Stefan Grosskinsky
www.warwick.ac.uk/~masgav/teaching/co905.html

CO905
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# Stochastic Processes

## Problem sheet 3 – Part 1

**3.1** Let $\mathbf{B}$ be a standard Brownian motion in $\mathbb{R}^d$. Show the following:

(a) Scaling property:
If $\lambda > 0$, then $\mathbf{B}_\lambda = \big(\lambda^{-1/2} B_{\lambda t} : t \geq 0\big)$ is a standard Brownian motion in $\mathbb{R}^d$.

(b) Orthogonal transformations:
If $U \in O(d)$ is an *orthogonal* $d \times d$ matrix (i.e. $U^{-1} = U^T$), then $U\,\mathbf{B} = \big(U\,\mathbf{B}_t : t \geq 0\big)$ is a standard Brownian motion. In particular $-\mathbf{B}$ is a standard Brownian motion.

[4]

**3.2** Let $X = (X_n : n \in \mathbb{N})$ be a simple random walk on $\mathbb{Z}$ with transition probabilities

$$p_{i,i+1} = 1/2 + \epsilon\,, \quad p_{i,i-1} = 1/2 - \epsilon \quad \text{for all } i \in \mathbb{Z}\,.$$

Rescale time $t = \Delta t\, n$ and derive the Fokker-Planck equation for an appropriate scaling of space and $\epsilon$, analogous to the derivation of Section 2.1. What is the right scaling of the asymmetry $\epsilon(\Delta t)$ to get a limit with non-zero drift and diffusion?

[6]

**3.3** Fokker-Planck approximation of the Moran model:
Consider the Moran model $X = (X_t : t \geq 0)$ with population size $N$, including selection (characterized by $\alpha$) and mutation (characterized by $\epsilon$), i.e. for $i \in \{0, \ldots, N\}$

$$\begin{aligned} g_{i,i+1} &= \alpha(1-\epsilon) i \frac{N-i}{N+1} + \epsilon(N-i)\frac{N-i}{N+1} \\ g_{i,i-1} &= (1-\epsilon)(N-i)\frac{i}{N+1} + \alpha\epsilon i \frac{i}{N+1}\,. \end{aligned}$$

(a) Set $\pi_i(t) = f(t,x)$ with $x = i/N$ and write the master equation in terms of $f$ and $x$.

(b) For $\epsilon = 0$ expand the master equation up to the second derivative of $f$. It is (very!) useful to actually do the expansion not for $f$ but for the function $g(t,x) := f(t,x)x(1-x)$.
For which function $\alpha = \alpha(N)$ woulde the drift and diffusion term be of the same order in $1/N$?

(c) Compute the drift $a(x)$ and diffusion coefficient $b(x)$ as a function of $\alpha$, $\epsilon$ and $N$ according to the formula

$$\begin{aligned} \mathbb{E}\big(X_{t+h} - X_t \,\big|\, X_t = [xN]\big) &= a(x)\,h + o(h)\,, \\ \mathbb{E}\big((X_{t+h} - X_t)^2 \,\big|\, X_t = [xN]\big) &= b(x)\,h + o(h)\,. \end{aligned}$$

To compute the expectations, remember that $h$ is very small and use the interpretation of a jump rate $g$. Compare your result to the one from (b).
Write down the Fokker-Planck equation for general $\alpha$, $\epsilon$ and $N$. A derivation as in (b) is not necessary.

[10]