Graph Coloring Problems in the Two-Party Communication Model

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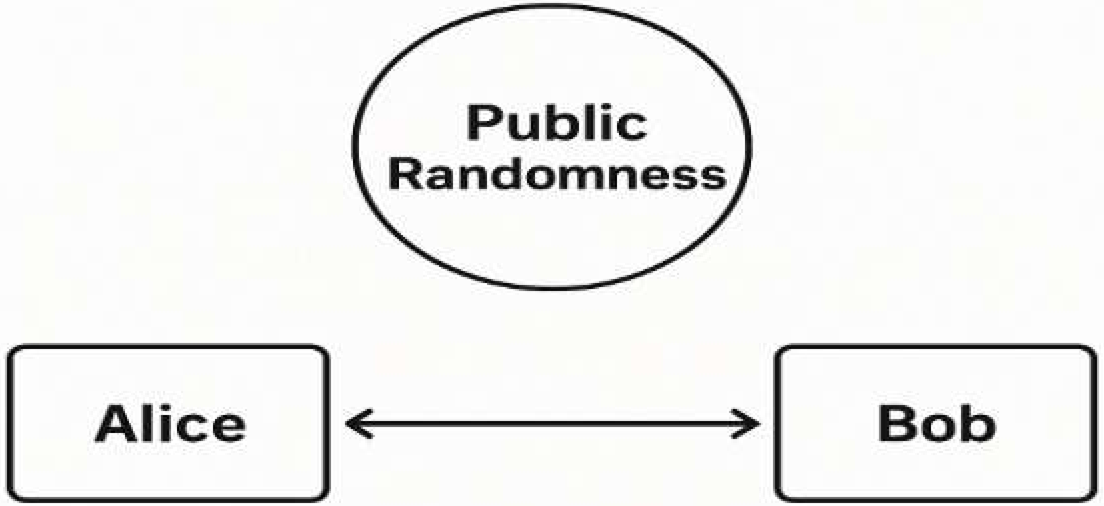
Two-party communication model



parameters:

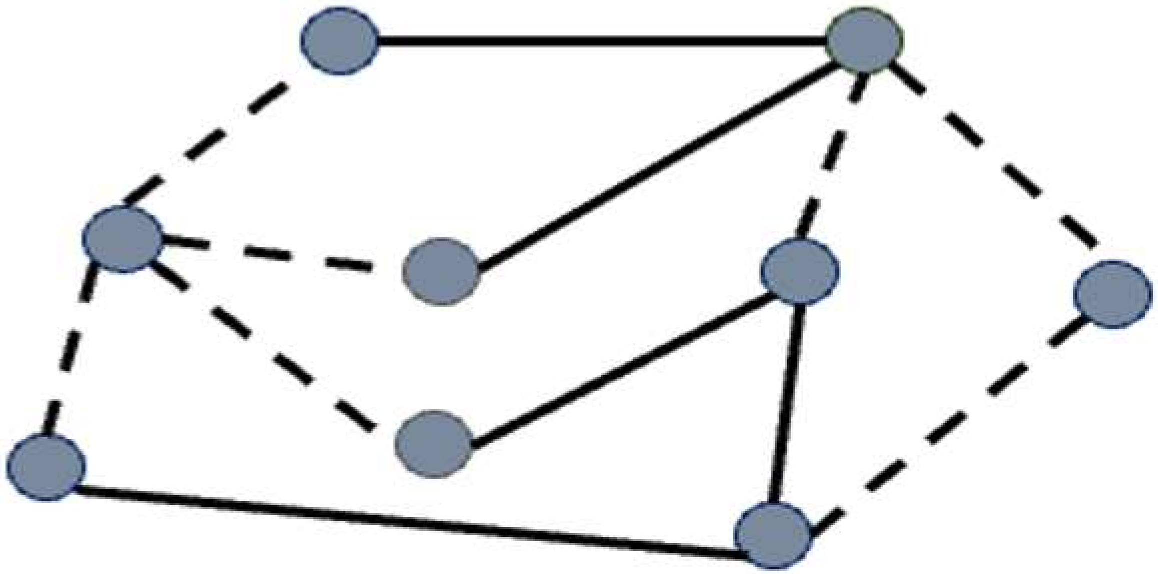
1. Communication complexity: # bits communicated
2. Round complexity: Total number of rounds

Private randomness

- Any public-coin protocol can be simulated using private coins.
- So, public and private randomness are equivalent up to a small additive overhead.
- For this talk, only think of public randomness.

The model for the graph problems

- The vertex set V is known to both Alice and Bob.
- Edges are partitioned among them.



Complexity of some graph problems

Graph Problem	Upper Bound	Lower Bound	Rounds	References
Connectivity	$O(n \log n)$	$\Omega(n)$	1	[HMT'88]
Minimum Cut	$O(n \log n)$ (randomized)	$\Omega(n \log\log n)$	2	[AD'21]
Maximum Matching	$O(n \log^2 n)$ (deterministic)	$\Omega(n)$	$O(n)$	[BVMEN'22, BFS'86; Raz'92]

$O(n)$ vs. almost $O(n)$ communication cost!

Problems and key take away

- $(\Delta + 1)$-Vertex Coloring
- $(2\Delta - 1)$-Edge Coloring
- Maximal Independent Set
- Maximal Matching

n:#vertices
Δ:the maximum degree

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n:#vertices
Δ:the maximum degree

The communication complexities of all the problems is $\Theta(n)$ bits with some caveats on round complexities.

We give communication optimal round efficient algorithms!

Problems and key take away

Problem	Communication Complexity	Round complexity	Reference
$(\Delta+1)$-Vertex Coloring	$\Theta(n)$	$O(\log\log n \cdot \log\Delta)$	[FM’25] & [CMNS’25a]
$(2\Delta-1)$-Edge Coloring	$\Theta(n)$	$O(1)$	[CMNS’25a]
Maximal Independent Set	$\Theta(n)$	$O((\log n)^{1/2})$	[CMNS’25b]
Maximal Matching	$\Theta(n)$	1	[CMNS’25b]

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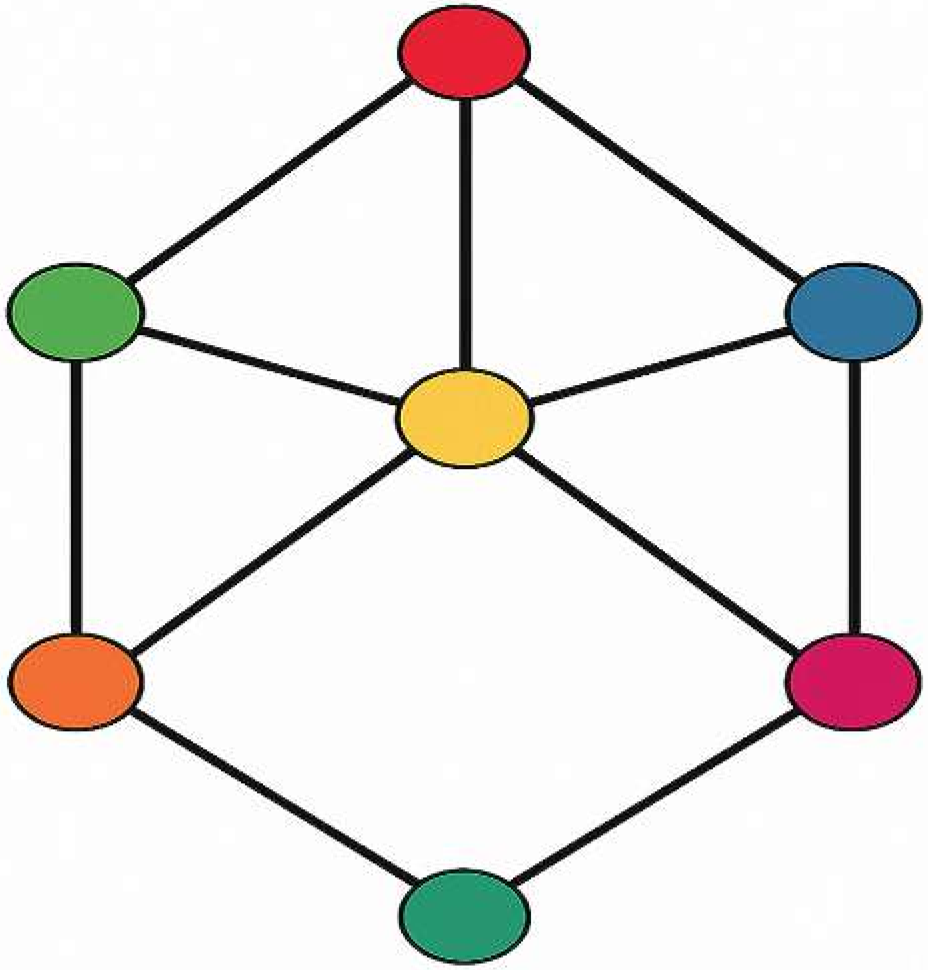
Maximal Independent Set	$\Theta(n)$	$O((\log n)^{1/2})$	[CMNS'25b]
Maximal Matching	$\Theta(n)$	1	[CMNS'25b]

All the algorithms are communication optimal irrespective of the number of rounds.

$(\Delta + 1)$-Vertex Coloring

$(\Delta + 1)$-Vertex Coloring

- Each node is to use colors from $\{1, \ldots, \Delta + 1\}$.
- No two adjacent vertices can use the same color.
- Objective of each party is to report the colors of all vertices.



Trivial algorithm: $O(\min\{n\Delta \log n, n^2\})$ bits and 1 round.

Our algorithm for $(\Delta + 1)$-Vertex Coloring

- $O(n)$ bits of communication and $O(\log\log n \cdot \log\Delta)$ rounds.

- This improves the previous results by [FM'25] with the same communication complexity of but of $\Omega(n)$ rounds.

Simulating the greedy algorithm

- Pick vertices one by one in some arbitrary order.
- For each vertex, find an available color.
- For a vertex v and color c, whether v can be colored with color c can be verified with 2 bits of communication.

Communication cost= $O(n\Delta)$ bits

Number of rounds = $O(n\Delta)$

Can we find an available color efficiently?

Using binary search!

- The total number of already used colors is at most Δ.
- Divide the color set $\{1, \dots, \Delta + 1\}$ into two equal sets L and R.
- Alice and Bob communicate the number of used colors in L and R.
- For at least one of L and R, the total number of used colors must be less than $\Delta/2$
- Then recurse.

Communication cost= $O(\log^2 \Delta)$.

Number of rounds = $O(\log \Delta)$.

Implication for $(\Delta + 1)$-Vertex Coloring

A deterministic algorithm

Communication cost= $O(\mathrm{n} \cdot \log^2 \Delta)$.

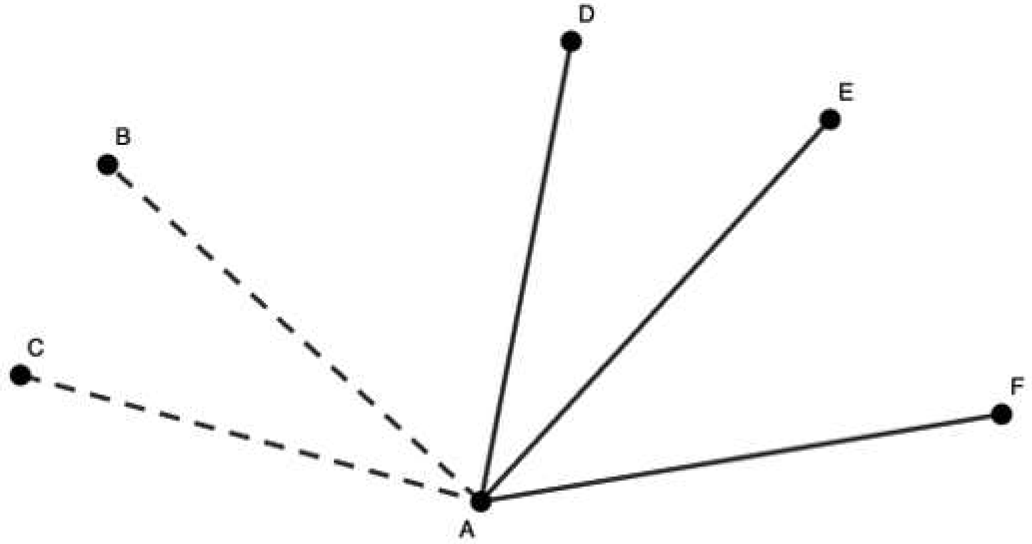
Number of rounds = $O(\mathrm{n} \cdot \log \Delta)$.

Can we find an available color efficiently?

Can we find an available color efficiently?

[FM'25] k_v is the number of available colors, one can be found/sampled with (in expectation):

- $O(\log^2(\Delta/k_v))$ bits
- $O(\log(\Delta/k_v))$ rounds



The final algorithm by [FM'25]

- Alice and Bob use public randomness to order the vertices uniformly at random.

- Each vertex v finds an available color using $O(\log^2 \Delta / k_v)$ bits of communication and in $O(\log \Delta / k_v)$ rounds: both in expectation!

- k_v is uniformly distributed over $\{\Delta + 1, \dots, \Delta + 1 - d(v)\}$.

- The expected communication cost to color v is

$$\frac{1}{d(v)+1} \sum_{k_v = \Delta+1-d(v)}^{\Delta+1} \log^2(\Delta / k_v) = O(1).$$

The final algorithm by [FM'25]

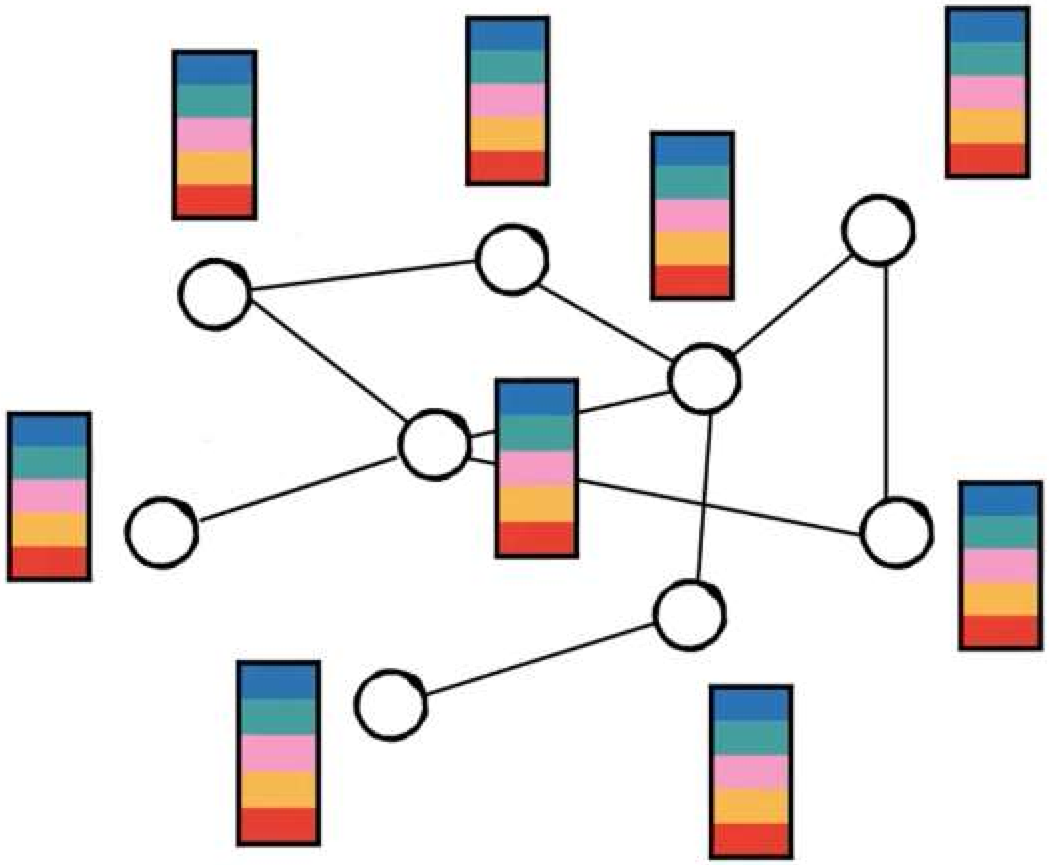
- Expected communication complexity is $O(n)$.
- Expected number of rounds is $O(n)$.
- Communication optimal, but round inefficient.

We improve the worst case round complexity to $O(\log\log n \cdot \log \Delta)$!

Ingredient One: Palette sparsification

- Introduced by [ACK'19] for streaming graph coloring.
- Allows for sparsifying the graph dramatically while still allowing for a valid $\Delta + 1$ coloring.

Ingredient One: Palette sparsification



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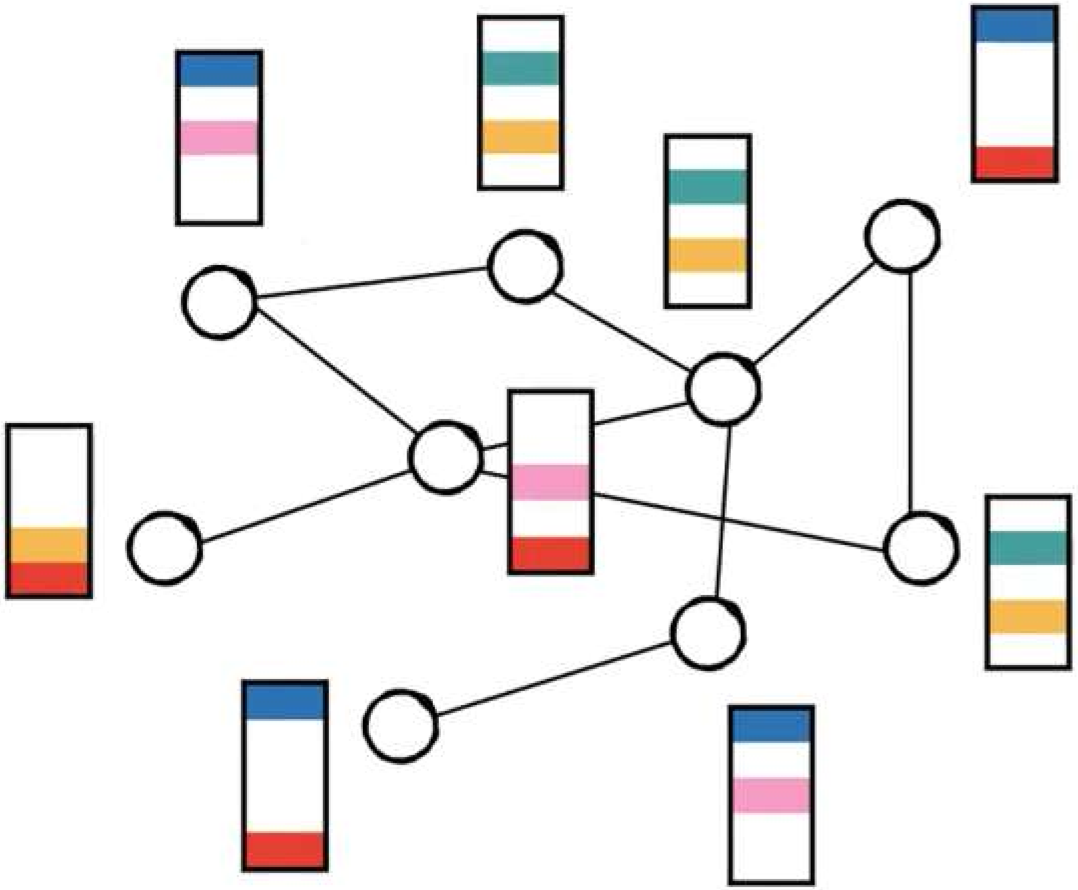

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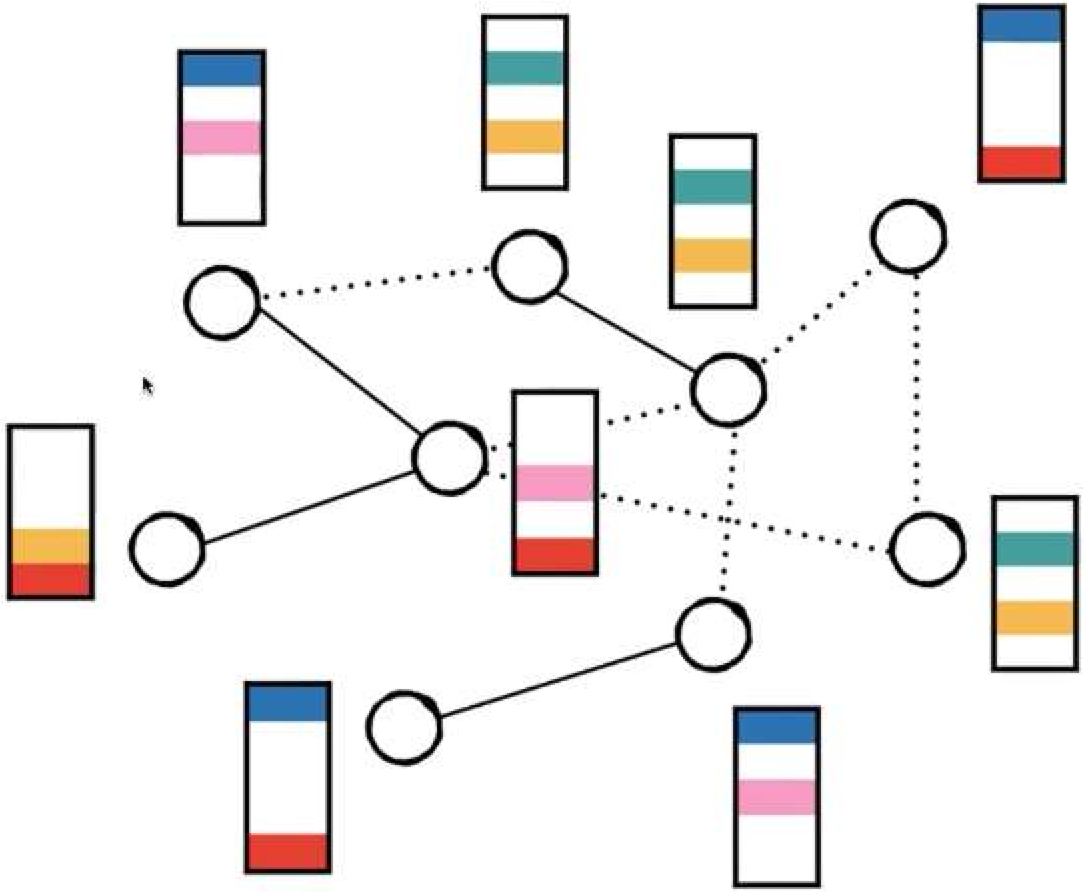


Ingredient One: Palette sparsification

Palette sparsification theorem [HKNT'22][ACK'19]

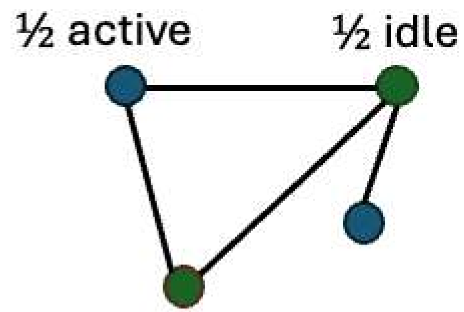
- Each vertex independently and uniformly sampling $O(\log\ n)$ colors from its palette.
- Remove edge (u, v) if the sampling lists are disjoint.
- Remaining $O(n \cdot \log^2 n)$ edges

- To sample colors, Alice and Bob can get them for free from public randomness.
- Implies an one-round algorithm with $O(n \log^3 n)$ bits of communication

Ingredient Two: Random Color Trial

Random Color Trial

- Every (uncolored) vertex participates with probability 1/2
  - Sample a color uniformly at random from available palette using [FM'25].
- Vertex v is colored with the sampled color if none of the neighbors have selected it



[FM'25] k_A is the number of available colors, one can be sampled with

- $O(\log^2(\Delta/k_A))$ bits in expectation:
- $O(\log(\Delta))$ rounds in the worst case

Pr(v is colored)$\geq \frac{1}{24}$.

In expectation, constant fraction of vertices Will be colored in an iteration.

Our algorithm for $(\Delta + 1)$-Vertex Coloring

- Run random Color Trial for $O(\log \log n)$ iterations.

- Use palette sparsification theorem on remaining nodes.

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Expected # remaining nodes = $O(n/\log^3 n)$

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- Run random Color Trial for $O(\log \log n)$ iterations.

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1 round is enough to color the remaining nodes

Our algorithm for $(\Delta + 1)$-Vertex Coloring

- Run random Color Trial for $O(\log \log n)$ iterations.

Expected # remaining nodes = $O(n / \log^3 n)$

- Use palette sparsification theorem on remaining nodes.

1 round is enough to color the remaining nodes

Total number of rounds = $O(\log \log n \cdot \log \Delta)$ in the worst case.

Communication complexity: main challenge?

- In [FM'25], a vertex gets colored in a step based on the random permutation.
- In our case, a vertex can be colored in any iteration and in an iteration multiple vertices may be colored requiring different communication cost and rounds.

$(2\Delta - 1)$-Edge Coloring

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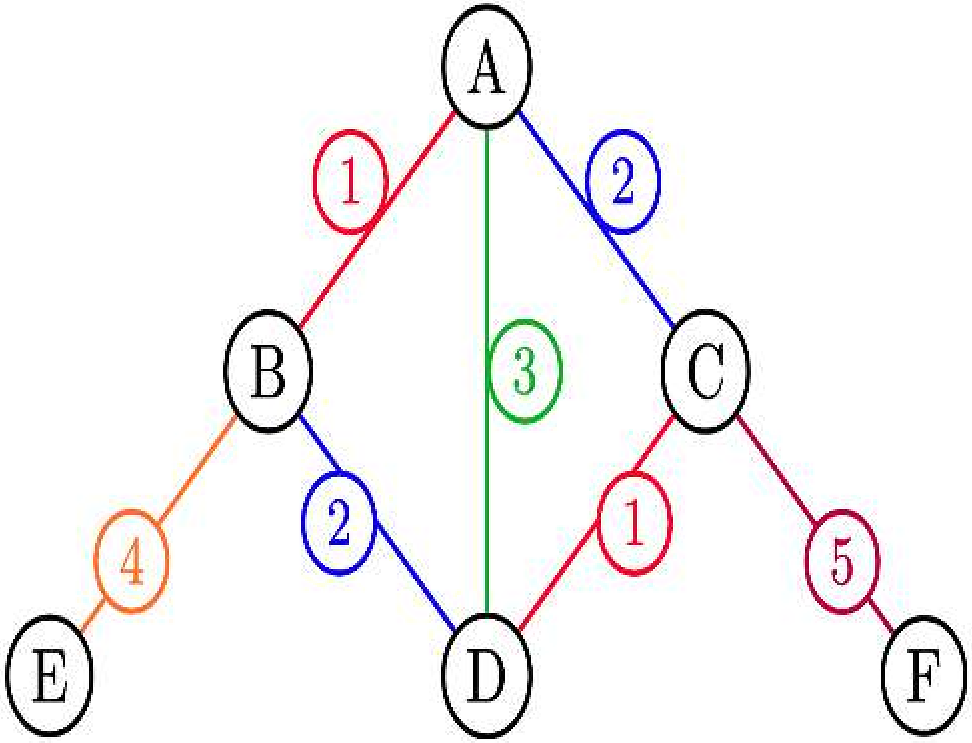
- Each edge is to use colors from $\{1, \ldots, 2\Delta - 1\}$.
- No two adjacent edges can use the same color.
- Objective of each party is to report the color of its own edges.



Trivial algorithm: $O(\min\{n\Delta \log n, n^2\})$ bits and 1 round.

Our result on $(2\Delta - 1)$-Edge Coloring

- Deterministic protocol.
- $O(n)$ bits of communication and $O(1)$ rounds.
- We give first edge coloring protocol in the two party communication model.

Our result on $(2\Delta - 1)$-Edge Coloring

- Deterministic protocol.
- $O(n)$ bits of communication and $O(1)$ rounds.
- We give first edge coloring protocol in the two party communication model.
- In this talk, we discuss a randomized $O(\log \log n)$ round protocol.

Simulating the greedy algorithm

- Alice can color the color of all of its own edges.
- Then, for each vertex, she can send a Δ-bit string denoting the list of available colors.
- Bob resume the algorithm with the information.

Communication cost= $O(n\Delta)$ bits
Number of rounds = 1

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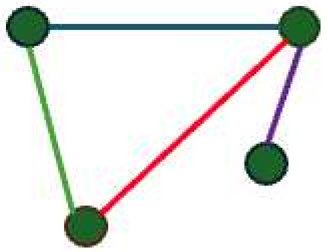
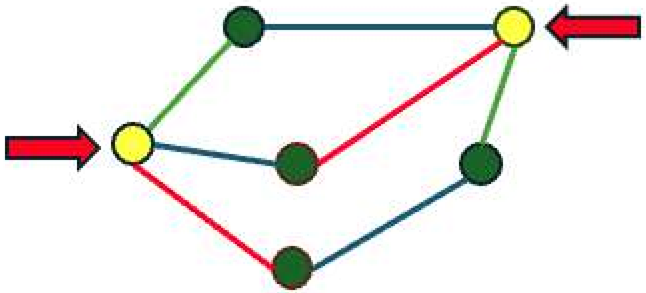
Background

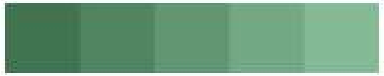
Vizing's Theorem [Viz64]: A simple graph requires either Δ or $(\Delta + 1)$ colors for proper edge coloring.



Fournier's Extension [Fou73]: If the maximum-degree vertices form an independent set, then only Δ colors are needed.



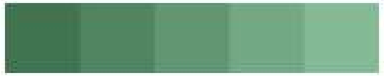
(2Δ)-Edge Coloring Protocol



Alice's palette of size Δ

Bob's palette of size Δ

(2Δ)-Edge Coloring Protocol



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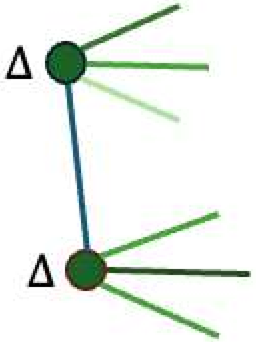
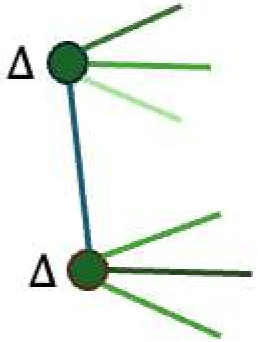


Consider an edge with Alice whose two vertices have Δ neighbors in Alice. Use any color from Bob.

(2Δ)-Edge Coloring Protocol

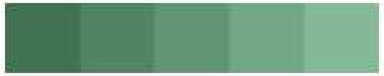
Alice's palette of size Δ

Bob's palette of size Δ



In the remaining subgraph of Alice and Bob, the vertices of degree Δ form independent set.

(2Δ)-Edge Coloring Protocol



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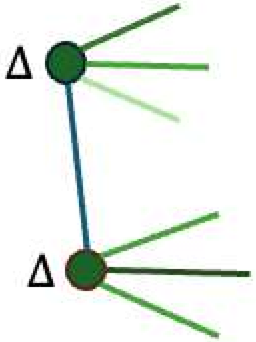

In the remaining subgraph of Alice and Bob, the vertices of degree Δ form independent set. Hence, edge with both end points degree Δ can use any color from the other party.

(2Δ)-Edge Coloring Protocol

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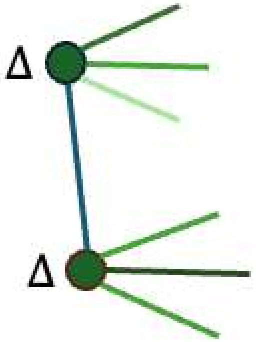
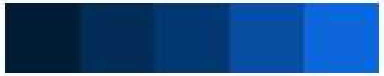


In the remaining subgraph of Alice and Bob, the vertices of degree Δ form independent set. Hence, edge with both end points degree Δ can use any color from the other party.

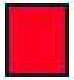
Zero communication required for (2Δ)-edge coloring.

$(2\Delta - 1)$-Edge Coloring Protocol

Alice's palette of size $\Delta - 1$



Bob's palette of size $\Delta - 1$



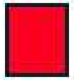
Special color

Fournier's Extension [Fou73]: If the maximum-degree = $\Delta - 1$ vertices form an **independent set**, then only $\Delta - 1$ colors are needed.

Deferred edges

Alice's palette of size $\Delta - 1$

Bob's palette of size $\Delta - 1$



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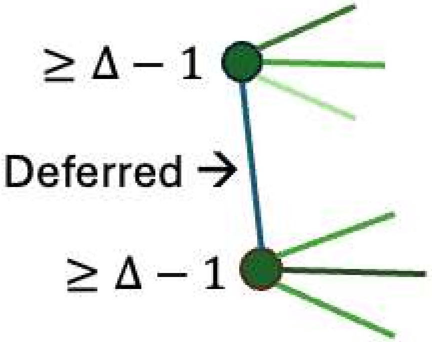

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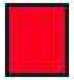
$(2\Delta - 1)$-Edge Coloring Protocol

Alice's palette of size $\Delta - 1$

Bob's palette of size $\Delta - 1$



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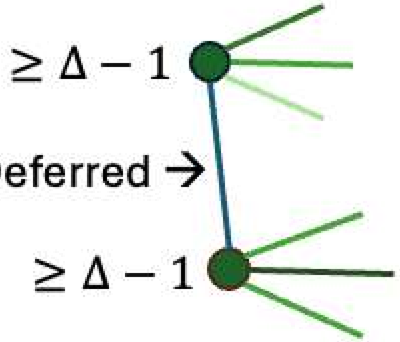
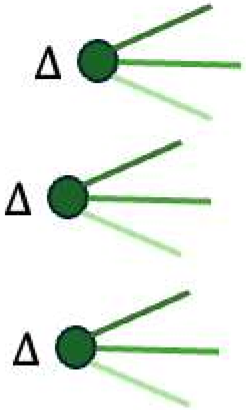
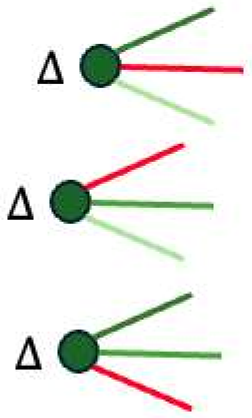




Fournier's Extension [Fou73]: If the maximum-degree = $\Delta - 1$ vertices form an **independent set**, then only $\Delta - 1$ colors are needed.

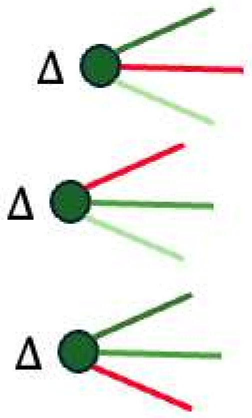
Δ-degree vertices are "independent"

Δ-Matching



Δ-degree vertices are originally "independent"

Δ-Matching



Δ-degree vertices are originally "independent"

A *Matching* covers these vertices. We call it Δ-Matching

$(2\Delta - 1)$-Edge Coloring Protocol

We started with deferring the coloring of the edges with both the vertices having degree at least $\Delta - 1$.

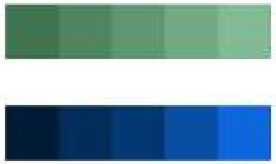
Now we have deferred the coloring of the Δ-matching edges. So, **Fournier's Extension** holds.

So, Alice and Bob can color the remaining graph with the set of own $\Delta - 1$ colors.

Fournier's Extension [Fou73]: If the maximum-degree = $\Delta - 1$ vertices form an **independent set**, then only $\Delta - 1$ colors are needed.

We are left with only $O(n)$ edges. Not Done yet!

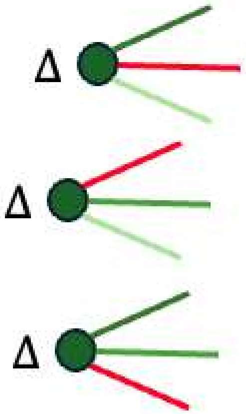
Use of special color



Alice's palette of size $\Delta - 1$

Bob's palette of size $\Delta - 1$

Special color

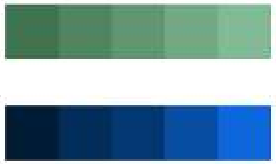


Δ-degree vertices are originally "independent"

A Δ-Matching covers these vertices

Color the matching by special color

Use of special color



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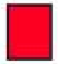

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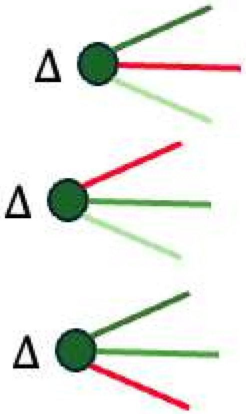

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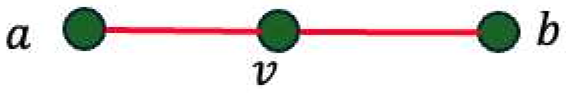
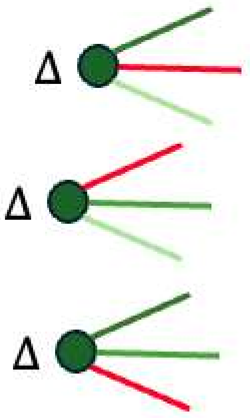


An matching edge of Alice and another of Bob may be adjacent.

An observation



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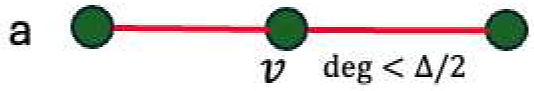
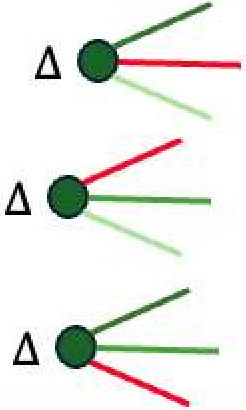


An matching edge of Alice and another of Bob may be adjacent.

Special color and random sampling



Color vb with special color and av with a color from Bob using random sampling.

Δ-degree vertices are originally "independent"

A Δ-Matching covers these vertices.

Color the matching by special color

An matching edge of Alice and another of Bob may be adjacent.

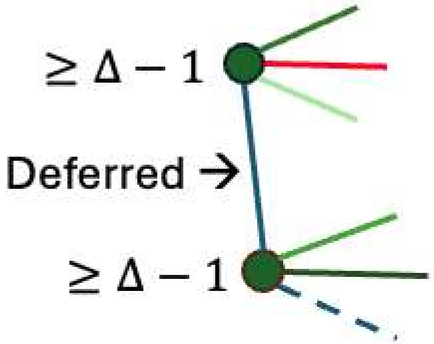
Coloring the Δ-matching edges

- Alice and communicate $O(n)$ bits regarding the match vertices.
- They color the Δ-matching edges that need to be colored using special color.
- For $O(\log\log n)$ rounds they try to color the other Δ-matching edges.
- In each round, the probability that a desired edges gets colored is at east a constant.
- After that, in expectation only $O(n/\log n)$ matching edges left, which can be communicate explicitly using $O(n)$ expected bits.

Coloring the deferred edges

- The subgraph induced by the deferred edges has max degree 2.

- We can run essentially $O(n\,\Delta)$ bit and $O(1)$ round algorithm discussed earlier.



$(2\Delta - 1)$-Edge Coloring Protocol

- Each party defer coloring of some edges incident to high degree vertices to apply

Fournier's Extension [Fou73]: If the maximum-degree = $\Delta - 1$ vertices form an **independent set**, then only $\Delta - 1$ colors are needed.

- Then color the Δ-matching edges using expected O(n) bits of communication and O(log log n) rounds: Only randomization step.

- Color the deferred subgraph using $O(n)$ bits of communication and $O(1)$ rounds.

$(2\Delta - 1)$-Edge Coloring Protocol

- Each party defer coloring of some edges incident to high degree vertices to apply

Fournier's Extension [Fou73]: If the maximum-degree = $\Delta - 1$ vertices form an **independent set**, then only $\Delta - 1$ colors are needed.

- Then color the Δ-matching edges using O(n) bits of communication and O(1) rounds deterministically.

- Color the deferred subgraph using $O(n)$ bits of communication and $O(1)$ rounds.

Maximal Independent Set

Maximal Independent Set

Algorithm	Communication Complexity	Rounds Complexity	Random Greedy?
Prior Works			
Greedy MIS	$O(n)$	$O(n)$	Yes
Luby's Algorithm	$O(n \log n)$	$O(\log n)$	No
Parallel Random Greedy MIS	$O(n \log n)$	$O(\log n)$	Yes
Batched processing	$O(n \cdot \text{poly}(\log n))$	$O(\log \log \Delta)$	Yes
New Results			
Lower Bound	$\Omega(n)$	Any	Any
Main MIS Result	$O(n)$	$O(\sqrt{\log n})$	No
Random Greedy MIS	$O(n)$	$O(\log n)$	Yes
Round-Efficient MIS-1	$O(n \log n)$	$O(\log \log \Delta)$	Yes
Round-Efficient MIS-2	$O(n \log \log n)$	$O(\log \log n)$	No

Summary and open problems

Problem	Communication Complexity	Round complexity	Reference
$(\Delta+1)$-Vertex Coloring	$\Theta(n)$	$O(\log\log n \cdot \log \Delta)$	[FM'25] & [CMNS'25a]
$(2\Delta - 1)$-Edge Coloring	$\Theta(n)$	$O(1)$	[CMNS'25a]
Maximal Independent Set	$\Theta(n)$	$O((\log n)^{1/2})$	[CMNS'25b]
Maximal Matching	$\Theta(n)$	1	[CMNS'25b]

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Maximal Independent Set	$\Theta(n)$	$O((\log n)^{1/2})$	[CMNS'25b]
Maximal Matching	$\Theta(n)$	1	[CMNS'25b]

Open problems:

1. Improve the round complexities or prove lower bounds.

2. Consider other problems in this model, for example, Δ-veretex-coloring