THE UNIVERSITY OF WARWICK

Fourth Year Examinations: Summer 2009

STATISTICAL MECHANICS OF COMPLEX SYSTEMS

Answer TWO questions.

Time allowed: 1.5 hours.

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

A maximum of 25 credit marks will be awarded for each question. The nominal mark assigned to each part of a question is indicated by means of a bold figure enclosed by curly brackets, e.g. $\{\mathbf{2}\}$, immediately following that part. This mark scheme is for guidance only and may be adjusted by the examiner.

Calculators may be used for this examination.

1. (a) What is the information entropy? In your answer include the formula and the meaning of the quantity. $\{\mathbf{4}\}$

   (b) The formula for the information entropy can be derived just from imposing three fundamental requirements which it has to satisfy. One of them is that it is a continuous function of the probabilities; what are the other two? $\{\mathbf{4}\}$

   (c) A (simple) text message can contain up to 160 characters, where each character can take any of 128 possible values (including space).

      (i) How many distinct text messages are there? $\{\mathbf{2}\}$

      (ii) Give an estimate of the number of weeks necessary to write all possible messages at a rate of 1.5×10^9 text messages per week. (This rate is an estimate for the UK in 2008.) $\{\mathbf{2}\}$

      (iii) What is the upper limit for the Shannon entropy of a text message? $\{\mathbf{2}\}$

   (d) (i) Every Friday morning you are sent a text message to tell you in which pub your friends will gather that evening. There are three choices: pub A and C are equally likely, but pub B is twice as likely as A or C. Suppose the wording of the messages is unimportant, only the the choice of $A/B/C$ matters. What is the lower bound for the Shannon entropy for these messages? $\{\mathbf{2}\}$

      (ii) Devise an optimal encoding (with 0's and 1's) of the important part of the messages. (Optimal: use only as many bits as necessary.) What is the expected length of one encoded message? $\{\mathbf{4}\}$

      (iii) What is the minimum number of 8-bit characters which you will need to store one year's worth of these messages? $\{\mathbf{2}\}$

   (e) What do we mean by the Maximum Entropy Principle? $\{\mathbf{3}\}$

2. (a) Particle A has mass m, is subject to gravity (denote the gravitational acceleration by g), and is in thermal equilibrium with the surrounding air, which can be considered to be at a uniform temperature T. Assume that the particle can only move in the vertical direction, and that it only has a translational degree of freedom.

(i) Which ensemble describes this particle? {**1**}

(ii) Write down an expression for the partition function for the particle. Show that it evaluates to the form $Ag^{\alpha}(k_{\mathrm{B}}T)^{\gamma}$. What are the values of A, α and γ? {**5**}

(iii) State a general expression for $\langle E\rangle$, the average energy of the particle, in terms of (operations on) the partition function. Hence or otherwise evaluate $\langle E\rangle$. {**4**}

(iv) State the analogous general expression for Var(E), the variance of the energy of the particle. Hence or otherwise evaluate Var(E). {**4**}

(v) Calculate $\langle z\rangle$, the average vertical position of the particle. {**3**}

(vi) Show that

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial g \partial \beta} = C_1\langle z\rangle + C_2\beta\langle Ez\rangle \ ,$$

[where $\beta = 1/(k_{\mathrm{B}}T)$] and give the values of C_1 and C_2. Hence or otherwise calculate Cov(z, E), the covariance of the vertical position and the energy of the particle. {**5**}

(b) Particle B has two internal states of energy ϵ_0 and ϵ_1, otherwise it is identical to particle A. Find how $\langle E\rangle_B$ differs from case A, explaining your reasoning. {**3**}

You may use the following integral:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int\limits_{x=-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \ = \ 1$$

3. (a) What is a phase transition? What are the fundamental types of phase transition? How do you decide the type of a given phase transition? {**4**}

(b) Sketch the phase diagram of water. Label the regions with the names of the phases. Where can 2 phases coexist? Where can 3 phases coexist? {**5**}

(c) Draw paths corresponding to each fundamental type of phase transition (if water can exhibit that type of phase transition). {**2**}

(d) What is the relationship between thermodynamic phases and symmetries? Give examples. {**2**}

(e) What is the order parameter in the following examples? What kind of mathematical object is the order parameter? (Eg. scalar, vector, ...; but take care to specify any special properties or restrictions.) {**5**}

(i) uniaxial magnet

(ii) liquid-gas transition

(iii) nematic liquid crystal

(iv) crystal

(v) superconductor

(f) (i) What is a topological defect? {**3**}

(ii) Give two examples each for order parameters where topological line defects can exist, and where they cannot exist. {**4**}