Smoothed Analysis of Multiobjective Optimization

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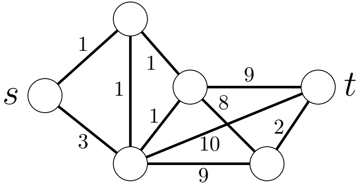
based on joint work with
Rene Beier, Shang-Hua Teng, and Berthold Vöcking

Single-criterion Optimization Problem: min $f(x)$ subject to $x \in \mathcal{S}$.

Example:
Shortest Path Problem



Optimization Problems

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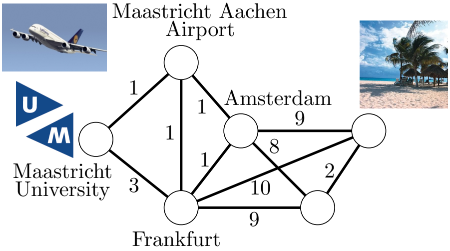

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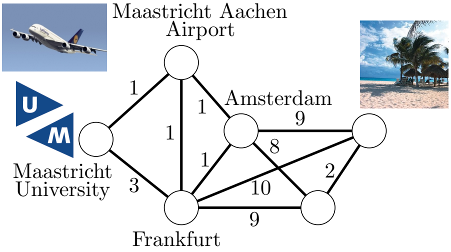

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(travel time, fare, departure time, etc.)

Multiobjective Opt. Problem: min $f_1(x), \ldots,$ min $f_d(x)$ s.t. $x \in \mathcal{S}$.
Usually, there is no solution that is simultaneously optimal for all f_i.

Question

What can we do algorithmically to support the decision maker?

Pareto-optimal Solutions

Multiobjective Opt. Problem: $\min w^1(x), \ldots, \min w^d(x)$ s.t. $x \in \mathcal{S}$

$x \in \mathcal{S}$ **dominates** $y \in \mathcal{S} \iff$
$\forall i\colon w^i(x) \le w^i(y)$ and
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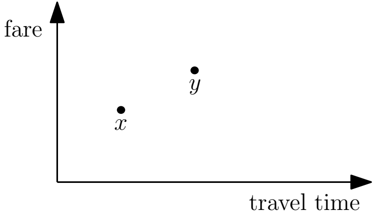

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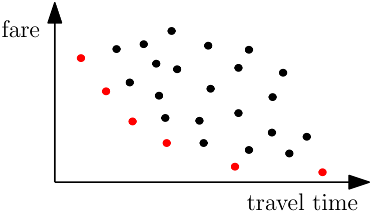

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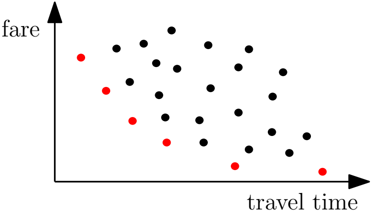

Often the **Pareto curve** is generated:

- Pareto curve limits options for decision maker.
- Monotone functions are optimized by Pareto-optimal solutions, e.g., $\lambda_1 w^1(x) + \ldots + \lambda_d w^d(x)$ or $w^1(x) \cdot \cdots \cdot w^d(x)$.

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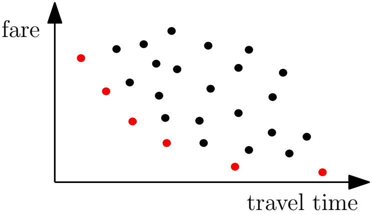

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Central Question

How large is the Pareto curve?

Model

Linear Binary Optimization Problem

- set of feasible solutions $\mathcal{S} \subseteq \{0,1\}^n$
  solution $x = (x_1, \ldots, x_n) \in \mathcal{S}$ consists of n binary variables
- d linear objective functions:
  $\forall i \in \{1, \ldots, d\}\colon\ \min w^i(x) = w^i_1 x_1 + \cdots + w^i_n x_n$

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How large is the Pareto curve?

- Exponential in the worst case for almost all problems.
- In practice, often few Pareto optimal solutions.

Example: Train Connections
w.r.t. travel time, fare, number of train changes
[Müller-Hannemann, Weihe 2001]

Smoothed Analysis

- $\mathcal{S} \subseteq \{0,1\}^n$, d objectives: min $w^i(x) = w^i_1 x_1 + \cdots + w^i_n x_n$
- Every coefficient w^i_j is an independent random variable following a probability density $f^i_j \colon [-1,1] \to [0,\phi]$.

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- each w^i_j uniformly at random from interval of length $1/\phi$
- w^i_j are Gaussians, adversary specifies means, $\phi \sim 1/\sigma$
- ϕ **large** $\approx$ **worst case** ϕ **small** $\approx$ **average case**

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- **ϕ large $\approx$ worst case** **ϕ small $\approx$ average case**

$$Q_d(n,\phi) = \max_{\mathcal{S}, f^i_j} \mathbf{E}\left[\text{number of Pareto-optimal sol. for } \mathcal{S} \text{ and } f^i_j\right]$$

Bicriteria Optimization ($d = 2$):

Beier, Vöcking (STOC 2003)

For any $\mathcal{S}$ and any f_j^i, $Q_2(n, \phi) = O(n^4 \phi)$.
There are $\mathcal{S}$ and f_j^i such that $Q_2(n, \phi) = \Omega(n^2)$.

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Multiobjective Optimization (d arbitrary constant):

R., Teng (FOCS 2009)

For any $\mathcal{S}$ and any f_j^i, $Q_d(n, \phi) = O((n\phi)^{h(d)})$ for some function h.
For any $c \in \left[1, \sqrt{\log(n)}\right]$, $Q_d(n, \phi)^c = O((n\phi)^{c \cdot h(d)})$.

Generalized Loser Gap

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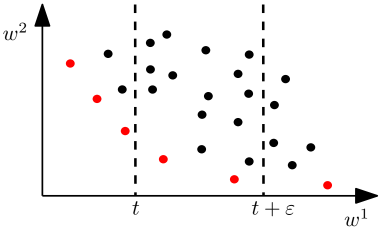

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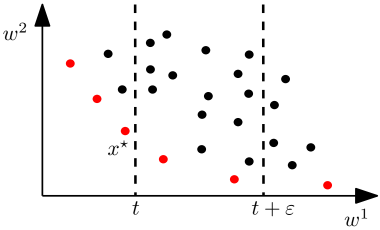

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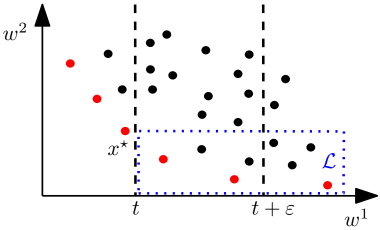

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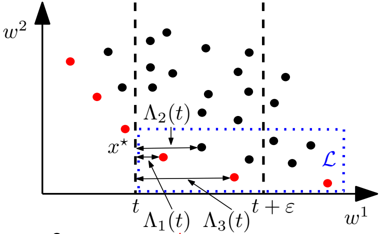

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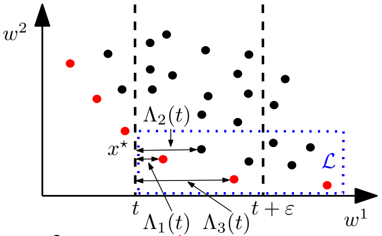

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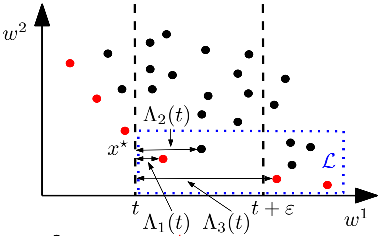

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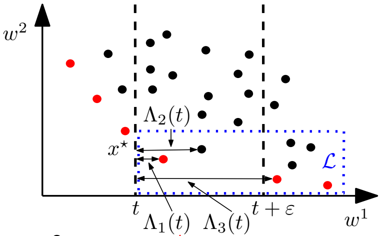

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Lemma [Beier, Vöcking (STOC 2004)]

For every $\varepsilon \geq 0$ and $t \in \mathbb{R}$, $\mathbf{Pr}\left[\Lambda^1(t) \leq \varepsilon\right] \leq n\phi\varepsilon$.

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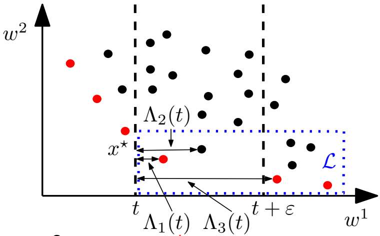

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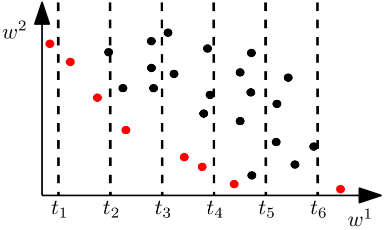
Lemma [R., Teng (FOCS 2009)]

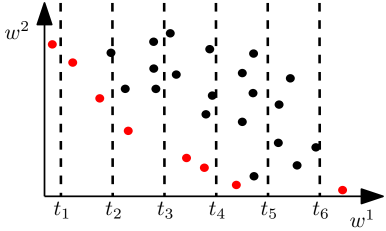
For every $\varepsilon \geq 0$, $z \in \mathbb{N}$, and $t \in \mathbb{R}$,

$$\mathbf{Pr}\left[\Lambda^{2^{z-1}}(t) \leq \varepsilon\right] \leq 2^{z^2+z} n^z \phi^z \varepsilon^{z-1}.$$

Higher Moments

divide $[0, n]$ into k intervals:
$[0, n] = [t_0, t_1], \ldots, [t_{k-1}, t_k]$





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$$\mathbf{Pr}\left[Q_2(n,\phi)^c \geq (2^{z-1} \cdot k)^c\right]$$
$$\leq \mathbf{Pr}\left[\exists i\colon\ [t_i, t_{i+1}] \text{ contains more than } 2^{z-1} \text{ PO solutions}\right]$$
$$\leq \mathbf{Pr}\left[\exists i\colon \Lambda^{2^{z-1}}(t_i) \leq \frac{2n}{k}\right] \leq \frac{2^{z^2+2z-1} n^{2z-1} \phi^z}{k^{z-2}}$$

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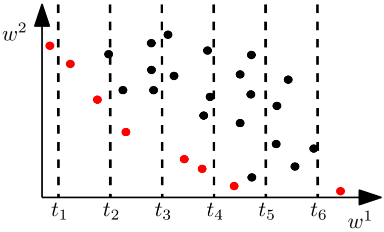

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For any $\mathcal{S}$ and any f_j^i and every $c \in \left[1, \sqrt{\log(n)}\right]$,

$$\mathbf{Q_d(n, \phi)^c = (n^2\phi)^{c(1+o(1))}}.$$

Proof Idea for $d \geq 3$

How many PO $x \in \mathcal{S}$ with $w^1(x) \in [t, t + \varepsilon]$?

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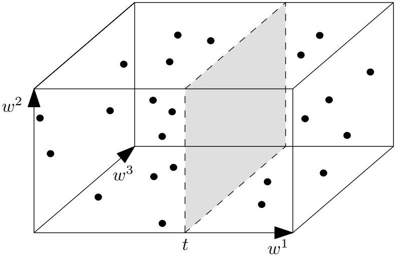

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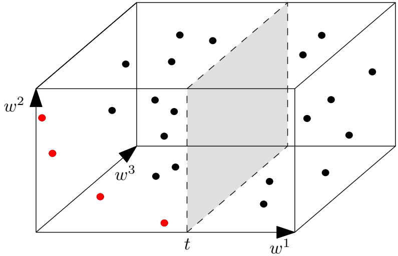

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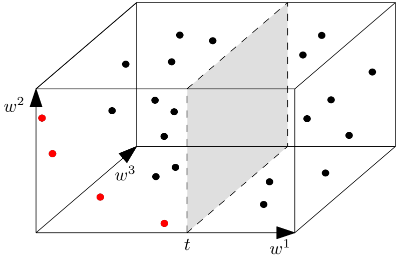

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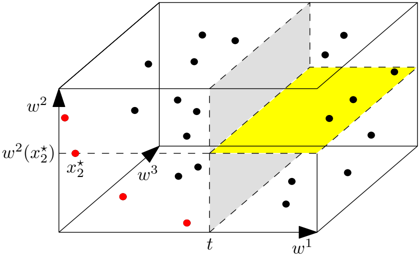

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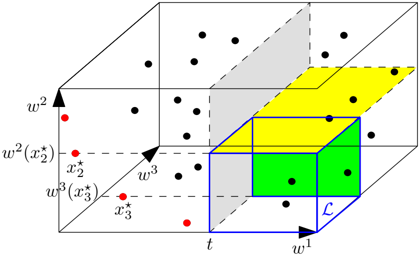

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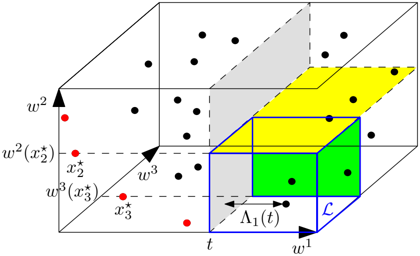

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$(x_2^\star, \ldots, x_d^\star) \in (\mathcal{P}^\star)^{d-1}$

$\mathcal{L} = \{x \in \mathcal{S} \mid \forall i\colon w^i(x) < w^i(x_i^\star)\}$

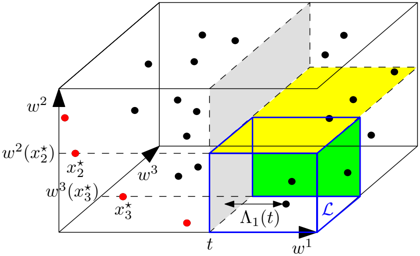
Every $(d-1)$-tuple from $\mathcal{P}$ defines a loser gap.

Induction: Use bound for $Q_{d-1}(n, \phi)^{d-1}$ to bound $Q_d(n, \phi)$.

Proof Idea for $d \geq 3$

How many PO $x \in \mathcal{S}$ with $w^1(x) \in [t, t+\varepsilon]$?

$\min w^2(x), \ldots, \min w^d(x)$
s.t. $w^1(x) \leq t$ and $x \in \mathcal{S}$



$d = 2$

optimal solution $x^\star$

$\mathcal{L} = \{x \in \mathcal{S} \mid w^2(x) < w^2(x^\star)\}$

$d \geq 3$

$\mathcal{P}^\star$ = set of PO solutions

$(x_2^\star, \ldots, x_d^\star) \in (\mathcal{P}^\star)^{d-1}$

$\mathcal{L} = \{x \in \mathcal{S} \mid \forall i\colon w^i(x) < w^i(x_i^\star)\}$

Every $(d-1)$-tuple from $\mathcal{P}$ defines a loser gap.

Induction: Use bound for $Q_{d-1}(n, \phi)^{d-1}$ to bound $Q_d(n, \phi)$.

Theorem

$\forall d\ \forall c \in [1, \sqrt{\log(n)}]$: $Q_d(n, \phi)^c = O((n\phi)^{c \cdot h(d)})$ for $h(d) = 2^{d-3} d!$.

Extensions and Open Questions

Further Results

- polynomial bound for $Q_d(n, \phi)^c$ for any constants c and d
  $\Rightarrow$ first concentration bounds for $|\mathcal{P}|$
- extension to integer case $\mathcal{S} \subseteq \{-m, -m+1, \ldots, m-1, m\}^n$
- polyn. smoothed complexity $=$ expected polynomial running time

Extensions and Open Questions

Further Results

- polynomial bound for $Q_d(n, \phi)^c$ for any constants c and d
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Open Questions

- $h(d) = 2^{d-3}d!$: $(n\phi)^{O(d)}$ possible?
- lower bounds?
- higher moments, stronger concentration?

Thank you for your attention!

Questions?