Stochastic Processes

Problem sheet 1

1.1 A dice is rolled repeatedly. Which of the following are Markov chains?
For those that are, supply the state space and the transition matrix.

(a) The largest number X_n shown up to the nth roll.
(b) The number N_n of sixes in n rolls.
(c) At time n, the time B_n since the most recent six. [6]
(d)* At time n, the time C_n until the next six.

1.2 (a) Consider a simple symmetric random walk on $\{1, \ldots, L\}$ with
- periodic boundary conditions, i.e. $p_{L,L-1} = p_{L,1} = p_{1,L} = p_{1,2} = 1/2$,
- closed boundary conditions, i.e. $p_{L,L-1} = p_{L,L} = p_{1,1} = p_{1,2} = 1/2$,
- reflecting boundary conditions, i.e. $p_{L,L-1} = p_{1,2} = 1$,
- absorbing boundary conditions, i.e. $p_{L,L} = p_{1,1} = 1$.

(All transition probabilities which are not specified above are 0.)
In each case, sketch the transition matrix $P = (p_{ij})_{ij}$ of the process, decide whether the process is irreducible, and give at least one stationary distribution $\boldsymbol{\pi}^*$.
(Hint: Use detailed balance.) [8]

(b)* Consider a symmetric connected graph (G, E) without loops and double edges. A simple random walk on (G, E) has transition probabilities $p_{i,j} = e_{i,j}/c_i$, where c_i is the number of outgoing edges in vertex i, and $e_{i,j} \in \{0, 1\}$ denotes the presence of an edge (i, j).
Find a formula for the stationary distribution $\boldsymbol{\pi}^*$.
Does your formula also hold on a non-symmetric, strongly connected graph?

1.3 Let $Z = (Z_n : n \in \mathbb{N})$ be a branching process, defined recursively by

$$Z_0 = 1\ , \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \qquad \text{for all } n \geq 0\ ,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum i in generation n.

(a) Consider a geometric offspring distribution $X_i^n \sim Geo(p)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = p\,(1-p)^k\ , \quad p \in (0, 1)\ .$$

Compute the prob. generating function $G(s) = \sum_k p_k s^k$ as well as $\mathbb{E}(X_i^n)$ and $Var(X_i^n)$. Sketch $G(s)$ for (at least) three (wisely chosen) values of p and compute the probability of extinction as a function of p.

(b) Consider a Poisson offspring distribution $X_i^n \sim Poi(\lambda)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = \frac{\lambda^k}{k!}\, e^{-\lambda}\ , \quad \lambda > 0\ .$$

Repeat the same analysis as in (a). [11]

(c)* For geometric offspring with $p = 1/2$, show that $G_n(s) = \frac{n-(n-1)s}{n+1-ns}$ and compute $\mathbb{P}(Z_n = 0)$. If T is the (random) time of extinction, what is its distribution and its expected value?