Stochastic Models of Complex Systems

Problem sheet 2

Sheet counts $40/100$ homework marks, [x] indicates weight of the question.
* Questions do not enter the mark.

2.1 Birth-death processes [10]

A birth-death process X is a continuous-time Markov chain with state space $S = \mathbb{N} = \{0, 1, \ldots\}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S\ , \qquad i \xrightarrow{\beta_i} i-1 \quad \text{for all } i \geq 1\ .$$

(a) Write the generator G and the master equation. Under which conditions is X irreducible? Using detailed balance, find a formula for the stationary probablities π^*_k in terms of π^*_0.

(b) Suppose $\alpha_i = \alpha$ for $i \geq 0$ and $\beta_i = \beta$ for $i > 0$. This is called an $M/M/1$ **queue**.
- Under which conditions on α and β can the stationary distribution be normalized? Give a formula for π^*_k in that case.
- Derive a differential equation for $\mu_t = \mathbb{E}(X_t)$ (involves $\pi_0(t)$ on the right-hand side), and show that with $\boldsymbol{\pi}(0) = \boldsymbol{\pi}^*$ the right-hand side vanishes.

(c) Suppose $\alpha_i = \alpha$ and $\beta_i = i\beta$ for $i \geq 0$. This is called an $M/M/\infty$ **queue**.
- Under which conditions on α and β can the stationary distribution be normalized? Give a formula for π^*_k in that case.
- Derive a closed equation for μ_t and solve it for general initial condition μ_0.

(d) Suppose $\alpha_i = i\alpha$, $\beta_i = i\beta$ for $i \geq 0$ and $X_0 = 1$.
- Discuss qualitatively the behaviour of X_t as $t \to \infty$.
- Derive a closed equation for μ_t and solve it for general initial condition μ_0.

2.2 Contact process [10]

Consider the CP $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (all sites connected) with state space $S = \{0, 1\}^L$ and transition rates

$$c(\eta, \eta^x) = \eta(x) + \lambda\big(1 - \eta(x)\big) \sum_{y \neq x} \eta(y)\ ,$$

where $\eta, \eta^x \in S$ are connected states such that $\eta^x(y) = \begin{cases} 1 - \eta(x) & , y = x \\ \eta(y) & , y \neq x \end{cases}$,

(η with site x flipped).

(a) Let $N_t = \sum_{x \in \Lambda_L} \eta_t(x) \in \{0, \ldots, L\}$ be the number of infected sites at time t. Show that $(N_t : t \geq 0)$ is a Markov chain with state space $\{0, \ldots, L\}$ by computing the transition rates $c(n, m)$ for $n, m \in \{0, \ldots, L\}$.
Write down the master equation for the process $(N_t : t \geq 0)$.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states?
What are the stationary distributions?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\,\rho(t) = f\big(\rho(t)\big) = -\rho(t) + L\lambda\big(1-\rho(t)\big)\rho(t)\ .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $f(\rho^*) = 0$. What is the prediction for the stationary density ρ^* depending on λ?

2.3 Simulation of CP (Sample code on the course webpage) [20]

Consider the contact process $(\eta_t : t \geq 0)$ as defined in Q2.2, but now on the one-dimensional lattice $\Lambda_L = \{1, \ldots, L\}$ with connections only between nearest neighbours and periodic boundary conditions.
The critical value λ_c is defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, depends on the dimension, and lies in the interval $[1, 2]$ in our case.
All simulations of the process should be done with initial condition $\eta_0(x) = 1$ for all $x \in \Lambda$.

(a) To get a general idea, simulate the process for e.g. $L = 256$ for several values of $\lambda \in [1, 2]$. Plot the number of infected individuals $N_t = \sum_{x\in\Lambda_L} \eta_t(x)$ as a function of time up to time $10 \times L$, averaging over 100 realizations in a double-logarithmic plot. What is the expected behaviour of N_t depending on λ?
For a given system size L, find the window of interest choosing $\lambda = 1, 1.2, \ldots, 1.8, 2$ averaging over 100 realizations with times up to $10 \times L$. Then use fine increments of 0.01 for λ and averages of at least 500 realizations to find an estimate of the critical value $\lambda_c(L) \in [1, 2]$.
Repeat this for different lattice sizes, e.g. $L = 128, 256, 512, 1024$, and plot your estimates of $\lambda_c(L)$ against $1/L$. Extrapolate to $1/L \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$. This approach is called **finite size scaling**, in order to correct for systematic **finite size effects** which influence the critical value.

(b) Let T be the hitting time of state $\boldsymbol{\eta} = \mathbf{0}$, i.e. the lifetime of the infection. Measure the lifetime of the infection for $\lambda = 1$ and $\lambda = 2$ by running the process until extinction of the epidemic.
For $\lambda = 1 < \lambda_c$ we expect $T \propto C \log L$+small fluctuations for some $C > 0$. So use large system sizes e.g. $L = 128, 256, 512, 1024$ (or larger), confirm that $\mathbb{E}(T)$ scales like $\log L$ and determine C by averaging at least 200 realizations of T for each L. Then shift your data T_i for each L by $T_i - \mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the shifted data (use log-scale on the y-axis).
For $\lambda = 2 > \lambda_c$ we expect $T \sim Exp(1/\mu)$ to be an exponential random variable with mean $\mu = \mathbb{E}(T) \propto e^{CL}$ for some $C > 0$. So use *small* system sizes e.g. $L = 8, 10, 12, 14$ (see how far you can go), confirm that $\mathbb{E}(T)$ scales like e^{CL} and determine C by averaging at least 200 realizations of T. Then rescale your data T_i for each L by $T_i/\mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the rescaled data (use log-scale on the y-axis).
The **empirical tail** of data $T = (T_1, \ldots, T_M)$ is the statistic $tail_t(T) = \frac{1}{M}\sum_{i=1}^{M} \mathbb{1}_{T_i > t}$. This decays from 1 to 0 as a (random) function of time t.
BINNING DESTROYS INFORMATION $\to$ AVOID HISTOGRAMS!

(c)* Repeat the analysis of (a) on the fully connected graph Λ_L, and compare your estimate of λ_c with the mean-field prediction from Q2.2.

(d)* For $\lambda = 0$ and $\eta_0(x) = 1$ for all $x \in \Lambda$, derive a formula for the distribution of the lifetime T of the infection. (Hint: google 'extreme value statistics' and 'Gumbel distribution'.)