THE UNIVERSITY OF WARWICK

Fourth Year Examinations: Summer 2008

STATISTICAL MECHANICS OF COMPLEX SYSTEMS

Answer TWO questions.

Time allowed: 1.5 hours.

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

A maximum of 25 credit marks will be awarded for each question. The nominal mark assigned to each part of a question is indicated by means of a bold figure enclosed by curly brackets, e.g. $\{\mathbf{2}\}$*, immediately following that part. This mark scheme is for guidance only and may be adjusted by the examiner.*

Calculators may be used for this examination.

1. A vending machine sells coke bottles for £1. It accepts 20p, 50p and £1 coins, exact change only: one can pay with $5 \times$20p, $2 \times$50p or $1 \times$£1 coins. We know the average number of coins collected by the machine per bottle, and ask for the breakdown of this number for each type of coin.

   We claim complete ignorance about any further information (eg. ratio of coins in circulation, size of customer's wallet, etc).

   (a) Based on the 3 ways one can pay for a bottle, formulate the problem in the Maximum Entropy framework. $\{\mathbf{4}\}$

   (b) (i) Show that the partition function is of this form:

   $$Z(\lambda) = e^{-5\lambda} + e^{-2\lambda} + e^{-\lambda}. \qquad \{\mathbf{4}\}$$

   (ii) Express the "best guess" (in the Maximum Entropy sense) for the number of 20p, 50p and £1 coins per bottle in terms of λ. $\{\mathbf{3}\}$

   (iii) Comment on the following limits: $\lambda \to -\infty, \quad \lambda = 0, \quad \lambda \to \infty.$ $\{\mathbf{3}\}$

   (c) After selling a large number of bottles, the machine collects twice as many coins as number of bottles sold. Calculate λ. $\{\mathbf{5}\}$

   (d) The machine sells 100 bottles and collects 200 coins.

   (i) Show that the number of £1 coins collected is expected to be around 48. $\{\mathbf{2}\}$

   (ii) The actual number of £1 coins can deviate from the value above: give a rough estimate of its standard deviation (without going into detailed calculations). $\{\mathbf{4}\}$

2. (a) Are the following characteristic for abrupt or continuous phase transitions or both? {**6**}

   (i) transition occurs via nucleation of droplets;
   (ii) coexisting phases;
   (iii) large fluctuations;
   (iv) scale invariance;
   (v) metastable phases;
   (vi) can be described by renormalisation theory.

   (b) Explain what is meant by universality. (Your answer should include the type of phase transition in question, universality classes, and its relevance to describing transitions in systems with complex interactions.) {**4**}

   (c) (i) Calculate the difference between the Gibbs free energy per molecule of a liquid and a gas, $\Delta G/N$, at small undercooling $\Delta T = T_{\text{v}} - T$ below the phase transition temperature T_{v} at fixed pressure. {**3**}
   Use the following facts:
   - at $T = T_{\text{v}}$ the free energies are equal;
   - $\partial G/\partial T = -S$ at fixed pressure and particle number;
   - the entropy difference between the two phases is $\Delta S = LN/T_{\text{v}}$, where L is the latent heat per molecule.

   (ii) Show that the Gibbs free energy of a liquid droplet of radius R in gas which is undercooled by ΔT is of the form

$$G(R) = aR^2 - bR^3.$$

   Calculate the coefficients a and b, and comment on the behaviour for small and large R. {**3**}
   [*Hint:* your answer might include the surface tension γ (free energy cost of liquid–gas interface per unit area), and the number density n (number of liquid molecules in unit volume). Also, you calculated the free energy gain per molecule above.]

   (iii) Calculate the critical droplet radius (where the free energy is maximal) and the nucleation barrier (which is the corresponding free energy). {**3**}

   (iv) How does the droplet nucleation rate depend on the nucleation barrier? (No need to give prefactors.) Comment on the limit $\Delta T \to 0$. {**3**}

   (v) Suppose that a droplet forms on the wall of the container. Assume that the wall–gas and wall–liquid surface tension both negligible compared to the liquid–gas surface tension, so a hemispherical droplet forms. How does the nucleation barrier in this case relate to the nucleation barrier for spherical droplets? What are the implications? {**3**}

3. (a) Explain what is meant by a jamming transition. {**4**}

(b) In a certain traffic model of single lane roads the cars are represented by line segments of length L, moving on a line in one direction. The velocity v of each car satisfies

$$\begin{aligned} \frac{dv}{dt} &= a\left(1-\left(\frac{v}{v^*}\right)^{\delta}\right), \\ v^* &= \min\left(v_0, \frac{s}{T}\right), \end{aligned}$$

where s is the distance to the car in front. a, δ, v_0 and T are parameters, their value is positive. An additional quantity is the global density of cars, ρ, which is the number of cars per unit length of road, averaged over a very long road segment.

(i) Explain the meaning of the parameters a, v_0 and T. {**3**}

(ii) Calculate and sketch the flux of cars (number of cars passing a given point in unit time) as a function of ρ in the stationary state. {**5**}
[*Hint:* In the stationary state all cars move with velocity independent of time. Calculate the flux for small and for large ρ, and where the two behaviours cross over.]

(iii) Still under the assumption of stationary state, calculate the maximum flux, and discuss the limit $v_0 \to \infty$. {**3**}

(c) In the above traffic model and similar traffic models, often a non-stationary solution emerges. Explain the nature of transition from stationary to non-stationary solutions. What phases emerge? {**4**}

(d) (i) Describe a model of self-propelled particles which exhibits a dynamic phase transition. {**3**}

(ii) Describe the nature of the dynamic phase transition. In your answer include what the order parameter is, and whether the transition is abrupt or continuous. {**3**}