An $O(\log \log m)$ Prophet Inequality for Subadditive Combinatorial Auctions

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Joint work with
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Online Combinatorial Auction

- *n* buyers, arriving one by one

- *m* items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
Online Combinatorial Auction

- $n$ buyers, 

\[ \begin{align*}
  v_1(\{1\}) &= 1 \\
  v_1(\{2\}) &= 2 \\
  v_1(\{1, 2\}) &= 3
\end{align*} \]

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
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Online Combinatorial Auction

- $n$ buyers, arriving one by one
  - $v_1(\{1\}) = 1$
  - $v_1(\{2\}) = 2$
  - $v_1(\{1, 2\}) = 3$
  - $v_2(\{1\}) = 0$
  - $v_2(\{2\}) = 10$
  - $v_2(\{1, 2\}) = 10$
  - $v_3(\{1\}) = 5$
  - $v_3(\{2\}) = 5$
  - $v_3(\{1, 2\}) = 5$
  - $v_4(\{1\}) = 20$
  - $v_4(\{2\}) = 50$
  - $v_4(\{1, 2\}) = 60$

- $m$ items

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Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
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\[
\begin{align*}
v_2(\{1\}) &= 0 \\
v_2(\{2\}) &= 10 \\
v_2(\{1, 2\}) &= 10
\end{align*}
\]
Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
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$v_3(\{1\}) = 5$
$v_3(\{2\}) = 5$
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Online Combinatorial Auction

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Online Combinatorial Auction

- \(n\) buyers, arriving one by one

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\(v_4(\{1\}) = 20\)
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Online Combinatorial Auction

- $n$ buyers, arriving one by one

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
- $v_i \sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance

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An $O(\log \log m)$ Prophet Inequality for Subadditive Combinatorial Auctions
Definition

A valuation function \( v_i : 2^{[m]} \rightarrow \mathbb{R}_{\geq 0} \) is subadditive if

\[
v_i(S \cup T) \leq v_i(S) + v_i(T) \quad \text{for all } S, T \subseteq [m]
\]
Subadditive Valuations

Definition

A valuation function \( v_i : 2^{[m]} \rightarrow \mathbb{R}_{\geq 0} \) is subadditive if

\[
v_i(S \cup T) \leq v_i(S) + v_i(T) \quad \text{for all } S, T \subseteq [m]
\]

Definition

A valuation function \( v_i : 2^{[m]} \rightarrow \mathbb{R}_{\geq 0} \) is XOS if

\[
v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j} \quad \text{for all } S \subseteq [m]
\]
Prior Work

If all valuation functions are XOS (for example submodular):

- 2-approximation of welfare via static, anonymous item prices (generalizes classic prophet inequality) [Feldman, Gravin, Lucier SODA 2015]

- $O(1)$-approximation of revenue via simple mechanism [Cai and Zhao STOC 2017]
If all valuation functions are XOS (for example submodular):

- 2-approximation of welfare via static, anonymous item prices (generalizes classic \textit{prophet inequality})
  
  \[ [\text{Feldman, Gravin, Lucier SODA 2015}] \]

- \(O(1)\)-approximation of revenue via simple mechanism
  
  \[ [\text{Cai and Zhao STOC 2017}] \]

**Our question:** Valuations are only subadditive (i.e. \( v_i(S \cup T) \leq v_i(S) + v_i(T) \))

**So far:** Only \( \Theta(\log m) \)-approximations
Our Results

If all valuation functions are subadditive (i.e. $\nu_i(S \cup T) \leq \nu_i(S) + \nu_i(T)$):

- $O(\log \log m)$-approximation of welfare
  via static, anonymous item prices

- $O(\log \log m)$-approximation of revenue
  via simple mechanism
If all valuation functions are \textit{subadditive} (i.e. \( v_i(S \cup T) \leq v_i(S) + v_i(T) \)):

- \(O(\log \log m)\)-approximation of welfare via static, anonymous item prices
- \(O(\log \log m)\)-approximation of revenue via simple mechanism
- Both run in polynomial time given access to demand oracles
Follow-Up Work

- [Assadi, Kesselheim, Singla SODA’21] use our key lemma to design a truthful prior-free $O((\log \log m)^3)$-approximation for XOS and subadditive combinatorial auctions
1 The balanced prices approach
2 Our new argument
3 Summary and open problems
The Balanced Prices Approach
The Classic Prophet Inequality

Theorem (Samuel-Cahn ’84; Kleinberg & Weinberg STOC’12)

For the single-item problem,

\[ E[ALG(v)] \geq \frac{1}{2} \cdot E[OPT(v)]. \]
Analysis

\[ \nu_1 \sim D_1 \quad \nu_2 \sim D_2 \quad \nu_3 \sim D_3 \quad \nu_4 \sim D_4 \quad \nu_5 \sim D_5 \]
Analysis

Set any price $p$. 

$v_1 \sim D_1$, $v_2 \sim D_2$, $v_3 \sim D_3$, $v_4 \sim D_4$, $v_5 \sim D_5$
Set any price $p$. Let $q =$ probability that item is sold.
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How much money do we collect?

$$E[\text{revenue}] = p \cdot q$$
Analysis

Set any price $p$. Let $q =$ probability that item is sold.

How much money do we collect?

$$\mathbf{E}[\text{revenue}] = p \cdot q$$

What’s a buyer’s utility (value minus payment)?

$$\mathbf{E}[u_i] = \mathbf{E}[(v_i - p)^+ \cdot 1_{\text{nobody before } i \text{ buys}}]$$

$$= \mathbf{E}[(v_i - p)^+] \cdot \mathbf{P}[\text{nobody before } i \text{ buys}]$$

$$\geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$
Putting the Pieces Together

So far:

\[ E[\text{revenue}] = p \cdot q \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]
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So far:

\[ E[\text{revenue}] = p \cdot q \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:

\[ E[\text{welfare}] = E[\text{revenue}] + \sum_i E[u_i] \]

\[ \geq p \cdot q + \sum_i E[(v_i - p)^+] \cdot (1 - q) \]

\[ \geq p \cdot q + E[\max_i (v_i - p)] \cdot (1 - q) \]
Putting the Pieces Together

So far:

\[ E[\text{revenue}] = p \cdot q \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:

\[ E[\text{welfare}] = E[\text{revenue}] + \sum_{i} E[u_i] \]

\[ \geq p \cdot q + \sum_{i} E[(v_i - p)^+] \cdot (1 - q) \]

\[ \geq p \cdot q + E[\max_i (v_i - p)] \cdot (1 - q) \]

For \( p = \frac{1}{2} \cdot E[\max_i v_i] \) this yields

\[ E[\text{welfare}] \geq \frac{1}{2} \cdot E[\max_i v_i] \cdot q + \frac{1}{2} \cdot E[\max_i v_i] \cdot (1 - q) = \frac{1}{2} \cdot E[\max_i v_i] \]
Consider full information.

\[ p = \max_k v_k \]

Let \( v_i = \max_k v_k \)

Case 1:
- Somebody \( i' < i \) buys item
  \[ \text{revenue} \geq \frac{1}{2} v_i \]

Case 2:
- Nobody \( i' < i \) buys item
  \[ u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i \]

In either case:

\[ \text{welfare} = \text{revenue} + \text{utilities} \geq \frac{1}{2} v_i \]
Consider full information.

Price $p = \frac{1}{2} \cdot \max_k v_k$ is “balanced”
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Price $p = \frac{1}{2} \cdot \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
Consider full information.

Price \( p = \frac{1}{2} \cdot \max_k v_k \) is “balanced”

Let \( v_i = \max_k v_k \)

- **Case 1:** Somebody \( i' < i \) buys item
  \[ \Rightarrow \text{revenue} \geq \frac{1}{2} v_i \]
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Price \( p = \frac{1}{2} \cdot \max_k v_k \) is “balanced”

Let \( v_i = \max_k v_k \)

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Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$

- **Case 1:** Nobody $i' < i$ buys item
  $\Rightarrow$ $u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$
Consider full information.

Price \( p = \frac{1}{2} \cdot \max_k v_k \) is “balanced”

Let \( v_i = \max_k v_k \)

- **Case 1**: Somebody \( i' < i \) buys item
  \[ \Rightarrow \text{revenue} \geq \frac{1}{2} v_i \]

- **Case 1**: Nobody \( i' < i \) buys item
  \[ \Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i \]

**In either case**: welfare = revenue + utilities \( \geq \frac{1}{2} v_i \)
Posted Prices in Combinatorial Auctions

- \(n\) buyers, arriving one by one

- \(m\) items

- Precompute item prices \(p_1, \ldots, p_m\)
- At each arrival: Arriving buyer purchases bundle maximizing utility \(v_i(S) - \sum_{j \in S} p_j\)
- Maximize social welfare \(\sum_{i=1}^{n} v_i(X_i)\)
Posted Prices in Combinatorial Auctions

- $n$ buyers, $\infty$

- $v_1(\{1\}) = 1$
- $v_1(\{2\}) = 2$
- $v_1(\{1, 2\}) = 3$

- $m$ items

- 4
- 5

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Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one

\[ v_2(\{1\}) = 0 \]
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- At each arrival: Arriving buyer purchases bundle maximizing utility $\nu_i(S) - \sum_{j \in S} p_j$

- Maximize social welfare $\sum_{i=1}^n \nu_i(X_i)$
Posted Prices in Combinatorial Auctions

- \( n \) buyers, arriving one by one
  
  - Precompute item prices \( p_1, \ldots, p_m \)
  - At each arrival: Arriving buyer purchases bundle maximizing utility \( v_i(S) - \sum_{j \in S} p_j \)
  - Maximize social welfare \( \sum_{i=1}^{n} v_i(X_i) \)

- \( m \) items
Theorem (Feldman, Gravin, Lucier SODA’15)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$
\mathbb{E} \left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{2} \cdot \mathbb{E}[OPT(v)].
$$

Recall: XOS $\Leftrightarrow v_i(S) = \max_\ell \sum_{j \in S} v_{i,j}^\ell$
Balanced Prices: Definition

**Definition (Dütting, Feldman, Kesselheim, Lucier FOCS’17)**

A valuation function $v_i$ admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices $p_j$ for $j \in U$ such that for all $T \subseteq U$:

\[
\begin{align*}
\sum_{j \in U \setminus T} p_j &\leq v_i(U \setminus T) \quad \text{(prices are not too high)} \\
\sum_{j \in T} p_j &\geq v_i(U) - v_i(U \setminus T) \quad \text{(prices are not too low)}
\end{align*}
\]
Balanced Prices: Definition

Definition (Dütting, Feldman, Kesselheim, Lucier FOCS’17)

A valuation function \( v_i \) admits balanced prices if for every set of items \( U \subseteq [m] \) there exist item prices \( p_j \) for \( j \in U \) such that for all \( T \subseteq U \):

- \[ \sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \] (prices are not too high)
- \[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \] (prices are not too low)

Observation: XOS functions admit balanced prices

Let \( \ell^* \) be such that \( v_i(U) = \sum_{j \in U} v_{i,j}^{\ell^*} \)
Let \( p_j = v_{i,j}^{\ell^*} \)
Balanced Prices: Examples

$$\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$U = \{1, 2, 3\}$$
Balanced Prices: Examples

\[ \sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \quad (\forall T \subseteq U) \]
\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive

\[ v_1(S) = |S| \]
Balanced Prices: Examples

\[ \sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

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Balanced Prices: Examples

\[ \sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \quad (\forall T \subseteq U) \quad \checkmark \quad \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

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\(U = \{1, 2, 3\}\)

Example 1: Additive

\(v_1(S) = |S|\)

Example 2: Unit-Demand

\(v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}\)
Balanced Prices: Examples

\[
\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \quad (\forall T \subseteq U)
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\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)
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Balanced Prices: Examples

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Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 
0 & \text{if } S = \emptyset \\
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\end{cases} \]
If a class of valuations admits balanced prices, then for any distributions $D_1, \ldots, D_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$
E \left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{2} \cdot E[OPT(v)].
$$
Setting the Prices

Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\tilde{v})\}$

For $j \in U_i$ set $p_j^{\tilde{v}}$ to balanced price for item $j$ in $\tilde{v}_i, U_i$

Price for item $j$: $\bar{p}_j = \frac{1}{2} \cdot E_{\tilde{v} \sim D}[p_j^{\tilde{v}}]$
Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$
Proof Sketch Full Information

Let \( U_i = \{ j \mid i \text{ gets } j \text{ in } OPT(\nu) \} \)

Set price \( \bar{p}_j = \frac{p_j}{2} \) for \( j \in U \)

Let \( T_i = \{ j \in U_i \text{ sold to buyers } i' \neq i \} \)

Because prices are balanced:

(a) \( \sum_{j \in U_i \setminus T_i} \bar{p}_j \leq \frac{1}{2} \nu_i(U_i \setminus T_i) \)

(b) \( \sum_{j \in T_i} \bar{p}_j \geq \frac{1}{2} (\nu_i(U_i) - \nu_i(U_i \setminus T_i)) \)
Proof Sketch Full Information

Let $U_i = \{ j \mid i \text{ gets } j \text{ in } \text{OPT}(v) \}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U$

Let $T_i = \{ j \in U_i \text{ sold to buyers } i' \neq i \}$

Because prices are balanced:

(a) $\sum_{j \in U_i \setminus T_i} \bar{p}_j \leq \frac{1}{2} v_i(U_i \setminus T_i)$

(b) $\sum_{j \in T_i} \bar{p}_j \geq \frac{1}{2} (v_i(U_i) - v_i(U_i \setminus T_i))$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left( v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

$$\geq \left( v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left( v_i(U_i) - v_i(U_i \setminus T_i) \right)$$

$$= \frac{1}{2} v_i(U_i)$$
Proof Sketch Full Information

Let \( U_i = \{ j \mid i \text{ gets } j \text{ in } \text{OPT}(\nu) \} \)

Set price \( \bar{p}_j = \frac{p_j}{2} \) for \( j \in U \)

Let \( T_i = \{ j \in U_i \text{ sold to buyers } i' \neq i \} \)

Because prices are balanced:
(a) \( \sum_{j \in U_i \setminus T_i} \bar{p}_j \leq \frac{1}{2} \nu_i(U_i \setminus T_i) \)
(b) \( \sum_{j \in T_i} \bar{p}_j \geq \frac{1}{2} (\nu_i(U_i) - \nu_i(U_i \setminus T_i)) \)

Then, for the allocation \( X_1, \ldots, X_n \), we have:

\[
\sum_{i=1}^{n} \nu_i(X_i) \geq \sum_{i=1}^{n} \left[ u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \right] \geq \sum_{i=1}^{n} \left[ \left( \nu_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j \right] \\
\geq \sum_{i=1}^{n} \left[ \left( \nu_i(U_i \setminus T_i) - \frac{1}{2} \nu_i(U_i \setminus T_i) \right) + \frac{1}{2} \left( \nu_i(U_i) - \nu_i(U_i \setminus T_i) \right) \right] \\
= \sum_{i=1}^{n} \frac{1}{2} \nu_i(U_i)
\]
Subadditive functions admit approximately balanced prices
This way we can get a $\Theta(\log m)$ approximation
But we cannot do better than this
Our New Argument
Lemma (Dütting, Kesselheim, Lucier FOCS’20)

For any subadditive valuation $v_i$ and any set $U \subseteq [m]$ there exist prices $p_j$ for $j \in U$ and a probability distribution $\lambda$ such that for all $T \subseteq U$

$$\sum_{S \subseteq U} \lambda_S \left( v_i(S \setminus T) - \sum_{j \in S \setminus T} p_j \right) + \sum_{j \in T} p_j \geq \frac{v_i(U)}{\gamma},$$

where $\gamma \in O(\log \log m)$. 

Key Lemma
Lemma (Dütting, Kesselheim, Lucier FOCS’20)

For any subadditive valuation $v_i$ and any set $U \subseteq [m]$ there exist prices $p_j$ for $j \in U$ and a probability distribution $\lambda$ such that for all $T \subseteq U$

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Paul Dütting
An $O(\log \log m)$ Prophet Inequality for Subadditive Combinatorial Auctions

23
**Key Lemma**

**Lemma (Dütting, Kesselheim, Lucier FOCS’20)**

For any subadditive valuation $v_i$ and any set $U \subseteq [m]$ there exist prices $p_j$ for $j \in U$ and a probability distribution $\lambda$ such that for all $T \subseteq U$

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where $\gamma \in O(\log \log m)$.
Lemma

For every subadditive function $v_i$ and set $U$ there exists a probability distribution $\lambda$ on $S \subseteq U$ so that for every probability distribution $\mu$ on $T \subseteq U$ with $\sum_{T : j \in T} \mu_T \leq \sum_{S : j \in S} \lambda_S$ for all items $j$, it holds that

$$\sum_{S, T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{1 \gamma} \cdot v_i(U).$$
**Equivalent to Key Lemma**

**Lemma**

For every subadditive function $v_i$ and set $U$ there exists a probability distribution $\lambda$ on $S \subseteq U$ so that for every probability distribution $\mu$ on $T \subseteq U$ with $\sum_{T:j \in T} \mu_j \leq \sum_{S:j \in S} \lambda_S$ for all items $j$, it holds that

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Lemma

For every subadditive function $v_i$ and set $U$ there exists a probability distribution $\lambda$ on $S \subseteq U$ so that for every probability distribution $\mu$ on $T \subseteq U$ with $\sum_{T: j \in T} \mu_T \leq \sum_{S: j \in S} \lambda_S$ for all items $j$, it holds that

$$\sum_{S, T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{\gamma} \cdot v_i(U).$$
Claim: There is $\lambda$ such that for all $\mu$:  
\[ \sum_{S, T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U). \]
Claim: There is $\lambda$ such that for all $\mu$: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U)$.

For $q = \frac{1}{2}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$
subject to $\sum_{S:j \in S} \lambda_S \leq q$
Claim: There is \( \lambda \) such that for all \( \mu \):

\[
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For $q = \frac{1}{2}$:

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For $q = \frac{1}{2}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S : j \in S} \lambda_S \leq q$

By subadditivity:

If $E[v_i(S \setminus T)]$ is small then $E[v_i(S \cap T)]$ is large.
Claim: There is $\lambda$ such that for all $\mu$: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U)$.

For $q = \frac{1}{2}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S:j \in S} \lambda_S \leq q$

By subadditivity:
If $\mathbb{E}[v_i(S \setminus T)]$ is small then $\mathbb{E}[v_i(S \cap T)]$ is large.

Furthermore: $\mathbb{Pr}[j \in S \cap T] = q^2$. 
Claim: There is \( \lambda \) such that for all \( \mu \):

\[
\sum_{S, T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U).
\]

For \( q = \frac{1}{2}, \frac{1}{4} \):

Take \( \lambda \) that maximizes \( \sum_S \lambda_S \cdot v_i(S) \)
subject to \( \sum_{S:j \in S} \lambda_S \leq q \)

By subadditivity:
If \( E[v_i(S \setminus T)] \) is small then \( E[v_i(S \cap T)] \) is large.

Furthermore: \( \Pr[j \in S \cap T] = q^2. \)
Claim: There is \( \lambda \) such that for all \( \mu \):

\[
\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U).
\]

For \( q = \frac{1}{2}, \frac{1}{4} \):

Take \( \lambda \) that maximizes \( \sum_S \lambda_S \cdot v_i(S) \)

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By subadditivity:

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Claim: There is $\lambda$ such that for all $\mu$: \[
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\]

For $q = \frac{1}{2}, \frac{1}{4}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S:j \in S} \lambda_S \leq q$

By subadditivity:
If $E[v_i(S \setminus T)]$ is small then $E[v_i(S \cap T)]$ is large.

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Paul Dütting
An O(\log \log m) Prophet Inequality for Subadditive Combinatorial Auctions
Claim: There is $\lambda$ such that for all $\mu$: \[ \sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U). \]

For $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S:j \in S} \lambda_S \leq q$

By subadditivity:
If $\mathbb{E}[v_i(S \setminus T)]$ is small then $\mathbb{E}[v_i(S \cap T)]$ is large.

Furthermore: $\Pr[j \in S \cap T] = q^2$. 
Claim: There is $\lambda$ such that for all $\mu$: \[
\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U).
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For $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$:

Take $\lambda$ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S:j \in S} \lambda_S \leq q$

By subadditivity:
If $E[v_i(S \setminus T)]$ is small then $E[v_i(S \cap T)]$ is large.

Furthermore: $\Pr[j \in S \cap T] = q^2$.

$\Rightarrow$ One of $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$ will be good.
Additional Results in the Paper
The $O(\log \log m)$ bound is tight for the equal marginals approach taken here.

An alternative proof of key lemma based on configuration LP, which yields an efficient algorithm.

A simple, DSIC mechanism that yields a $O(\log \log m)$ approximation to the optimal revenue.
Conclusion and Open Questions
Summary

- Major progress on one of the main frontiers in the posted pricing/prophet inequalities literature
- Technique for dealing with subadditive valuations that goes beyond “approximate with XOS functions”
- Big open question: Can we get $O(1)$?

Thanks! Questions?