Algorithmic Meta-Theorems for Distributed Computing

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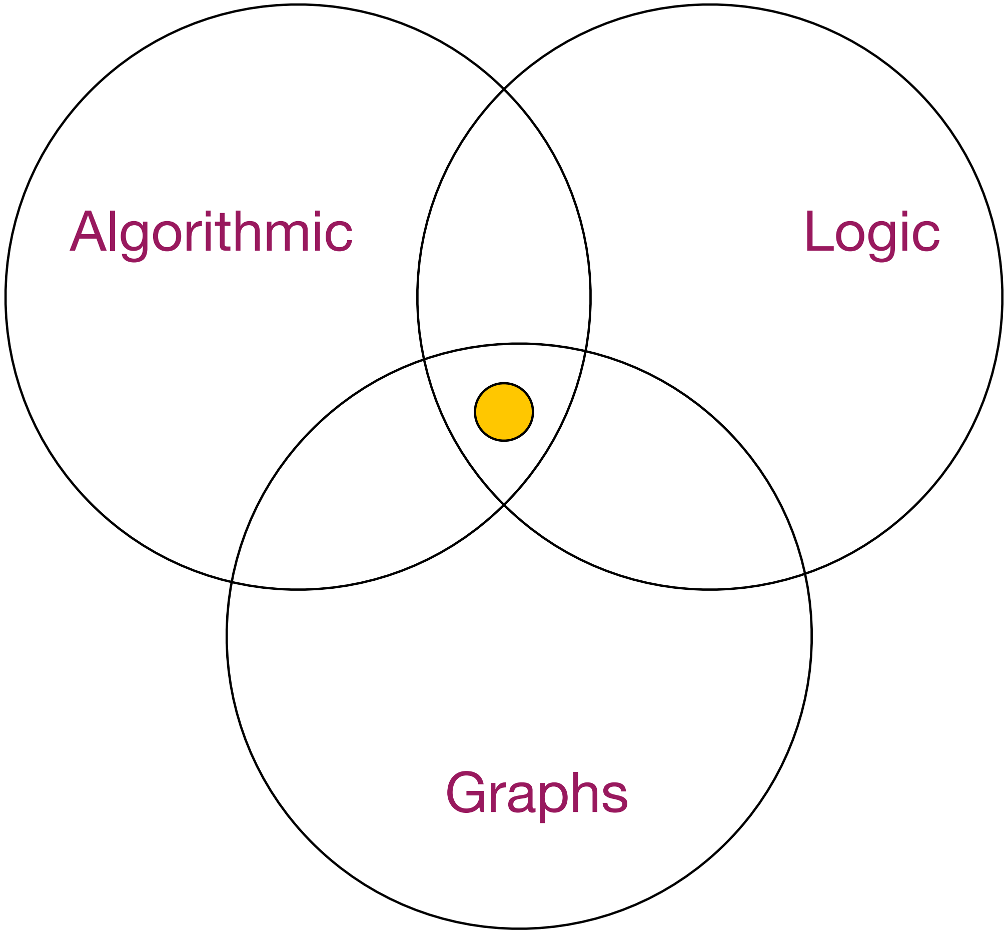
Algorithmic Meta-Theorems

Meta-Theorems

- **Theorems** are proved within a given formal system
- **Metatheorems** are statements about formal systems
- **Example:** Gödel's completeness theorem states that *first-order logic is complete*.

Algorithmic Meta-Theorems

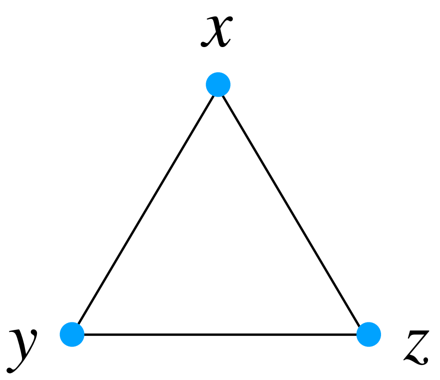


First-Order Logic on (labeled) Graphs

- Objects: Vertices
- Boolean predicates:
  - Equality: $x = y$
  - Adjacency: $\mathrm{adj}(x, y)$
  - Labels: $\mathrm{red}(x)$
- Quantifiers: $\forall \quad \exists$
- Connectors: $\neg \quad \wedge \quad \vee$

Example: triangle-freeness

$$\neg\Big(\exists x \exists y \exists z \; \big(\mathrm{adj}(x, y) \wedge \mathrm{adj}(y, z) \wedge \mathrm{adj}(z, x)\big)\Big)$$



Second-Order Logic on (labeled) Graphs

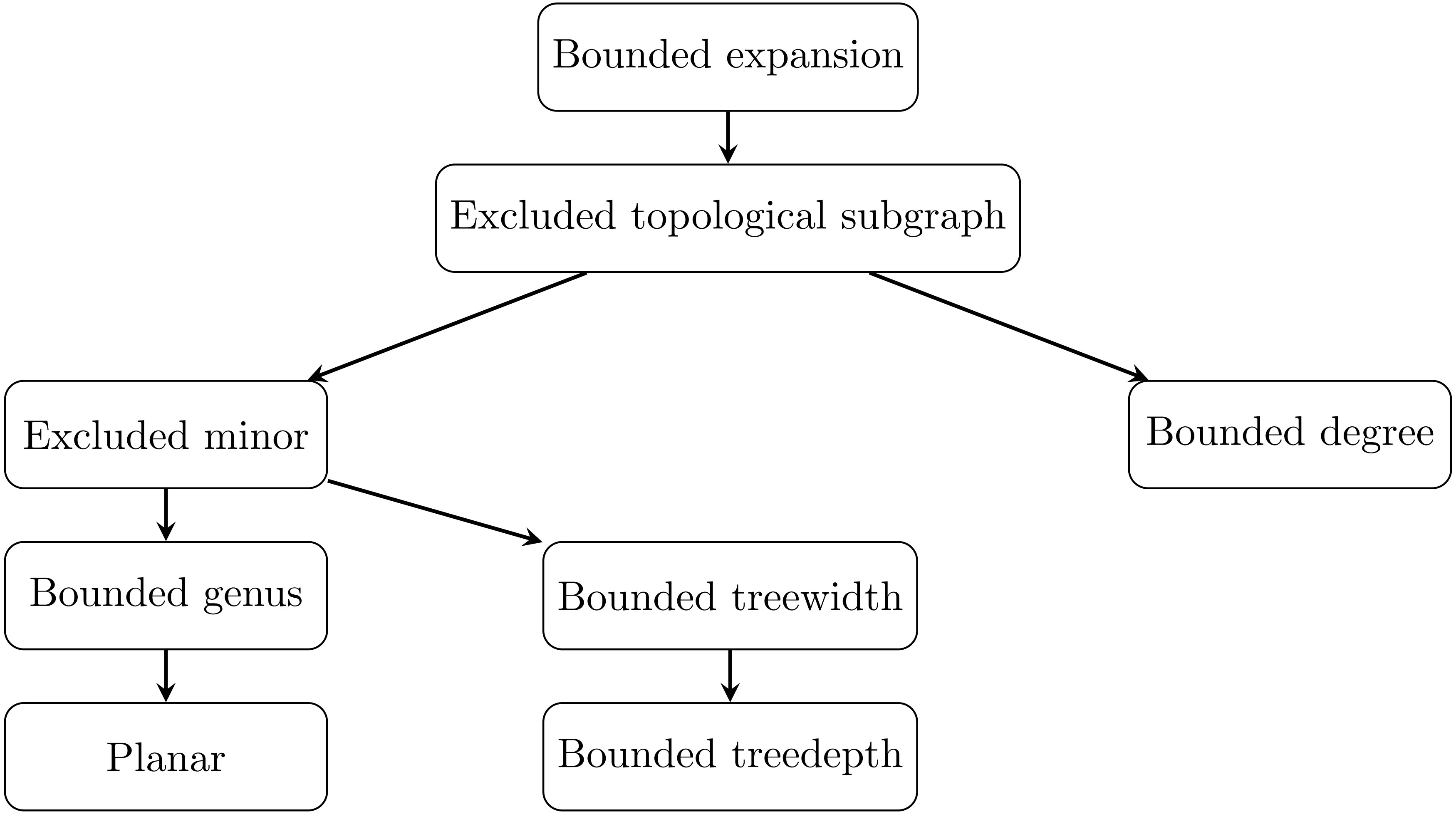
- Second-order logic quantifies over relations
- Monadic second-order logic (MSO): quantification is over sets.
- Example: Bipartiteness

$$\begin{aligned}\exists A\,\exists B\,\Big(&\big(\forall x\,(x \in A \land x \notin B) \lor (x \notin A \land x \in B)\big) \land \\ &\big(\forall x \forall y\ \neg\mathsf{adj}(x, y) \lor (x \in A \land y \in B) \lor (x \in B \land y \in A)\big)\Big)\end{aligned}$$

- Example: Path(x,y)

$$\begin{aligned}\exists S\Big(&\big(x \in S \land \exists z \in S\ \mathsf{adj}(x, z)\big) \land \big(y \in S \land \exists z \in S\ \mathsf{adj}(y, z)\big) \land \\ &\big(\forall z \in S \setminus \{x, y\}\ \exists w\,\exists w'\ \mathsf{adj}(z, w) \land \mathsf{adj}(z, w') \land w \neq w'\big)\Big)\end{aligned}$$

Graph Classes



Bounded expansion

Excluded topological subgraph

Excluded minor

Bounded degree

Bounded genus

Bounded treewidth

Planar

Bounded treedepth

Another Parameter: Clique-Width

- Grammar for constructing graphs
  - NewVertex_i : Creation of a new vertex with color i
  - $G \mid\mid H$: Disjoint union of two colored graphs G and H
  - $i \bowtie j$: Joining by an edge every vertex colored i to every vertex colored $j \neq i$
  - $\text{recolor}_{i,j}$: Recolor i into color j

- **Definition** Clique-Width of a graph G, denoted by $\text{cw}(G)$, is the minimum number of colors used for constructing G.

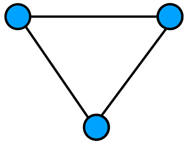
Example: Cliques

- NewVertex$_{blue}$

$n-1$ times

- NewVertex$_{red}$
- $blue \bowtie red$
- recolor$_{red,blue}$

- NewVertex$_{red}$
- $blue \bowtie red$
- recolor$_{red,blue}$

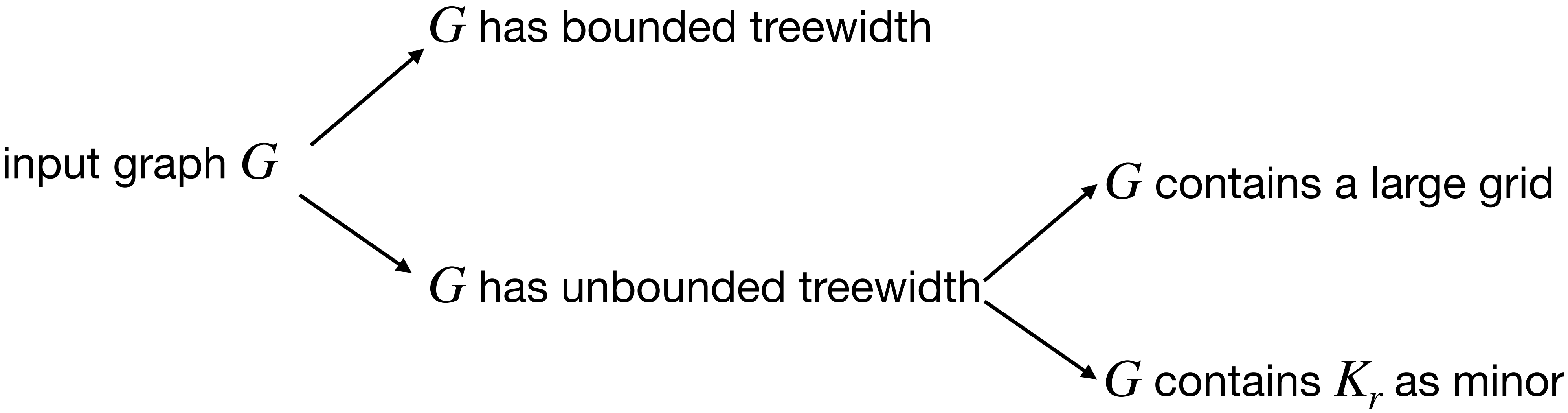


cw$(K_n) \leq 2$

Graph-Minor Theory

Theorem (Robertson, Seymour)
Every graph class closed under minor has a finite number of obstructions

Impact on algorithm design (parametrized complexity):



input graph G

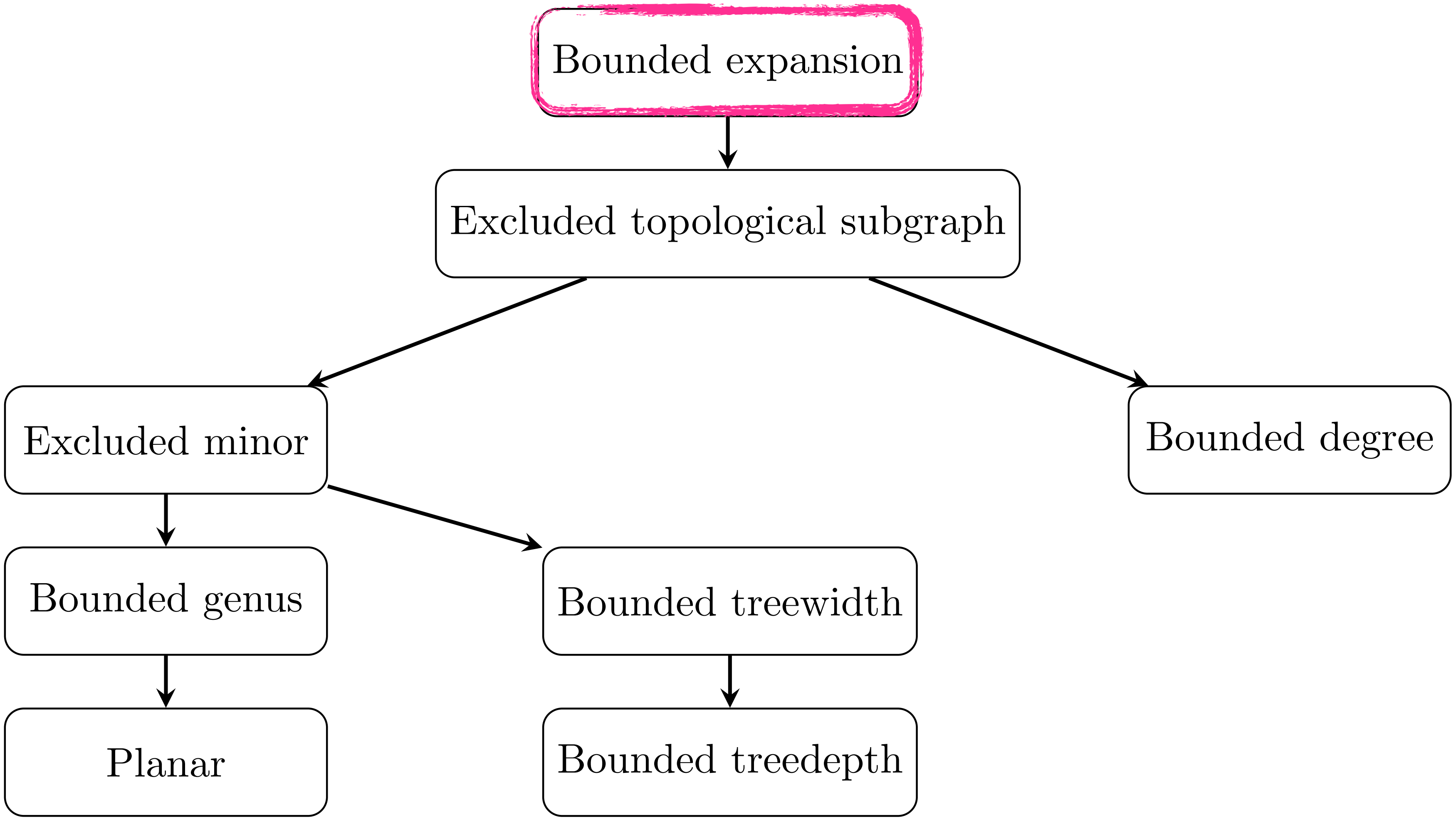
G has bounded treewidth

G has unbounded treewidth

G contains a large grid

G contains K_r as minor

Graph Classes



Bounded expansion

Excluded topological subgraph

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Classes of Bounded expansion

Definition A class $\mathscr{G}$ of graphs is of bounded expansion if there exists a function

$$f : \mathbb{N} \to \mathbb{N}$$

such that, for every $G \in \mathscr{G}$ and every $r \geq 0$, any graph H obtained from G by

1. contracting vertex-disjoint balls of radius at most r, and
2. deleting the remaining vertices

satisfies

$$\frac{|E(H)|}{|V(H)|} \leq f(r)$$

Examples

Bounded degree graphs:

$$\mathcal{G}_\Delta = \{G = (V, E) \mid \forall v \in V, \deg(v) \leq \Delta\}$$

Expansion $f(r) = \Delta^{r+1}$

Planar graphs:

Expansion $f(r) = 6$

Other classes:

- graphs of bounded genus
- graphs of bounded treewidth
- graphs that exclude a fixed (topological) minor
- graphs of bounded stack number
- a.a.s., graphs drawn from $G_{n,p}$ with $p = O(1/n)$

Algorithmic Meta-Theorems

Theorem (Courcelle 1990)

Every graph property definable in the MSO_2 logic of graphs can be decided in linear time on graphs of bounded treewidth*.*

Theorem (Courcelle, Makowsky, Rotics, 2000)

Every graph property definable in the MSO_1 logic of graphs can be decided in linear time on graphs of bounded cliquewidth*.*

Theorem (Dvorák, Král, Thomas, 2010)

Every graph property definable in the FO logic of graphs can be decided in linear time on graph classes of bounded expansion*.*

Distributed
Algorithmic Meta-Theorems

Distributed Decision

Definition A distributed algorithm A decides a graph property Π if, for every graph $G = (V, E)$,

$$G \models \Pi \iff A \text{ accepts at all nodes.}$$

Can be extended to labeled graphs (G, ℓ) where $\ell : V(G) \to L$

- Is the graph properly 3-colored?
- Is the set of pointers forming a (minimum-weight) spanning tree?

FO on Bounded Expansion

Theorem (Blin, Fomin, F., Gay, Golovach, Montealegre, Rapaport, Todinca, 2025)

For every FO formula φ on graphs, and for every class of graphs $\mathcal{G}$ of bounded expansion, there exists a deterministic algorithm that, for every n-node network $G \in \mathcal{G}$ of diameter D, decides whether $G \models \varphi$ in $O(D + \log n)$ rounds in the CONGEST model.

Remark: $\Omega(D)$ rounds are necessary for formulas such as

$$\exists x \exists y \;\; \deg(x) = \deg(y) = 3$$

Local FO on Bounded Expansion

Theorem (Blin, Fomin, F., Gay, Golovach, Montealegre, Rapaport, Todinca, 2025)

For every local FO formula $\varphi(x)$ on graphs, and for every class of graphs $\mathcal{G}$ of bounded expansion, there exists a deterministic algorithm that, for every n-node network $G \in \mathcal{G}$, marks all vertices v of G satisfying $G \models \varphi(v)$, in $O(\log n)$ rounds in the CONGEST model.

Treedepth

Definition For any $d \geq 0$, a graph G has treedepth at most d if the n vertices of G can be embedded onto an n-node rooted forest F of depth at most d such that, for any edge of G, one of its end-points is mapped to an ancestor of the image of its other end-point in F.

Treedepth of G, $\text{td}(G)$, is the minimum d such that such a forest exists.

Lemma (Fomin, F., Montealegre, Rapaport, Todinca, 2024)
For every FO formula φ on graphs, and for every $d \geq 0$, there exists a deterministic algorithm that, for every n-node network G with $\text{td}(G) \leq d$, decides whether $G \models \varphi$, in $O(1)$ rounds in the CONGEST model.

Low Treedepth Coloring

Definition A graph class $\mathscr{G}$ admits a low treedepth coloring if there exists a function

$$f : \mathbb{N} \to \mathbb{N}$$

satisfying that, for every integer $p \geq 1$, the vertices of every graph $G \in \mathscr{G}$ can be properly colored using $f(p)$ colors such that, for every set

$$S \subseteq \{1, \ldots, f(p)\}$$

of at most p colors, the subgraph $G[S]$ of G induced by the vertices with colors in S satisfies $\mathrm{td}(G[S]) \leq |S|$.

Theorem (Nešetřil,Ossona de Mendez, 2008)

$\mathscr{G}$ has bounded expansion $\iff$ $\mathscr{G}$ admits a low treedepth coloring.

Remark $f(p) = O(\Delta^2 p)$ for $\mathscr{G}_\Delta$ and $f(p) = O(p \log p)$ for $\mathscr{G}_{\text{planar}}$

Efficient Distributed Algorithm for Low Treedepth Coloring

Theorem (Nešetřil,Ossona de Mendez, 2008)

Let $\mathcal{G}$ be a graph class of bounded expansion. There exists an $O(\log n)$-round algorithm in CONGEST that, for every given p, computes a treedepth coloring of any $G \in \mathcal{G}$ with $f(p)$ colors.

This may not be the 'best' f
not even the one describing $\mathcal{G}$

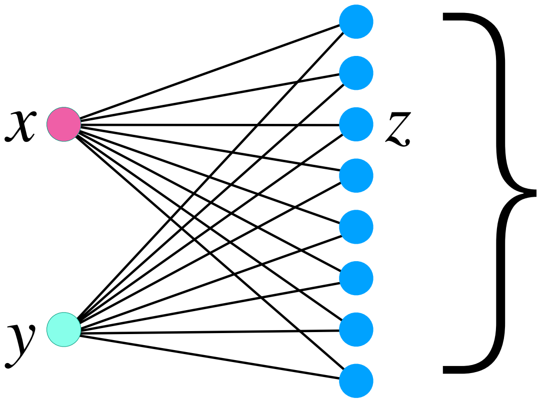
Application: Deciding H-Freeness

- Let $\mathscr{G}$ be a class of bounded expansion (e.g., planar graphs)
- Let $f : \mathbb{N} \to \mathbb{N}$ be such that $(p, f(p))$-treedepth coloring exists for every $p \geq 1$.
- Let $p = |V(H)|$
- Let $G \in \mathscr{G}$, and let $c : V(G) \to [f(p)]$ be a low-treedepth coloring.
- If G contains H as a subgraph, then there is a set S of at most p colors such that $G[S]$ contains H.
- For every $S \subseteq [f(p)]$ of p colors, testing whether $G[S]$ contains H is easy as $G[S]$ has treedepth at most p.

Algorithm: Try all $\binom{f(p)}{p}$ sets $S \subseteq [f(p)]$ of $p = |V(H)|$ colors.

Deciding Twins?

$$\mathsf{twin}(x) = \exists y \forall z \ \Big((\mathsf{adj}(x, z) \wedge \mathsf{adj}(y, z)) \vee (\neg \mathsf{adj}(x, z) \wedge \neg \mathsf{adj}(y, z)) \Big)$$



arbitrarily large number of common neighbors

Main Tool: Quantifier Elimination

$$\varphi(x) = Q_1 y_1 \; Q_2 y_2 \; \ldots \; Q_k y_k \; \psi(x, y_1, \ldots, y_k)$$

Goal: Constructing a *quantifier-free* formula $\widehat{\varphi}(x)$ such that, for every $G \in \mathscr{G}$

$$G \models \varphi(v) \iff \widehat{G} \models \widehat{\varphi}(v)$$

where $\widehat{G}$ is G enriched with additional unary predicates (i.e., labels).

Technique:

Bounded expansion

$\updownarrow$

Bounded treedepth

$\updownarrow$

Forests

Two Main Difficulties

1. Implementing all steps of quantifier-elimination in CONGEST, i.e.,
    (a) communications must remain local, and
    (b) communication are limited to $O(\log n)$ bits per round on each edges

2. Preserving the locality of the formula throughout the transformation

$$\begin{aligned}\text{twin}(x) &= \exists y \forall z \ \big((\text{adj}(x,z) \wedge \text{adj}(y,z)) \vee (\neg\text{adj}(x,z) \wedge \neg\text{adj}(y,z))\big) \\ &= \exists y \neg \exists z \ \big((\text{adj}(x,z) \wedge \neg\text{adj}(y,z)) \vee (\neg\text{adj}(x,z) \wedge \text{adj}(y,z))\big)\end{aligned}$$

This is addressed by adopting a redundant relativization of quantifiers.

Distributed Certification

Distributed Certification (1)

- Let Π be a Boolean predicate on (labeled) graphs
- **Certification scheme for Π**
  - **Prover =** centralized computationally-unlimited non-trustable oracle providing every node of the input labeled graph (G, ℓ) with a *certificate*
  - **Verifier =** 1-round distributed algorithm exchanging certificates between neighbors for checking whether the set of certificates is a distributed proof that $(G, \ell) \models \Pi$

Distributed Certification (2)

Completeness: $(G, \ell) \models \Pi \implies$ prover can give certificates such that verifier accepts at every node

Soundness: $(G, \ell) \not\models \Pi \implies$ for any set of certificates given by prover, verifier rejects in at least one node

Complexity measure = **size** of the certificates (for legal instances)

Examples

Theorem (Korman, Kutten, Peleg, 2005)
Certifying cycle-freeness can be done with certificates on $O(\log n)$ *bits.*

Theorem (Drucker, Kuhn, Oshman, 2014)
Certifying C_4*-freeness requires certificates on* $\tilde{\Omega}(\sqrt{n})$ *bits.*

Meta-Theorems for Certification

Theorem (Blin, Fomin, F., Gay, Golovach, Montealegre, Rapaport, Todinca, 2025)
For every FO formula φ, and for every graph class $\mathscr{G}$ of bounded ***expansion,*** *certifying $G \models \varphi$ can be implemented with certificates on $O(\log n)$ bits in n-node graphs $G \in \mathscr{G}$.*

Theorem (Cook, Kim, Masarik, 2025)
For every MSO_2 formula φ, certifying $G \models \varphi$ can be implemented with certificates on $O(\log n)$ bits in any n-node graph of bounded ***treewidth.***

Theorem (F., Montealegre, Todinca, Rapaport, 2022)
For every MSO_1 formula φ, certifying $G \models \varphi$ can be implemented with certificates on $O(\log^2 n)$ bits in any n-node graph of bounded ***cliquewidth.***

Conclusion

Open Problem

- Pushing FO Distributed Model Checking to nowhere dense graphs
- Compact distributed certification of H-minor-freeness
- Links beween sequential algorithms and CONGEST algorithms

⸮ poly time → polylog-round CONGEST → polylog space certification ?

Thank you!