Stochastic Models of Complex Systems

Problem sheet 3

Sheet counts $35/100$ homework marks, [x] indicates weight of the question.
* Questions do not enter the mark.

3.1 Moran model [10]

We consider the Moran model in continuous time, which is a simple model for evolution:
In a population of size N each individuum can be of type A or B. Each individuum independently reproduces at rate 1 passing on its type to the offspring. When this happens, one of the now $N+1$ individuals is chosen uniformly at random and dies instantaneously, to keep the population size constant to N.
Let X_t be the number of type A individuals at time t. Then $X = (X_t : t \geq 0)$ is a continuous-time Markov chain with state space $\{0, \ldots, N\}$.

(a) Give transition rates of X, write down the generator $(\mathcal{L}_N g)(n)$ and the master equation. Is X ergodic? Does it have absorbing states? Give all stationary distributions.

(b) Consider the rescaled process $X_t/N \in [0,1]$ and Taylor expand the generator $(\mathcal{L}_N g)(x)$ up to second order in $x \in [0,1]$.

(c) Rescale time appropriately ($t \mapsto t\,\Delta t$) and derive the generator $(\mathcal{L}g)(y)$ of the limiting process $(Y_t : t \geq 0)$ where $X_{t\Delta t}/N \to Y_t$ as $N \to \infty$. Show that it fulfills the FPE

$$\frac{\partial}{\partial t} f(t,y) = \frac{\partial^2}{\partial y^2}\Big(y(1-y) f(t,y) \Big) \quad \Big(= (\mathcal{L}^* f)(t,y) \Big) .$$

How are Δt and N related?

(d) The limiting process from (c) is called **Wright-Fisher diffusion**.
Show that $\mathbb{E}(Y_t) = \mathbb{E}(Y_0)$ for all $t > 0$, that $(Y_t : t \geq 0)$ is a martingale and derive the limit of Y_t as $t \to \infty$.

3.2 Kingman's coalescent [10]

Consider a system of N well mixed, coalescing particles. Each of the $\binom{N}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of N individuals in a population model, tracing back to a single common ancestor.

(a) Let N_t be the number of particles at time t. Give the transition rates of the process $(N_t : t \geq 0)$ on the state space $\{1, \ldots, N\}$ and write down the generator $(\mathcal{L}_N f)(n)$ for $n \in \{1, \ldots, N\}$ and the master equation.
Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

(b) Show that the mean time to absorption is given by $\mathbb{E}(T) = 2\big(1 - \frac{1}{N}\big)$.

(c) Write down the generator of the rescaled process N_t/N and do a Taylor expansion up to second order.

Show that the slowed down, rescaled process $X_t^N := \frac{1}{N} N_{t/N} \to X_t$ converges to the process $(X_t : t \geq 0)$ with generator

$$\mathcal{L}_X f(x) = -\frac{x^2}{2} f'(x) \quad \text{and state space} \quad (0,1] \quad \text{with} \quad X_0 = 1 \,.$$

Convince yourself that this process is 'deterministic', i.e. $X_t = \mathbb{E}(X_t)$ for all $t \geq 0$, and compute X_t explicitly. How is your result compatible with the result from (b)?

(d) Show that the fluctuations to leading order as $N \to \infty$ are given by

$$X_t^N = X_t + \Xi_{t/N} + o(1/N) \,,$$

where $(\Xi_t : t \geq 0)$ is a diffusion process with generator

$$\mathcal{L}_\Xi f(\xi) = \frac{1}{2}(\xi + X_t)\, f'(\xi) + \frac{1}{4}(\xi + X_t)^2 f''(\xi) \quad \text{and} \quad \Xi_0 = 0 \,.$$

HINT: Use that for the independent processes $Y_t := X_t^N$ and X_t with generators $\mathcal{L}_Y$ and $\mathcal{L}_X$, the generator $\mathcal{L}_\Xi$ of the difference $X_t^N - X_t$ is given by

$$\mathcal{L}_\Xi f(y-x) = (\mathcal{L}_Y f)(y-x) - (\mathcal{L}_X f)(y-x) \,,$$

so that the leading order term cancels.

(e) Compute the expected size of the correction $\mathbb{E}(\Xi_{t/N})$ explicitly. What is the leading order behaviour as $N \to \infty$?

3.3 Simple traffic model [15]

A simple sample code is on the course webpage (to be adapted).
Simulate a totally asymmetric exclusion process with periodic boundary conditions on the lattice $\Lambda = \{1, \ldots, L\}$ with uniform initial condition. The jump rates should depend on the neighbourhood configuration in the following way:

$$0100 \xrightarrow{1} 0010 \,, \quad 1101 \xrightarrow{\alpha} 1011 \,, \quad 0101 \xrightarrow{\beta} 0011 \,, \quad 1100 \xrightarrow{\gamma} 1010 \,.$$

For this model, the average stationary current is a function of the number N of particles (cars), or the density $\rho = N/L$. It is defined by $j(\rho) = \mathbb{E}\big(c(\eta, \eta^{x,x+1})\big)$, where $c(\eta, \eta^{x,x+1})$ is the jump rate of a particle from x to $x+1$ as given above.

(a) Measure the **fundamental diagram**, i.e. $j(\rho)$ as a function of the density. The easiest way is to just count all jumps up to a given time and normalize properly.
For fixed lattice size L (e.g. 500) vary N to get j for $\rho = 0, 0.1, \ldots, 0.9, 1$. Do this for $\alpha = \beta = \gamma = 1$ (usual TASEP) and at least three other choices of rates.
Interpret your choices in terms of driver behaviour if used as a traffic model.

(b) Calculate the stationary current using a mean-field approximation for the parameters you chose in (a), and compare this to your measurement results in a plot. Discuss how the different shapes of the fundamental diagram are related to your choice of rates.

(c) For $\alpha = \beta = \gamma = 1$ initialize the system of size $L = 500$ with 250 particles in the left half. Output the full configuration in regular time intervals in one file. Visualize the time evolution (e.g. by using e.g. *imagesc* in MATLAB) for a single realization, and after averaging 200 realizations.
NEW: Repeat the same initializing with densities ρ_l in the left and ρ_r in the right half of the system. Use the pairs $(\rho_l, \rho_r) = (0.8, 0.2)$ and $(0.8, 0.4)$.