An Elementary Construction of Constant-Degree Expanders

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Graphs are finite, undirected, may contain loops and multiple edges.

Definitions

A $d$-regular graph $G = (V, E)$ is a $\delta$-expander, if for every set of vertices $S \subseteq V$ with $|S| \leq \frac{1}{2}|V|$, we have $e(S, V \setminus S) \geq \delta d|S|$. A $[n, d, \delta]$-expander is a $n$-vertex $d$-regular $\delta$-expander.

The replacement product $G \circ H$, for a $d$-regular graph $H$ and a $D$-regular graph $G$ with a $D$-edge-colouring, is as illustrated below.

For $q = 2^t$ and $r \in \mathbb{N}$, we define a graph $LD(q, r)$ as follows. Vertices of $LD(q, r)$ are elements of $\mathbb{F}_q^{r+1}$ and vertex $a = (a_0, a_1, \ldots, a_r)$ has neighbours $a + y \cdot (1, x, x^2, \ldots, x^r)$, for $(x, y) \in \mathbb{F}_q^2$. The edges $a, a + y \cdot (1, x, x^2, \ldots, x^r)$ is coloured by colour $(x, y)$. $LD(q, r)$ is a $q^2$-regular graph on $q^{r+1}$ vertices that is $q^2$-colourable.

Theorems

Theorem 1 (Main Theorem). There exists a fixed $\delta > 0$ such that any integer $q = 2^t$ and for any $q^4/100 \leq e \leq q^4/2$ there exists a polynomial time constructible $[q^{4r+12}, 12, \delta]$-expander.

Theorem 2. If $E_1$ is an $[n, D, \delta_1]$-expander and $E_2$ is a $[D, d, \delta_2]$-expander, then $E_1 \circ E_2$ is an $[nD, 2d, \frac{\delta_1 \delta_2}{\delta_2}]$-expander.
Theorem 3. [Pinsker] There exists a fixed $\delta > 0$ such that for any $d \geq 3$ and any even integer $n$, there is an $[n,d,\delta]$-expander, which is $d$-edge-colourable.

Theorem 4. [Alon, Roichman] For any $q = 2^t$ and integer $r < q$ we have $\lambda_2(LD(q,r)) \leq rq$.

Let $\lambda_2$ be the second largest eigenvalue of the incidence matrix of a $\delta$-expander, then
\[
\frac{1}{2}(1-\lambda/d) \leq \delta.
\] (1)

Proofs

I’ll prove Theorems 1 and 2. The proof of Theorem 1 relies on Theorems 2, 3, 4 and Inequality (1).

The idea of the proof of Theorem 2 is that every set $S \subseteq V(E_1 \circ E_2)$ with $|S| \leq nD/2$ either intersects copies of $E_2$ “sparingly”, or substantially intersects only a “few” copies of $E_2$. In the first case, we use the expansion properties of $E_2$ and show that there are a lot of edges between $S$ and $\bar{S}$ within the copies of $E_2$. In the second case, we use the expansion property of $E_1$ and show that there are many edges between copies of $E_2$ running between $S$ and $\bar{S}$.