Cogos – Handout ~~8~~ 4

Thm 3.6: Let X be a diff. process with drift $a(t,x)$ and diffusion $b(t,x)$. Then the pdf $f(t,x)$ satisfies the FP-equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(a(t,x)\, f\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(t,x)\, f\right) \quad \forall t\geq 0,\ x\in\mathbb{R}.$$

Proof. The state space is $S=\mathbb{R}$. Let $g:\mathbb{R}\to\mathbb{R}$ be a twice diff'able observable with compact support, i.e. $\exists_{[a,b]\in\mathbb{R}}$ s.th. $g(x)=0$ if $x\notin[a,b]$.

Then:

$$\mathbb{E}(g(X_{t+h})) = \int_{\mathbb{R}} g(x)\, f(t+h,x)\,dx = \int_{\mathbb{R}} g(x)\left[f(t,x) + h\,\partial_t f(t,x)\right] dx \quad (*)$$

On the other hand, given that $X_t = x$ the increment $\Delta X = X_{t+h} - x$ is small with some PDF $h_{\Delta X}$ and by the definition of diff. proc.

$$\mathbb{E}(\Delta X) = \int_{\mathbb{R}} y\, h_{\Delta X}(y)\,dy = a(t,x)\,h\,, \quad \mathbb{E}(\Delta X^2) = \int_{\mathbb{R}} y^2 h_{\Delta X}(y)\,dy = b(t,x)\,h$$

Thus (independently of the precise form of $h_{\Delta X}$):

$$\mathbb{E}(g(X_{t+h})) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} g(x+y)\,h_{\Delta X}(y)\,dy\right) f(t,x)\,dx =$$

$$= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left(g(x) + y\,\partial_x g(x) + \frac{y^2}{2}\,\partial_x^2 g(x)\right) h_{\Delta X}(y)\,dy\right] f(t,x)\,dx =$$

$$= \int_{\mathbb{R}}\left(g(x) + a(t,x)\,h\,\partial_x g(x) + \frac{1}{2} b(t,x)\,h\,\partial_x^2 g(x)\right) f(t,x)\,dx \quad (**)$$

Equating $(*)$ and $(**)$ ~~and~~ up to terms of order h gives the generator $\mathcal{L} = a(t,x)\,\partial_x + \frac{1}{2} b(t,x)\,\partial_x^2$ of the diffusion process, and by partial integration also the FP-equation (note that due to compact support of g boundary terms vanish), since the above holds for arbitrary g. $\square$