STATISTICAL MECHANICS OF COMPLEX SYSTEMS – SOLUTIONS 2009

---

1. (a) The information entropy of a random variable measures how much information is obtained by observing the random variable. (or: how much uncertainty is associated with it). $\{\mathbf{2}\}$

$$H(p_1, p_2, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \log p_i$$

where $p_1, p_2, \ldots, p_n$ are the probabilities of the states of the random variable, and K is an abritrary multiplicative constant (defining the unit of entropy). $\{\mathbf{2}\}$

[Bookwork]

(b) "sense of direction": if $n < m$, then

$$H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) < H\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$$

$\{\mathbf{2}\}$

and the addition rule:

$$H(p_1, p_2, p_3) = H(p_1, p_2 + p_3) + (p_2 + p_3)\, H\left(\frac{p_2}{p_2 + p_3}, \frac{p_3}{p_2 + p_3}\right)$$

$\{\mathbf{2}\}$

[Bookwork]

(c) (i) number of distinct text messages: $128^{160} = 2^{1120}$. $\{\mathbf{2}\}$

(ii) number of weeks: $2^{1120}/1.5 \times 10^9 \approx 2^{1120}/2^{30} = 2^{1090}$ $\{\mathbf{2}\}$

(iii) $H_{\text{Sh}} = -\sum_{i=1}^{2^{1120}} 2^{-1120} \log_2 2^{-1120} = 1120$ bits $\{\mathbf{2}\}$

(d) (i) $P_A = P_C = 1/4$, $P_B = 1/2$. So $H_{\text{Sh}} = -\left(2 \cdot \frac{1}{4} \log_2 \frac{1}{4} + \frac{1}{2} \log_2 \frac{1}{2}\right) = \frac{3}{2}$ bits $\{\mathbf{2}\}$

(ii) $A \rightarrow$"00", $C \rightarrow$"01", $B \rightarrow$"1". $\{\mathbf{3}\}$

Expected length: $\frac{1}{4} \cdot 2 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = \frac{3}{2}$ $\{\mathbf{1}\}$

(iii) number of 8-bit characters: $52 \cdot \frac{3}{2}/8 = 9.75$, so we need 10 characters. $\{\mathbf{2}\}$

[Unseen]

(e) Suppose we have a random variable with unknown probablities for the states, and (possibly) some constraints in the form of expectations of some functions of the random variable. Then the Maximum Entropy principle gives a guess for the unknown probabilities; which is the set of probabilities maximising the information entropy while satisfying the constraints. $\{\mathbf{3}\}$

[Bookwork]

2. (a) (i) canonical ensemble $\{\mathbf{1}\}$
[Bookwork]

(ii)

$$Z(\beta) = \frac{1}{h}\int_{z=0}^{\infty} dz \int_{p_z=-\infty}^{\infty} dp_z \exp\left\{-\beta\left(\frac{p_z^2}{2m} + mgz\right)\right\} \qquad \{\mathbf{2}\}$$

$$= \frac{1}{h}\frac{1}{\beta mg}\sqrt{\frac{2\pi m}{\beta}} = \frac{1}{hg}\sqrt{\frac{2\pi}{m}}\beta^{-3/2} \qquad \{\mathbf{3}\}$$

So using the question's notation $A = \frac{1}{h}\sqrt{\frac{2\pi}{m}}$, $\alpha = -1$, and $\gamma = 3/2$.
[Bookwork] formula applied to **[unseen]** situation.

(iii)

$$\langle E\rangle = \frac{\partial - \log Z}{\partial\beta} = \frac{3}{2}\frac{1}{\beta} = \frac{3}{2}k_{\mathrm{B}}T \qquad \{\mathbf{4}\}$$

(iv)

$$\mathrm{Var}(E) = -\frac{\partial^2 - \log Z}{\partial\beta^2} = \frac{3}{2}\frac{1}{\beta^2} = \frac{3}{2}(k_{\mathrm{B}}T)^2 \qquad \{\mathbf{4}\}$$

(v)

$$\langle z\rangle = \frac{\frac{1}{h}\int dz \int dp_z z e^{-\beta E}}{Z} = \frac{1}{-\beta m}\frac{\partial \log Z}{\partial g} = \frac{1}{\beta mg} = \frac{k_{\mathrm{B}}T}{mg} \qquad \{\mathbf{3}\}$$

In sections (iii)-(v) **[bookwork]** formulae have to be applied to **[unseen]** Z.

(vi) first calculate $\frac{\partial^2 Z}{\partial g \partial\beta}$:

$$\frac{\partial^2 \exp\left\{-\beta\left(\frac{p_z^2}{2m} + mgz\right)\right\}}{\partial g\partial\beta} = -\frac{\partial}{\partial g}\left(\frac{p_z^2}{2m} + mgz\right)e^{-\beta E}$$

$$= -\left(mz + E(-\beta mz)\right)e^{-\beta E}$$

so

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial g\partial\beta} = -m\langle z\rangle + \beta m\langle Ez\rangle\ . \qquad \{\mathbf{2}\}$$

Using the question's notation, $C_1 = -m$ and $C_2 = m$.
Thus

$$\langle Ez\rangle = \frac{1}{\beta m}\left(\frac{1}{Z}\frac{\partial^2 Z}{\partial g\partial\beta} + m\langle z\rangle\right) = \frac{1}{\beta m}\left(\frac{3}{2}\frac{1}{g\beta} + m\frac{k_{\mathrm{B}}T}{mg}\right) = \frac{5}{2}\frac{(k_{\mathrm{B}}T)^2}{mg} \qquad \{\mathbf{2}\}$$

finally

$$\mathrm{Cov}(z, E) = \langle Ez\rangle - \langle z\rangle\langle E\rangle = \frac{5}{2}\frac{(k_{\mathrm{B}}T)^2}{mg} - \frac{k_{\mathrm{B}}T}{mg}\frac{3}{2}k_{\mathrm{B}}T = \frac{(k_{\mathrm{B}}T)^2}{mg} \qquad \{\mathbf{1}\}$$

[Unseen]

(b)

$$Z(\beta) = \sum_{i=0}^{1}\frac{1}{h}\int_{z=0}^{\infty} dz \int_{p_z=-\infty}^{\infty} dp_z \exp\left\{-\beta\left(\frac{p_z^2}{2m} + mgz + E_i\right)\right\}$$

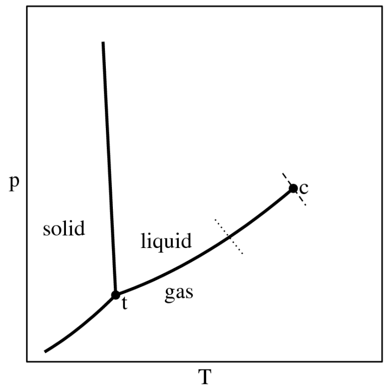
$$= Z_A \cdot \left(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}\right) \qquad \{\mathbf{2}\}$$

$$
\begin{aligned}
\langle E \rangle &= \frac{\partial - \log Z}{\partial \beta} = \frac{3}{2}\frac{1}{\beta} = \frac{3}{2}k_{\mathrm{B}}T + \frac{\epsilon_0 e^{-\beta\epsilon_0} + \epsilon_1 e^{-\beta\epsilon_1}}{e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}} \\
&= \frac{3}{2}k_{\mathrm{B}}T + \frac{\epsilon_0 + \epsilon_1 e^{-\beta(\epsilon_1 - \epsilon_0)}}{1 + e^{-\beta(\epsilon_1 - \epsilon_0)}} \qquad \{\mathbf{1}\}
\end{aligned}
$$

3. (a) Phase transitions are collective phenomena exhibited by some systems consisting of interacting components. In a phase transition a small change in a control parameter results in a qualitative change (from one phase to another) of the system, described by an order parameter. $\{\mathbf{2}\}$

There are two fundamental types of phase transitions: abrupt (where the order parameter changes discontinuously) and continuous (where the order parameter changer continuously). $\{\mathbf{2}\}$

(b) Diagram: $\{\mathbf{3}\}$



Thick lines represent phase coexistance (2 phase coexist), point t is the triple point where all 3 phases can coexist. $\{\mathbf{2}\}$

(c) The dotted line repesents an abrupt phase transition, and the dashed line (going through the critical point c) represents continuous phase transition. $\{\mathbf{2}\}$

(d) Different symmetries necessarily mean different phases (eg. crystaline for solids and statistically uniform and isotropic for liquids), but two different thermodynamic phases might have the same symmetry (statistically uniform and isotropic for both liquids and gases). $\{\mathbf{2}\}$

(e) (i) uniaxial magnet: amplitude of locally averaged magnetisation (scalar)

(ii) liquid-gas transition: density (scalar)

(iii) nematic liquid crystal: locally averaged direction of molecules (director: like a vector, but $\mathbf{d} = -\mathbf{d}$ identified)

(iv) crystal: wrapped vector to deform lattice back to perfect lattice (wrapped vector: $\mathbf{v} = \mathbf{v} + n_x\mathbf{e}_x + n_y\mathbf{e}_y + n_z\mathbf{e}_z$ identified with additon of integer number of the lattice vectors)

(v) superconductor: phase of the condensed quantum state (complex number)

(One mark for each case, total $\{\mathbf{5}\}$.)

(f) (i) A topological defect is a tear in the order parameter field which cannot be healed.

(ii) Topological line defects can exist: complex number (of fixed amplitude), director (of fixed amplitude), wrapped vector. $\{\mathbf{2}\}$

Topological line defects can not exist: scalar, vector (even if amplitude is fixed). $\{\mathbf{2}\}$

This question is mostly **[bookwork]** except last part of (b).