Chapter 2

Partial Differential Equations (PDE's)

2.1 Introduction to PDE's and their Mathematical Classification

The function to be determined, $v(x,t)$, is now a function of several variables (2 for us). The choice of notation suggests that one of these variables should be interpreted as a spatial variable and the other as a temporal variable. This is often, but certainly not always, the case in applications. A partial differential equation (PDE) relates partial derivatives of v. Many modelling problems lead to first or second order PDEs. We shall deal only with these two cases.

- **First order PDEs:**
  We shall consider first order pdes of the form

$$a(v,x,t)\frac{\partial v}{\partial t} + b(v,x,t)\frac{\partial x}{\partial t} = c(v,x,t). \tag{2.1}$$

  This is called a *quasi-linear* equation because, although the functions a,b and c can be nonlinear, there are no powers of partial derivatives of v higher than 1.

- **General second order linear PDE:**
  A general second order linear PDE takes the form

$$A\frac{\partial^2 v}{\partial t^2} + 2B\frac{\partial^2 v}{\partial x \partial t} + C\frac{\partial^2 v}{\partial x^2} + D\frac{\partial v}{\partial t} + E\frac{\partial v}{\partial x} + Fv + G = 0, \tag{2.2}$$

  where the coefficients, A to G are generally functions of x and t.

2.1.1 Classification of Second Order PDEs

Linear second order PDE's are grouped into three classes - elliptic, parabolic and hyperbolic - according to the following:

- $B^2 - 4AC < 0$: elliptic ("equilibrium")
- $B^2 - 4AC = 0$: parabolic ("diffusive propagation")
- $B^2 - 4AC > 0$: hyperbolic ("wave propagation")

Unlike for ODE's there are no general methods for solving PDEs. Identifying the class of problem will guide you in your choice of appropriate tools for solving it. This classification has limited usefulness in practice. General PDEs, particularly nonlinear ones, may not fall into any of these classes. Among linear second order PDEs, there are archetypal examples of each category of equation:

- **Poisson Equation (Elliptic)**
  Given $\rho(x,y)$, find $v(x,y)$ satisfying

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \rho(x,y) \tag{2.3}$$

in some region, $\Gamma \in \mathbb{R}^2$. Usually posed as a boundary value problem with behaviour specified on the boundary of Γ.

- **Diffusion Equation (Parabolic)**
  Find $v(x, t)$ satisfying
  $$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} \tag{2.4}$$
  in the domain $(x, t) \in [x_L, x_R] \times [0, \infty)$. This is solved as an initial value problem with initial data $v(x, 0) = V(x)$ and boundary conditions specified at $x = x_L$ and $x = x_R$.

- **Wave Equation (Hyperbolic)**
  Find $v(x, t)$ satisfying
  $$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2} \tag{2.5}$$
  in the domain $(x, t) \in [x_L, x_R] \times [0, \infty)$. This is also solved as an initial value problem with initial data $v(x, 0) = V(x)$ and boundary conditions specified at $x = x_L$ and $x = x_R$.

Here are some other examples which arise in applications. Some fit into the above classification and some don't:

- **The Advection Equation:**
  $$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0. \tag{2.6}$$

- **The Inviscid Burgers Equation:**
  $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0. \tag{2.7}$$

- **Burgers Equation:**
  $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}. \tag{2.8}$$

- **The Telegraph Equation:**
  $$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2} - k\frac{\partial v}{\partial t} - bv. \tag{2.9}$$

- **The Black-Scholes Equation:**
  $$\frac{\partial v}{\partial t} + rx\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} - rv = 0. \tag{2.10}$$

2.1.2 Types of Boundary Conditions

Determining the correct boundary conditions to impose on the solution of a PDE are as important a part of the modelling process as the derivation of the PDE itself. Some of the most common boundary conditions which arise in practice have names. Here we have in mind parabolic and hyperbolic problems in one spatial dimension having spatial domain $[x_L, x_R]$ (x_L and/or x_R could be ∞) where the boundary conditions should be imposed at $x = x_L$ and $x = x_R$. For elliptic problems, the boundary conditions should be specified along a line in the $x - y$ plane.

1. **Dirichlet Boundary Conditions:**
   The behaviour of $v(x, t)$ itself is specified on the boundaries:
   $$v(x_L, t) = \mathcal{V}_L(t) \qquad v(x_R, t) = \mathcal{V}_R(t). \tag{2.11}$$

2. **Neumann Boundary Conditions:**
   The behaviour of the partial derivative of $v(x,t)$ is specified on the boundaries:

$$\frac{\partial v}{\partial x}(x_L, t) = \mathcal{V}_L(t) \qquad \frac{\partial v}{\partial x}(x_R, t) = \mathcal{V}_R(t). \tag{2.12}$$

3. **Periodic Boundary Conditions:**
   The boundaries "wrap around" so that the solution satisfies:

$$v(x + L, t) = v(x, t) \tag{2.13}$$

   where $L = x_R - x_L$ is the size of the domain.

4. **Boundary Conditions at Infinity**
   In some problems involving infinite domains, it is sufficient to provide the asymptotic behaviour of the solution as $x \to \infty$. For example, $v(x,t) \to 0$ as $x \to \pm\infty$.

2.2 Non-Dimensionalisation

Modelling applications almost always produce PDEs where the quatities involved have actual physical units (dimensions). Mathematically we *never* want to work with dimensional quantities since functions of dimensional quantities make no sense: if x is a length, measured in metres, then e^x is meaningless. When presented with a PDE coming from a modelling problem, it is important to be able to regroup all dimensional quantities into dimensionless combinations thus identifying the essential combinations of parameters which control the solution. This process, known as *nondimensionalisation*, is almost entirely trivial from a theoretical point of view and and hence is usually neglected in textbooks.

There is a tremendous temptation, theoretically, to "choose units" in which various parameters in a problem are equal to one so as to simplify formulae and get going with the "real" work of solving the problem mathematically. Often this replaces a proper consideration of how to nondimensionalise the problem. The result is then that you may find it difficult to restore the original units. Nothing aggravates experimentalists more than to be presented with a theoretical curve without units!

Here is a simple example from physics which, from a mathematician's point of view, is littered with annoying physical units. A rod of length 2.0 m has the following temperature profile initially:

$$v(x,0) = \begin{cases} 10\,K & 0 < x < 1.0\,m \\ 0\,K & 1.0\,m < 2.0\,m \end{cases}$$

The left end is maintained at a constant temperature of 10 K and the right end is maintained at 0 K. If thermal diffusivity of the material is $\pi \times 10^{-2}\ m^2 s^{-1}$, what is the temperature profile in the rod after 25 s?

This problem is described by the diffusion equation, Eq. (2.4), on the interval $[0, 2.0]$ with $D = \pi \times 10^{-2}$ and an initial condition as described above. We want to know what is the solution at $t = 25$ s. However we never want to work with such dimensional quantities. Let us denote dimensional variables with overlines: $\bar{x}$, $\bar{v}$, $\bar{t}$ etc. The dimensional equation is

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \bar{D} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2}. \tag{2.14}$$

Let us introduce non-dimensional variables, x, t, v defined by:

$$\begin{aligned} \bar{x} &= x\,L \quad && L = \text{a characteristic length} \\ \bar{t} &= t\,T \quad && T = \text{a characteristic time} \\ \bar{v} &= V\,v \quad && V = \text{a characteristic temperature} . \end{aligned} \tag{2.15}$$

Eq. (2.14) then trivially leads to

$$\frac{\partial v}{\partial t} = \left(\frac{\bar{D}T}{L^2}\right)\frac{\partial^2 v}{\partial x^2}.$$

If we define the non-dimensional diffusivity,

$$D = \frac{\bar{D}T}{L^2}$$

then we are arrive at a version of the diffusion equation,

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2}, \tag{2.16}$$

where everything is dimensionless. This is the equation we would like to handle mathematically or numerically. We are free to choose L, T and V in Eqs. (2.15). Appropriate choices are suggested by the problem (not the only possibility):

$$\begin{aligned} V &= 10.0\,K \\ L &= 2.0\,m \\ T &= 25.0\,s. \end{aligned}$$

With these choices we then required to solve the nondimensional equation, Eq.(2.14), on the interval $x \in [0, 1]$, with $D = \frac{\pi\times10^{-2}\times25}{2^2} = 0.196$ and initial condition

$$v(x, 0) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & \frac{1}{2} \leq x < 1 \end{cases}$$

and find the solution at $t = 1$. Once we have done this, we use Eqs. (2.15) to restore the original units.

We now get to the business of solving some PDEs. As is the case for ODEs, some are exactly solvable by various analytic techniques but the vast majority are not. One could devote an entire course to the analytic tools which have been developed for certain PDEs. We shall not touch on this huge subject here and focus on numerical techniques which are appropriate for "typical" PDEs - those which are not analytically solvable. We shall start by looking at situations where PDEs can be reduced to ODEs which may then be tackled using the methods which we have already discussed in Chap 1. The first such situation occurs for first order PDEs which may be reduced to an ODE using the Method of Characteristics. The second such situation occurs when some symmetry of a problem means that the independent variables enter the solution only in a certain combination. This give rise to a class of solutions known as similarity solutions.

2.3 First Order PDEs: Method of Characteristics

Reminder: Geometry of Curves and Surfaces in $\mathbb{R}^3$:
Let us label the coordinates in $\mathbb{R}^3$ by (x, t, z) instead of the usual (x, y, z) to adapt our notation to the PDE application below.

- A function,
$$\mathcal{C} \;:\; \mathbb{R} \to \mathbb{R}^3 \;:\; s \to (x(s), t(s), z(s)),$$
defines a curve in $\mathbb{R}^3$.
- Given a curve, $\mathcal{C}$, the 3-vector, $\mathbf{T}(s)$ defined by
$$\mathbf{T}(s) = \left(\frac{dx}{ds}, \frac{dt}{ds}, \frac{dz}{ds}\right) \tag{2.17}$$
is tangent to $\mathcal{C}$ at the point $\mathcal{C}(s)$.
- A function of two variables, $v(x, t)$, can be used to define a surface, $\mathcal{S}$, in $\mathbb{R}^3$ as follows:
$$\mathcal{S} \;:\; \mathbb{R}^2 \to \mathbb{R}^3 \;:\; (x, t) \to (x, t, v(x, t)). \tag{2.18}$$
- The 3-vector, $\mathbf{N}(x, t)$, defined by
$$\mathbf{N}(x, t) = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, -1\right), \tag{2.19}$$
is normal to the surface $\mathcal{S}$ at the point $\mathcal{S}(x, t)$.
- Any 3-vector, $\mathbf{T}$ satisfying
$$\mathbf{T} \cdot \mathbf{N}(x, t) = 0$$
is in the tangent plane to the surface, $\mathcal{S}$, at the point $\mathcal{S}(x, t)$.

Consider the first order quasi-linear PDE
$$a(x, t, v)\frac{\partial v}{\partial x} + b(x, t, v)\frac{\partial v}{\partial t} = c(x, t, v). \tag{2.20}$$
given "initial data"
$$v(x(\sigma), t(\sigma)) = \mathcal{V}(\sigma), \tag{2.21}$$
along a curve, $\Gamma(\sigma) = (x(\sigma), t(\sigma))$ in the (x, t) in plane. If we could find the solution of Eq. (2.20), $v(x, t)$, then it would define a surface $\mathcal{S}(x, t)$ as in Eq. (2.18). Since the solution satisfies the boundary condition, the surface, $\mathcal{S}$ includes the curve $\mathcal{C}(\sigma) \;:\; \sigma \to (x(\sigma), t(\sigma), \mathcal{V}(\sigma))$. Therefore, iIf we can construct the surface, $\mathcal{S}$, starting from the curve, $\mathcal{C}(\sigma)$, then we solve Eq. (2.20).

The normal to $\mathcal{S}$ at the point (x,t) is
$$\mathbf{N}(x, t) = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, -1\right). \tag{2.22}$$
Now consider the vector,
$$\mathbf{T}(x, t) = (a(x, t, v(x, t)),\; b(x, t, v(x, t)),\; c(x, t, v(x, t)))\,. \tag{2.23}$$
Then
$$\begin{aligned}\mathbf{N} \cdot \mathbf{T} &= a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial t} - c\\ &= 0 \qquad\qquad \text{using Eq. (2.20).}\end{aligned}$$

Thus the vector $\mathbf{T}$ defined as in Eq. (2.23) is always tangent to the solution surface, $\mathcal{S}$. Here's the key to the Method of Characteristics: if we define a curve, $\mathcal{C}(s)$ in $\mathbb{R}^3$ by the equation

$$\frac{d\mathbf{x}}{ds} = \mathbf{T}(\mathbf{x}(s)). \tag{2.24}$$

then the tangent to this curve at a given point is $\mathbf{T}$ (by Eq. (2.17)). Thus if the $\mathbf{x}$ starts on the solution surface, $\mathcal{S}$ then it stays there. If we start such a curve from each point on the boundary curve, $\mathcal{C}(\sigma)$ then the union of all of these curves is the solution surface. There is a mathematical caveat here: the curve, $\mathcal{C}(\sigma)$, defined by the boundary data must be nowhere parallel to $\mathbf{T}(x(\sigma), t(\sigma))$. Boundary data satisfying this condition is called characteristic data.

This is somewhat abstract discussion shows that in principle one can construct the solution surface from the boundary data provided it is characteristic. In fact, we have a concrete method of computing the solution. Let us suppose, for concreteness that the boundary data is space-like: $\mathcal{C}(X) = (X, 0, V(X))$ so that it corresponds to an honest initial condition. The Method of Characteristics requires that we solve the equations

$$\begin{aligned}
\frac{dx}{ds} &= a(x(s), t(s), z(s)) \quad x(0) = X \\
\frac{dt}{ds} &= b(x(s), t(s), z(s)) \quad t(0) = 0 \\
\frac{dz}{ds} &= c(x(s), t(s), z(s)) \quad z(0) = V(X).
\end{aligned} \tag{2.25}$$

The curve in $\mathbb{R}^2$ given by $(x(s), t(s)$ emanating from the boundary point $(X, 0)$ is called a *characteristic curve*. Solving the characteristic equations, Eqs. (2.25), propagates the initial data along the characteristic curves. To find the solution at time T, we take a discrete set of initial values of x and for each such point, we integrate Eqs. (2.25) until $t(s) = T$. Examples of the results obtained by doing this are shown in Figs. 2.1, 2.2 and 2.3.

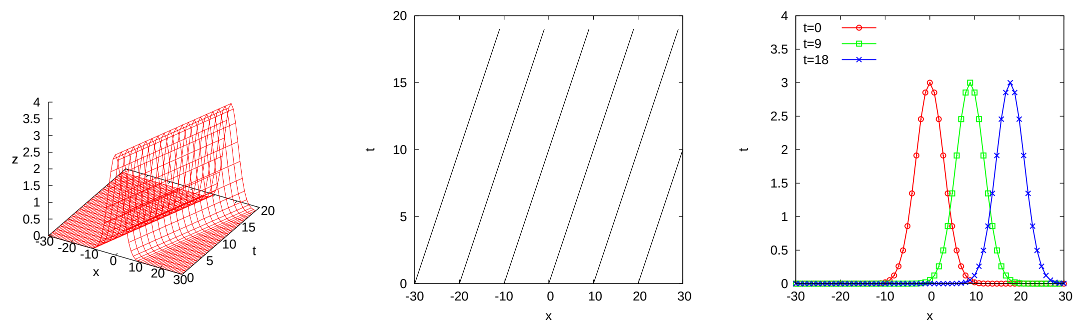

Figure 2.1: Solution of Eq. (2.26) over the time interval $[0 : 20]$ with $V(x) = 3.0\exp(-0.1 * x^2)$. Shown are the global solution, the characteristic curves in the $x - t$ plane and some snapshots of $v(x, t)$ at different times.

Example 8: **The Advection Equation**
Solve the equation,

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0. \tag{2.26}$$

with initial data $v(x, 0) = V(x)$. Comparing with Eq. (2.20),

$$\begin{aligned} a(x, t, v) &= c \\ b(x, t, v) &= 1 \\ c(x, t.v) &= 0. \end{aligned}$$

From Eq. (2.25), the characteristic curve emanating from $(X, 0)$ satisfies

$$\begin{aligned} \frac{dx}{ds} &= c \quad x(0) = X \\ \frac{dt}{ds} &= 1 \quad t(0) = 0 \\ \frac{dz}{ds} &= 0 \quad z(0) = V(X). \end{aligned} \tag{2.27}$$

These are easily solved:

$$\begin{aligned} x(s) &= cs + X \\ t(s) &= s \\ z(s) &= V(X). \end{aligned}$$

Eliminating s in favour of t, the characteristic curve emanating from $(X, 0)$ is $x = ct + X$. On this curve, $v(x, t) = V(X)$. Putting these together, the solution is given by

$$v(x, t) = V(x - ct). \tag{2.28}$$

Thus the initial data simply propagates along the characteristic curves. The numerical solution is shown in Fig. 2.1.

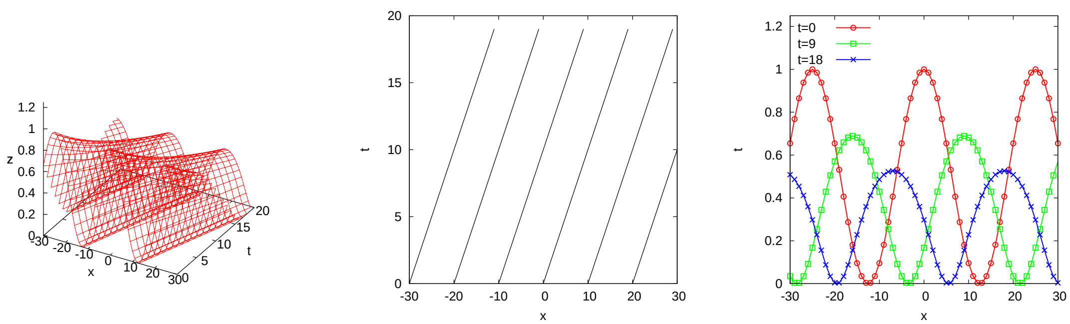

Figure 2.2: Solution of Eq. (2.29) over the time interval $[0:20]$ with $V(x) = \cos^2(\frac{4*\pi x}{x_R - x_L})$. Shown are the global solution, the characteristic curves in the $x-t$ plane and some snapshots of $v(x,t)$ at different times.

Example 9**: A Nonlinear Advection–Reaction Equation**
Solve the equation,

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = -\lambda v^2. \tag{2.29}$$

with initial data $v(x,0) = \cos^2(x)$. The characteristic curve emanating from $(X,0)$ satisfies

$$\begin{array}{lll} \frac{dx}{ds} & = c & x(0) = X \\ \frac{dt}{ds} & = 1 & t(0) = 0 \\ \frac{dz}{ds} & -\lambda z^2 & z(0) = \cos^2(X). \end{array} \tag{2.30}$$

Solving these equations gives:

$$\begin{aligned} x(s) &= cs + X \\ t(s) &= s \\ z(s) &= \frac{\cos^2(X)}{1 + \lambda t \cos^2(X)}. \end{aligned}$$

Eliminating s in favour of t, the characteristic curve emanating from $(X,0)$ is again $x = ct + X$. The solution is given by $\frac{\cos^2(X)}{1+\lambda t \cos^2(X)}$.

$$v(x,t) = \frac{\cos^2(x-ct)}{1 + \lambda t \cos^2(x-ct)} \tag{2.31}$$

Provided $\lambda > 0$, the initial data propagates along the characteristic curves as shown in Fig 2.2. If $\lambda < 0$, the initial data grows and forms a singularity at time $t^* = 1/\lambda$.

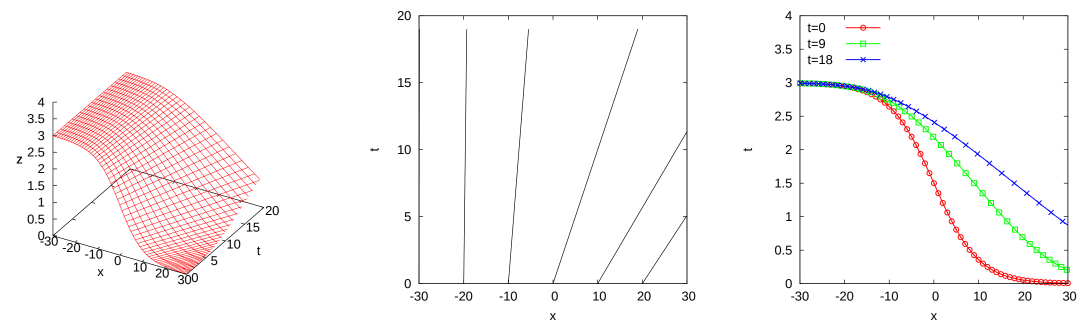

Figure 2.3: Solution of Eq. (2.32) over the time interval $[0:20]$ with $V(x) = 1.5 * (1 - \tanh(0.1 * x))$. Shown are the global solution, the characteristic curves in the $x - t$ plane and some snapshots of $v(x,t)$ at different times.

Example 10: **An Advection Equation with Density Dependent Speed**
Solve the equation,

$$\frac{\partial v}{\partial t} + c(v)\frac{\partial v}{\partial x} = 0. \tag{2.32}$$

with initial data $v(x,0) = V(x)$. The characteristic curve emanating from $(X,0)$ satisfies

$$\begin{aligned} \frac{dx}{ds} &= c(z) \quad x(0) = X \\ \frac{dt}{ds} &= 1 \quad t(0) = 0 \\ \frac{dz}{ds} & \lambda 0 \quad z(0) = V(X). \end{aligned} \tag{2.33}$$

We can easily integrate the last two equations:

$$\begin{aligned} t(s) &= s \\ z(s) &= V(X). \end{aligned}$$

We find that z is constant along the characteristic curve. Thus we can integrate the first equation to obtain

$$x(s) = c(V(X))s + X.$$

Eliminating s in favour of t, the characteristic curve emanating from $(X,0)$ is therefore $x = c(V(X))t + X$. On this curve, $v(x,t) =$. The solution is given by

$$v(x,t) = V(x - c(V(X))t. \tag{2.34}$$

The characteristic curves are again straight lines but the slope, $\frac{dt}{dx} = 1/c(V(X))$ depends on the initial point. We cannot say more without specifying $c(v)$. Numerical results are shown in Fig. 2.3 for

$$c(v) = 2.0(1 - \frac{v}{3}). \tag{2.35}$$

2.4 Similarity Solutions and Travelling Waves

Another way to reduce PDEs to ODEs is to look for solutions where the independent variables, x and t only enter the solution in some combination, ξ. If this occurs, then the equation written in terms of ξ is ordinary. We have already seen this happen ina simple way for the advection equation

2.4.1 Travelling waves

A travelling wave solution is of the form

$$v(x,t) = F(\xi) \qquad \text{where } \xi = x - u\,t. \tag{2.36}$$

for some u. If $u > 0$, this corresponds to a solution with a spatial profile described by $F(\xi)$ which translates to the right in time at a speed u. If $u < 0$, the profile translates to the left. We usually require $F(\xi)$ to be bounded. The issue is for what values of u, if any, is such a solution possible.

Travelling wave solution of the advection equation
Consider again the advection equation, Eq. (2.26), which we solved in Example 8 using the Method of Characteristics. Let us seek a solution in the form of Eq. (2.36). By the Chain Rule:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{dF}{d\xi}\frac{\partial \xi}{\partial t} = -u\,\frac{dF}{d\xi}\\ \frac{\partial v}{\partial x} &= \frac{dF}{d\xi}\frac{\partial \xi}{\partial x} = \frac{dF}{d\xi}.\end{aligned}$$

Eq. (2.26) then reduces to

$$(-u + c)F'(\xi) = 0.$$

Thus Eq. (2.36) solves Eq. (2.26) for *any* $F(\xi)$ provided we choose $u = c$. By choosing $F = V$, we match the initial data and obtain the solution

$$v(x,t) = V(x - c\,t),$$

as in Example 8.

Travelling wave solution of the linear wave equation
Above is the simplest example of a travelling wave solution of a PDE. A slightly more complicated example is the linear wave equation, Eq. (2.5). Let us seek a solution in the form of Eq. (2.36). Two applications of the chain rule gives

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2} &= u^2\,\frac{d^2 F}{d\xi^2}\\ \frac{\partial^2 v}{\partial x^2} &= \frac{d^2 F}{d\xi^2},\end{aligned}$$

so that Eq. (2.5) reduces to

$$(u^2 - c^2)\,F''(\xi) = 0.$$

Thus Eq. (2.36) solves Eq. (2.5) for *any* $F(\xi)$ provided we choose $u = \pm c$. We have two possible wave solutions, one travelling to the left and the other to the right. Eq. (2.5) is linear so the general solution involves both. One way to match the initial data is to choose

$$v(x,t) = \frac{1}{2}\left[V(x - c\,t) + V(x + c\,t)\right].$$

Travelling wave solutions of linear hyperbolic equations have a propagation speed determined from the equation and have a lot of freedom in choosing their shape. This allows us to match the initial data. In general, however, there is not really an "initial" condition for a travelling wave - it propagates from $t = -\infty$ to $t = +\infty$ over the whole real line. Travelling wave solutions of nonlinear equations, when they exist, are much more special as the following example shows.

A nonlinear travelling wave: Burgers' Equation

Consider Burgers' Equation, Eq. (2.8) with the boundary conditions at infinty,

$$\begin{aligned} v(x,t) &\rightarrow v_L \qquad x \rightarrow -\infty \\ v(x,t) &\rightarrow v_R \qquad x \rightarrow \infty. \end{aligned}$$

Assuming a solution in the form of Eq. (2.36) and using the Chain Rule reduces the equation to the following:

$$\begin{aligned} -uF' &+ F\,F' - \nu F'' = 0 \\ F(\xi) &\rightarrow v_L \qquad \xi \rightarrow -\infty \\ F(\xi) &\rightarrow v_R \qquad \xi \rightarrow \infty. \end{aligned} \tag{2.37}$$

Unlike in the previous linear problems, we now have an honest ODE to solve and the wave speed is to be determined as part of the solution. Eq. (2.37) is not an IVP like those we studied in Chapter 1. Rather it is a nonlinear eigenvalue problem - it is not obvious that there are values of u for which a trajectory leaving v_L at $\xi = -\infty$ will tend to v_R as $\xi \rightarrow \infty$. Determining whether such trajectories are possible is best done using the Dynamical Systems techniques which you learned in CO903 (this is a much easier example than the ones you looked at there!).

We can proceed a little further analytically before resorting to qualitative or numerical techniques. Observing that $\frac{d}{d\xi}\left(\frac{1}{2}\,F^2\right) = F\,\frac{dF}{d\xi}$ allows us to write Eq. (2.37) in a form which can be integrated once:

$$\begin{aligned} &\frac{d}{d\xi}\left[-uF + \frac{1}{2}F^2 - \nu F'\right] = 0 \\ \Rightarrow\quad & -uF + \frac{1}{2}F^2 - \nu F' = c_1 \\ \Rightarrow\quad & 2\nu\frac{dF}{d\xi} = F^2 - 2uF - 2c_1 \\ \Rightarrow\quad & 2\nu\frac{dF}{d\xi} = (F - F_+)(F - F_-) \end{aligned} \tag{2.38}$$

where

$$F_\pm = u \pm \sqrt{u^2 + 2c_1}.$$

Let us rescale, $\xi = 2\nu\xi'$, to remove ν:

$$\frac{dF}{d\xi} = (F - F_+)(F - F_-) \tag{2.39}$$

The fixed points are at $F = F_\pm$. Looking at the phase portrait, we can see the form of the solution:

The travelling wave solution is a heteroclinic orbit connecting the unstable fixed point at F_+ to the stable fixed point at F_-. It is now clear that we need to choose u and c_1 so that $F_+ = v_L$ and $F_- = v_R$. These conditions lead us to choose

$$\begin{aligned} u &= \frac{v_L + v_R}{2} \\ c &= -\frac{v_L v_R}{2}. \end{aligned} \tag{2.40}$$

Note that, unlike in the linear case, the propagation speed is determined by the boundary conditions, not by the equation itself. In the case of Burgers' equation, it is possible to solve Eq. (2.39) exactly

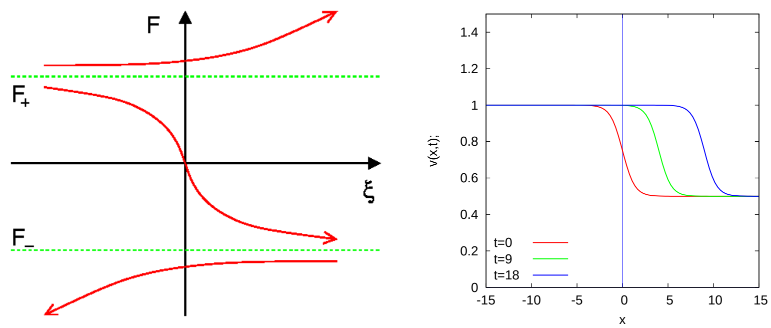

Figure 2.4: Phase portrait for Eq. (2.38) and plots of the resulting travelling wave for $v_R = 1.0$ and $v_L = 0.5$..

(the equation is separable):

$$
\begin{aligned}
& \frac{dF}{(F - v_L)(F - v_R)} = d\xi' \\
\Rightarrow\ & \left(-\frac{1}{(v_L - v_R)(F - v_R)} + \frac{1}{(v_L - v_R)(F - v_L)} \right) dF = d\xi' \\
\Rightarrow\ & \left(\frac{1}{(F - v_R)} - \frac{1}{(F - v_L)} \right) dF = -(v_L - v_R)\, d\xi' \\
\Rightarrow\ & \log\left(\frac{F - v_R}{F - v_L} \right) = -(v_L - v_R)\, \xi' + \log(c_2) \\
\Rightarrow\ & \frac{F - v_R}{F - v_L} = c_2\, \mathrm{e}^{-(v_L - v_R)\, \xi'} \\
\Rightarrow\ & F(\xi') = \frac{v_R - v_L\, c_2\, \mathrm{e}^{-(v_L - v_R)\, \xi'}}{1 - c_2\, \mathrm{e}^{-(v_L - v_R)\, \xi'}}
\end{aligned}
$$

If we now impose the condition that the trajectory should pass through the centre point, $F(0) = \frac{v_L + v_R}{2}$, then we are led to choose $c_2 = -1$ and restoring $\xi = \frac{\xi'}{2\nu}$ we obtain the travelling wave profile

$$
F(\xi) = \frac{v_R + v_L\, \mathrm{e}^{-\alpha \xi}}{1 + \mathrm{e}^{-\alpha \xi}} \qquad \text{with } \alpha = \frac{v_L - v_R}{2\nu}. \tag{2.41}
$$

In general, we are not so lucky and the ODE describing a travelling wave must be solved numerically. Note that for the nonlinear travelling wave, there is no freedom to adjust the solution to fit particular initial data.

2.4.2 Similarity Solutions

Travelling waves are solutions of PDEs which are invariant under translation in space. A related concept is a similarity solution which is invariant under a rescaling of space and time. As was the case for travelling waves, only certain equations admit such solutions, and even for those that do, they are only realisable when the initial and boundary conditions are compatible with the rescaling symmetry. The standard strategy is to seek a solution of the form

$$
v(x, t) = t^a\, F(\xi) \qquad \text{where } \xi = x t^b. \tag{2.42}
$$

for some exponents a and b which should be determined from the equation.

Let us consider the diffusion equation as an example:

$$
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} \tag{2.43}
$$

Look for a solution of the form Eq. (2.42). Then

$$\begin{aligned}
\frac{\partial v}{\partial t} &= at^{a-1}F(\xi) + t^a \frac{dF}{d\xi}\frac{d\xi}{dt} \\
&= t^{a-1}\left(aF + b\xi F'\right) \\
\frac{\partial v}{\partial x} &= t^a \frac{dF}{d\xi}\frac{d\xi}{dx} \\
&= t^{a+b}F' \\
\frac{\partial^2 v}{\partial x^2} &= t^{a+2b}F''
\end{aligned}$$

Comparing the left and right sides of the equation, for consistency we must choose

$$a - 1 = a + 2b.$$

Thus we must choose $b = -\frac{1}{2}$. a remains arbitrary and the profile of the similarity solution is determined from by solving

$$aF - \frac{1}{2}\xi F' - DF'' = 0.$$

We can hide the dependence on D by rescaling $\xi \to \sqrt{D}\xi'$ and dropping the primes immediately:

$$aF - \frac{1}{2}\xi F' - F'' = 0. \tag{2.44}$$

This is tricky to solve for general a. We shall look at a couple of special cases:

1. **A diffusing interface:** $a = 0$

   We can use an integrating factor, $\mathrm{e}^{\frac{\xi^2}{4}}$:

$$\begin{aligned}
& \frac{1}{2}\xi F' + F'' = 0 \\
\Rightarrow\ & \mathrm{e}^{\frac{\xi^2}{4}}\xi F' + \mathrm{e}^{\frac{\xi^2}{4}}F'' = 0 \\
\Rightarrow\ & \frac{d}{d\xi}\left[\mathrm{e}^{\frac{\xi^2}{4}}F'\right] = 0 \\
\Rightarrow\ & F'(\xi) = c_1 \mathrm{e}^{-\frac{\xi^2}{4}} \\
\Rightarrow\ & F(\xi) = c_1 \int_{-\infty}^{\xi} \mathrm{e}^{-\frac{\eta^2}{4}}\, d\eta + c_2
\end{aligned}$$

   If we recall the definition of $\mathrm{Erf}(x)$:

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\, dy,$$

   we obtain

$$F(\xi) = c_1\sqrt{\pi}\,\mathrm{Erf}\left(\frac{\xi}{2}\right) + c_2. \tag{2.45}$$

   Restoring $\xi = \sqrt{D}\xi'$ and recalling that $\xi = \frac{x}{\sqrt{t}}$ we finally get

$$v(x,t) = \sqrt{\pi}c_1 \mathrm{Erf}\left(\frac{x}{\sqrt{4Dt}}\right) + c_2. \tag{2.46}$$

   We can now choose c_1 and c_2 such that

$$\begin{aligned}
v(x,t) &\to v_L \qquad x \to -\infty \\
v(x,t) &\to v_R \qquad x \to \infty :
\end{aligned}$$

$$v(x,t) = \frac{v_L + v_R}{2} - \left(\frac{v_L - v_R}{2}\right)\mathrm{Erf}\left(\frac{x}{\sqrt{4Dt}}\right). \tag{2.47}$$

   This solution describes a diffusing interface as can be seen from Fig. 2.5.

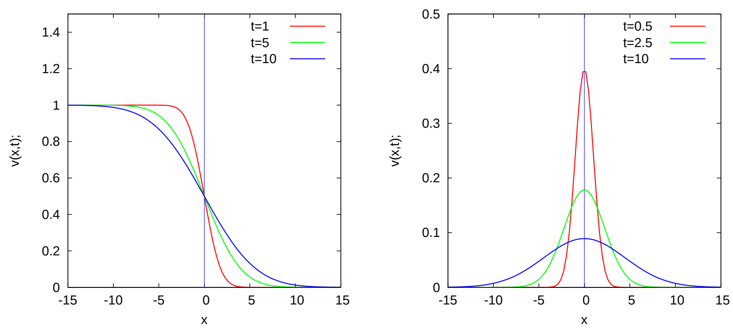

Figure 2.5: Two similarity solutions for the diffusion equation.

2. **A diffusing pulse:** $a = -\frac{1}{2}$

$$F'' + \frac{1}{2}\xi F' + \frac{1}{2}F = 0$$

has a Gaussian solution

$$A\mathrm{e}^{-\frac{\xi^2}{4}}$$

by inspection. Restoring the dependence on D we write

$$F(\xi) = A\mathrm{e}^{-\frac{\xi^2}{4D}}$$

We can fix A by setting the total mass equal to 1: $\int_{-\infty}^{\infty} F(\xi)d\xi = 1$. Noting that,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\xi^2}{4D}}\, d\xi = \sqrt{4\pi D}$$

we are led to choose $A = \frac{1}{2\sqrt{D\pi}}$. In terms of x and t we obtain the familiar solution describing a diffusing pulse:

$$v(x,t) = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-\frac{x^2}{4Dt}} \tag{2.48}$$

In general, similarity solutions of nonlinear PDEs lead to ODEs which cannot be solved explicitly as in these examples and must be solved numerically.