Stochastic Processes

Problem sheet 2

2.1 Birth-death processes

A birth-death process X is a generalized $M/M/1$ queue with state-dependent rates, i.e. a continuous-time Markov chain with state space $S = \mathbb{N} = \{0, 1, \ldots\}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S\ , \qquad i \xrightarrow{\beta_i} i-1 \quad \text{for all } i \geq 1\ .$$

(a) Write down the generator G. Under which conditions is X irreducible?
Using detailed balance, find a formula for the stationary probablities π_k^* in terms of π_0^*.

(b) Suppose $\alpha_i = \alpha$ and $\beta_i = i\beta$. This is called an $M/M/\infty$ queue.
Is X positive recurrent? If yes compute the stationary distribution.
What kind of situation is this a good model for?

(c) Suppose $\alpha_i = i\alpha$, $\beta_i = i\beta$ and $X_0 = 1$.
Discuss qualitatively the behaviour of X_t as $t \to \infty$.
What kind of situation is this a good model for? [8]

2.2 Contact process

The contact process is a basic stochastic model of an epidemic, where individuals $x \in \Lambda$ can have two states, $\eta(x) = 0$ (healthy/susceptible) and $\eta(x) = 1$ (infected). Λ is the set of individuals endowed with some geometric structure (in general a graph), for simplicity consider $\Lambda = \{1, \ldots, N\}$ with connections only between nearest neighbours and periodic boundary conditions. The following transitions are possible:

- Susceptibles get infected from infected neighbours independently with rate $\lambda > 0$,
- infected individuals recover independently with rate 1.

So the total rate of individual x to change its current state $\eta(x) \to 1 - \eta(x)$ is

$$\eta(x) + \lambda\big(1 - \eta(x)\big) \sum_{y \sim x} \eta(y) \qquad \text{(denote the change of state by } \eta \to \eta^x)\ ,$$

where in our case the sum is over nearest neighbours $y = x - 1, x + 1$.

(a) Give the state space of the process and its irreducible components, and write down the master equation.

(b) Simulate the process with initial condition $\eta(x) = 1$ for all $x \in \Lambda$ and several values of $\lambda \in [1, 2]$. Plot the number of infected individuals as a function of time averaging over 100 realizations, and find an estimate of the critical value $\lambda_c(N) \in [1, 2]$.
Repeat this for different lattice sizes, e.g. $N = 100, 200, 400, 800$, and plot your estimates of $\lambda_c(N)$ against $1/N$. Extrapolate to $1/N \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$.

The critical value λ_c is defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, and lies in the interval $[1, 2]$ for the one dimensional contact process.

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