unstable spiral)
- If $\alpha = 0 \Rightarrow$ a centre (periodic solution with period $T = 2\pi/\omega$), marginally stable.

Classification of fixed points

We classify the different types of behaviour according to the values of $\text{Tr}\, A$ and $\det A$.

- $\lambda_\pm$ are real if $(\text{Tr}\, A)^2 > 4\det A$.
- Real eigenvalues have the same sign if $\det A > 0$ and are positive if $\text{Tr}\, A > 0$ (negative if $\text{Tr}\, A < 0$) — **stable and unstable nodes**.
- Real eigenvalues have opposite signs if $\det A < 0$ — **saddle node**.
- Eigenvalues are complex if $(\text{Tr}\, A)^2 < 4\det A$ — **focus**.



1.4 Linear systems in $\mathbb{R}^n$

Consider the (autonomous) differential equation

$$\frac{dx}{dt} \equiv \dot{x} = Ax, \qquad x \in \mathbb{R}^n$$

where A is an $n \times n$ constant matrix. Given the initial condition $x(0) = x_0$, the solution is

$$x(t) = \mathrm{e}^{tA} x_0, \quad \mathrm{e}^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \tag{1}$$

Check this: use

$$\frac{d}{dt} \mathrm{e}^{tA} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k = A\mathrm{e}^{tA}$$

Thus

$$\frac{dx(t)}{dt} = \frac{d}{dt} \mathrm{e}^{tA} x_0 = A\mathrm{e}^{tA} x_0 = Ax(t)$$

The solution (1) also allows one to solve inhomogeneous equation

$$\dot{x} = Ax + g(t)$$

Multiplying both sides by e^{-tA} gives

$$\frac{d}{dt} \left[\mathrm{e}^{-tA} x(t) \right] = \mathrm{e}^{-tA} g(t)$$

Integrating wrt. t then gives

$$\mathrm{e}^{-tA} x(t) - x_0 = \int_0^t \mathrm{e}^{-t'A} g(t') dt'$$

or

$$x(t) = \mathrm{e}^{tA} x_0 + \mathrm{e}^{tA} \int_0^t \mathrm{e}^{-t'A} g(t') dt'$$

Normal forms

After classifying the fixed points (node, saddle or focus) can we determine what the flow looks like?

Consider linear change of variables $x = Py$, where P is an $n \times n$ invertible matrix ($\det P \neq 0$). Then if $\dot{x} = Ax$

$$\dot{y} = P^{-1} \dot{x} = P^{-1} A x = P^{-1} A P y$$

Choosing P such that $\Lambda = P^{-1} A P$ is a diagonal matrix we have that

$$\dot{y} = \Lambda y$$

If $x(0) = x_0$ then $y(0) = P^{-1} x_0$.

In the new coordinates solution is

$$y(t) = \mathrm{e}^{t\Lambda} y_0$$

Transforming back to original coordinates

$$x(t) = Py(t) = P\mathrm{e}^{t\Lambda} y_0 = P\mathrm{e}^{t\Lambda} P^{-1} x_0$$

Comparing equations (1) and (2) implies that

$$\mathrm{e}^{tA} = P\mathrm{e}^{t\Lambda} P^{-1} \tag{2}$$

Strategy: choose matrix P such that Λ takes a form which allows us to calculate $\mathrm{e}^{t\Lambda}$ and hence e^{tA}. The matrix Λ is then called a Normal Form whose particular structure depends on the eigenvalues of A.

Real distinct eigenvalues

Suppose that A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors e_i so that

$$Ae_i = \lambda_i e_i$$

Let $P = [e_1, ..., e_n]$ be the matrix with the eigenvectors of A as columns. Since the eigenvectors are real and linearly-independent, $\det P \neq 0$. Thus

$$AP = [Ae_1, ..., Ae_n] = [\lambda_1 e_1, ..., \lambda_n e_n] = [e_1, ..., e_n]\mathrm{diag}(\lambda_1, ..., \lambda_n) = P\mathrm{diag}(\lambda_1, ..., \lambda_n)$$

Hence for real, distinct eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. It follows that

$$\mathrm{e}^{tA} = P\mathrm{diag}(\mathrm{e}^{\lambda_1 t}, ..., \mathrm{e}^{\lambda_n t})P^{-1}$$

Example 2. $A = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$.

Characteristic equation $\det(A - \lambda I_2) = 0 \Rightarrow (\lambda + 2)(\lambda - 2) = 0$.

$$\lambda_1 = -2, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 2, \quad e_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \quad P^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathrm{e}^{tA} = P\begin{pmatrix} \mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{2t} \end{pmatrix}P^{-1} = \begin{pmatrix} \mathrm{e}^{-2t} & \frac{1}{4}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$$

Pair of complex eigenvalues

Consider a 2×2 matrix with a pair of complex eigenvalues $\rho \pm i\omega$. The associated complex eigenvector is q such that

$$Aq = (\rho + i\omega)q, \quad q \in \mathbb{C}^2$$

Let $q = u + iv$ where $u, v \in \mathbb{R}^2$ and equate real and imaginary parts:

$$Au = \rho u - \omega v$$
$$Av = \omega u + \rho v$$

or

$$A[v, u] = [v, u]\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$$

Hence, set

$$P = [v, u] = [\mathrm{Im}(q), \mathrm{Re}(q)], \quad \Lambda = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$$

to see that

$$AP = P\Lambda, \quad \text{or } \Lambda = P^{-1}AP$$

Having obtained the normal form, we need to solve the equation

$$\dot{x} = \rho x - \omega y, \quad \dot{y} = \omega x + \rho y, \quad x, y \in \mathbb{R}$$

Let $z = x + iy$. Then

$$\dot{z} = \dot{x} + i\dot{y} = (\rho + i\omega)z \tag{3}$$

Introduce polar coordinates $z = r\mathrm{e}^{i\theta}$ ($x = r\cos\theta, y = r\sin\theta$). Then an equivalent form for $\dot{z}$ is

$$\dot{z} = \dot{r}\mathrm{e}^{i\theta} + ir\dot{\theta}\mathrm{e}^{i\theta} \tag{4}$$

Comparing equations (3) and (4) we deduce that

$$\dot{r} + ir\dot{\theta} = (\rho + i\omega)r$$

which, on equating real and imaginary parts yields

$$\dot{r} = \rho r, \quad \dot{\theta} = \omega$$

Hence, we obtain the solution

$$r(t) = \mathrm{e}^{\rho t} r_0, \quad \theta(t) = \omega t + \theta_0$$

After writing $x(t) = r(t)\cos(\omega t + \theta_0)$ and $y(t) = r(t)\sin(\omega t + \theta)$ with $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$, it follows that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathrm{e}^{\rho t} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Stability dependent upon $\mathrm{Re}(\rho \pm i\omega) = \rho$.

Example 3. $A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$.

Characteristic equation $\det(A - \lambda I_2) = 0 \Rightarrow (\lambda - 2)\lambda + 2 = 0$.

$$\lambda = 1 + i, \quad q = \begin{pmatrix} 1 \\ -1 + i \end{pmatrix}, \quad \mathrm{Im}q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathrm{Re}q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathrm{e}^{tA} = \mathrm{e}^{t} P \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} P^{-1} = \mathrm{e}^{t} \begin{pmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{pmatrix}$$

Degenerate eigenvalues

Suppose that A has p distinct eigenvalues $\lambda_1, ..., \lambda_p$, $p \le n$. Then

$$\det(A - \lambda I_n) = \prod_{k=1}^{p} (\lambda - \lambda_k)^{n_k}$$

where $n_k \ge 1$ and $\sum_{k=1}^{p} n_k = n$. If all the eigenvectors are distinct then $p = n$ and $n_k = 1$ for all k. If $p < n$ then at least one $n_k > 1$ and the characteristic polynomial has repeated roots. Number n_k called the multiplicity of λ_k.

Consider 2-D case. Recall Cayley-Hamilton theorem: the matrix A satisfies its own characteristic equation. Therefore, $(A - \lambda I_2)^2 x = 0$ for all $x \in \mathbb{R}^2$. There are then two possibilities:

1. $(A - \lambda I_2)x = 0$ for all $x \in \mathbb{R}^2 \Rightarrow \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

2. $(A - \lambda I_2)e_2 \neq 0$ for some vector $e_2 \neq 0$. Define $e_1 = (A - \lambda I_2)e_2$. Then $(A - \lambda I_2)e_1 = 0$ so that

$$Ae_1 = \lambda e_1, \quad Ae_2 = e_1 + \lambda e_2 \Rightarrow A[e_1, e_2] = [e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Hence, we may set

$$P = [e_1, e_2], \quad \Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Solution of normal form equation (solve as an inhomogeneous system)

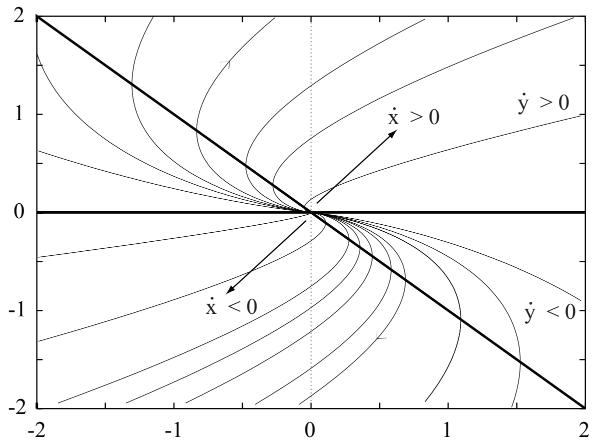
$$\dot{x} = \lambda x + y, \quad \dot{y} = \lambda y$$

is

$$x(t) = \mathrm{e}^{\lambda t}(x_0 + t y_0), \quad y(t) = \mathrm{e}^{\lambda t} y_0$$

Phase portrait. That is determine direction of trajectories at various points in phase-space to build up phase-portrait. Here

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\lambda x + y}$$



$\lambda = 1$

Solving linear systems

- Real eigenvalue $\lambda \quad \Rightarrow \quad C\mathrm{e}^{\lambda t}$
- Real eigenvalue λ of multiplicity $r \quad \Rightarrow \quad C_1\mathrm{e}^{\lambda t} + C_2 t\mathrm{e}^{\lambda t} + \cdots + C_r t^{r-1}\mathrm{e}^{\lambda t}$
- Pair of complex eigenvalues $\lambda = \rho \pm i\omega \quad \Rightarrow \quad \mathrm{e}^{\rho t}(B\cos\omega t + C\sin\omega t)$
- Pair of complex eigenvalues $\lambda = \rho \pm i\omega$, each with multiplicity $r \quad \Rightarrow$ $\mathrm{e}^{\rho t}(B_1\cos\omega t + C_1\sin\omega t + B_2 t\cos\omega t + C_2 t\sin\omega t + \cdots + B_r t^{r-1}\cos\omega t + C_r t^{r-1}\sin\omega t)$

1.5 Nonlinear systems in $\mathbb{R}^2$ (in $\mathbb{R}^n$)

We shall consider equations of the form

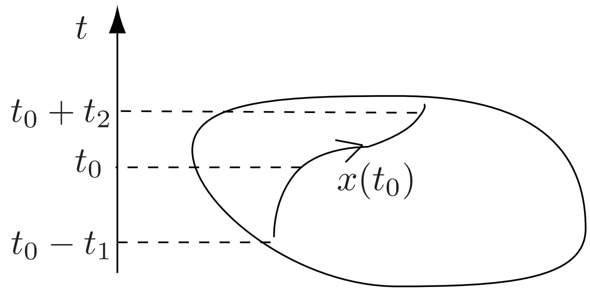
$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2).$$

This system can be wirtten in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x}(x_1, x_2)$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point.

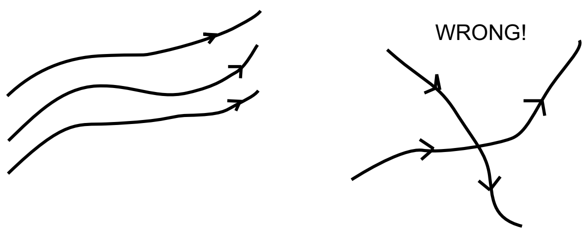
Existence and uniqueness theorem (in $\mathbb{R}^n$): Suppose $\dot{x} = f(x)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable (i.e. $\partial f_i/\partial x_j$, $i,\ j = 1, ..., n$ exist and are continuous for all x). Then there exits $t_1 > 0$ and $t_2 > 0$ such that the solution with $x(t_0) = x_0$ exists and is unique for all $t \in (t_0 - t_1, t_0 + t_2)$.



Phase-space and flows. Refer to local solution through x_0 as a *solution curve* or *trajectory*. Suppose that $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$. We define a flow $\phi(x, t)$ such that $\phi(x, t)$ is the solution of the ODE at time t with initial value x_0 at $t = 0$. The solution $x(t)$ with $x(0) = x_0$ is now written as $\phi(x_0, t)$

$$\frac{\mathrm{d}\phi(x, t)}{\mathrm{d}t} = f(\phi(x, t)), \qquad \phi(x, 0) = x_0$$

By varying initial condition x_0 we generate a family of trajectories called the *flow* generated by Φ.



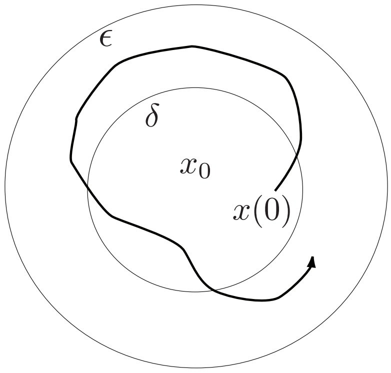
Note that uniqueness imples that trajectories cannot cross.

An **equilibrium** or fixed point satisfies $\Phi(x, t) = x$ for all t. Thus $f(x) = 0$. An important feature of nonlinearities is that there can exist more than one (isolated) fixed point.

Stability

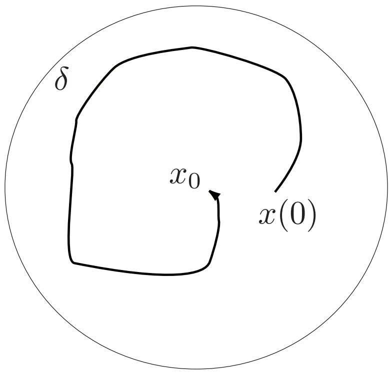
A fixed point x_0 is an attracting fixed point if all trajectories that start near x_0 approach it as $t \to \infty$. If x_0 attracts all trajectories it is called globally attracting.

A fixed point x_0 is **Lyapunov** (neutrally) stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0) - x_0| < \delta$ implies that $|x(t) - x_0| < \epsilon$ for all $t > 0$.



In other words, if a solution starts near an equilibrium x_0 then it stays near x_0 (for example harmonic oscillator).

A fixed point is **asymptotically stable** if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x(0) - x_0| < \delta$ then $|x(t) - x_0| \to 0$ as $t \to \infty$.



Linearisation

Consider the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is a fixed point. Considering a small disturbance from the fixed point

$$u = x - x^*, \qquad v = y - y^*$$

we have (by Taylor series expansion)

$$\dot{u} = \dot{x} = f(u + x^*, v + y^*) = f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

This leads to

$$\dot{u} = \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv)$$

and similarly

$$\dot{v} = \left.\frac{\partial g}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial g}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

Hence

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \quad - \text{ the linearised system}$$

with

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \quad - \text{ \underline{the Jacobian matrix}}$$

Theorem (linear stability): Suppose that $\dot{x} = f(x)$ has an equilibrium at x^* and the linearisation $\dot{x} = Ax$. If A has no zero or purely imaginary eigenvalues then the local stability of the fixed point (which is called **hyperbolic** in this case) is determined by the linear system. In particular, if all eigenvalues have a negative real part Re $(\lambda_i) < 0$ for all $i = 1, \ldots, n$ then the fixed point is asymptotically stable.

Hartman-Grobman theorem: The local phase-portrait near a hyperbolic fixed point is topologically equivalent to the phase-portrait of the linearisation.

Structural stability A phase portrait is structurally stable if its topology cannot be changed by an arbitrarily small perturbation to the vector field, i.e. a system is structurally stable if it is topologically equivalent to any ϵ-perturbation

$$\dot{x} = f(x) + \epsilon p(x)$$

where $\epsilon \ll 1$ and p is smooth enough. For example, the phase portrait of a saddle is structurally stable, but that of a centre is not: an arbitrarily small amount of damping converts the center to a spiral.

Exercise. Consider the system

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2) \\ \dot{y} &= x + ay(x^2 + y^2), \end{aligned}$$

where a is a parameter. Show that the linearised system incorrectly predicts that the origin is a centre for all values a. (Hint: rewite the system in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$)

Example 4. To illustrate some of the principles covered let us do a phase-plane analysis of the Lotka-Volterra model of population dynamics of two competing species. Assume i) each species grows in the absence of the other with logistic growth ($\dot{x} = x(1-x)$) and ii) when both species are present they compete for food such that one may go hungry. A particular model of rabbits (r) and sheep (s):

$$\begin{aligned} \dot{r} &= r(3 - r - 2s) \equiv f(r,s) \\ \dot{s} &= s(2 - r - s) \equiv g(r,s) \end{aligned}$$

Fixed points defined by $\dot{r} = \dot{s} = 0$. One finds $(\overline{r}, \overline{s}) = (0,0), (0,2), (3,0), (1,1)$. To classify them we compute

$$A = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{bmatrix} = \begin{bmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - r - 2s \end{bmatrix}$$

1. $(\overline{r}, \overline{s}) = (0, 0)$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues are both positive so $(0, 0)$ is an unstable node. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $(0, 1)$.

2. $(\overline{r}, \overline{s}) = (0, 2)$

$$A = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Hence $(0, 2)$ is a stable node. Slow eigendirection is $(1, -2)$.

3. $(\overline{r}, \overline{s}) = (3, 0)$

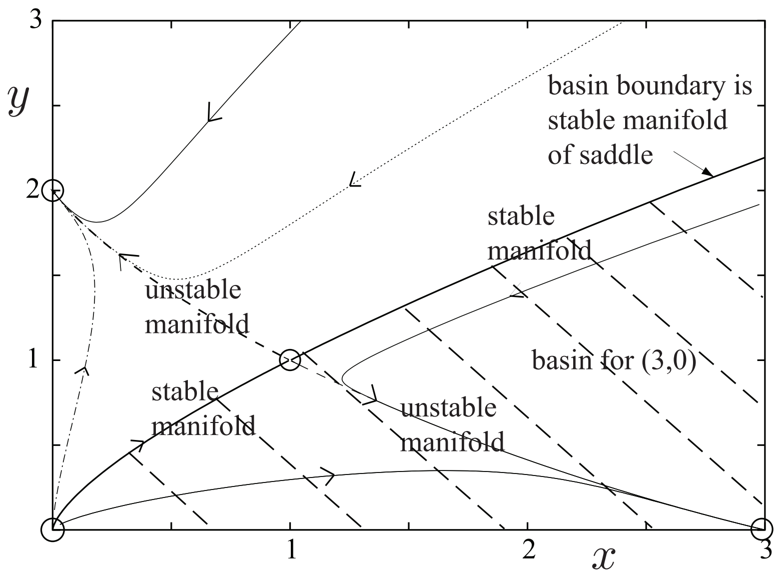
$$A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence $(3, 0)$ is a stable node. Slow eigendirection is $(3, -1)$.

4. $(\overline{r}, \overline{s}) = (1, 1)$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 + \sqrt{2} & 0 \\ 0 & -1 - \sqrt{2} \end{bmatrix}$$

Hence, $(1, 1)$ is a saddle



The above example nicely illustrates the notion of a **basin of attraction**. Given an attracting fixed point $\overline{x}$ we define its basin of attraction to be the set of initial conditions x_0 such that $x(t) \to \overline{x}$ as $t \to \infty$. For instance the basin of attraction for the node at $(3, 0)$ consists of all points lying below the stable manifold of the saddle. Because the stable manifold separates the basins of two nodes, it is called the **basin boundary**.

Lyapunov theorem: Suppose that x^* is a fixed point for the differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Then x^* is Lyapunov stable if there exists a (continuously differentiable) function $L(x)$ (called a Lyapunov function) with the following properties in some neighbourhood of x^*:

1. $L(x)$ and its partial derivatives are continuous

2. $L(x) > 0$ for all $x \neq x^*$ and $L(x^*) = 0$

3. $\dot{L} \leq 0$ for all $x \neq x^*$

Note that $\dot{L}$ is determined by the chain-rule

$$\dot{L} = \sum_i \frac{\partial L}{\partial x_i} \dot{x}_i = \sum_i \frac{\partial L}{\partial x_i} f(x_i)$$

Example 5. Show that $L(x, y) = x^2 + 4y^2$ is a Lyapunov function for

$$\dot{x} = -x + 4y, \qquad \dot{y} = -x - y^3$$

The fixed point is at $(0, 0)$.

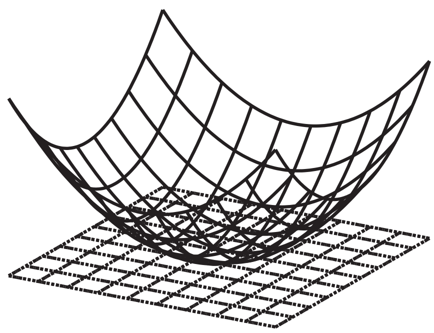
1. $L(x, y)$ is continuously differentiable.

2. $L(x, y) > 0$, $L(0, 0) = 0$.

3. $$\dot{L} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = -2x^2 - 8y^4 < 0$$

$\Rightarrow L(x, y)$ - Lyapunov function

Heuristic picture: sufficiently close to the fixed point, L forms a bowl and L decreases along trajectories.



Main difficulty of this method for checking stability is finding an appropriate Lyapunov function.