Triangle Factors in Randomly Perturbed Graphs

Julia Böttcher  Olaf Parczyk  Amedeo Sgueglia  Jozef Skokan

DIMAP Seminar, (not in Warwick), November 2020

The London School of Economics and Political Science
Department of Mathematics
Minimum degree thresholds in graphs

Problem
Determine the minimum degree threshold that ensures a graph $G$ contains a given (spanning) subgraph $H$. 

- Hamilton cycle (Dirac, 1952)
  - $\delta(G) \geq n/2$

- Perfect $K_r$-tiling (Hajnal and Szemerédi, 1970)
  - $\delta(G) \geq (1 - 1/r)n$

- Perfect $H$-tiling (Kühn and Osthus, 2009)
  - $\delta(G) \geq (1 - 1/\chi^\star(H))n + O(1)$

where $\chi^\star(H) \in \{\chi(H), (\chi(H) - 1)|H| - \sigma(H)\}$ and $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of $H$ with $\chi(H)$ colours.
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where $\chi^*(H) \in \{\chi(H), \frac{\chi(H) - 1}{|H| - \sigma(H)}\}$ and $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of $H$ with $\chi(H)$ colours.
The random graph $G(n, p)$ has vertex set $[n] := \{1, \ldots, n\}$ and each pair is an edge with probability $p$, independently of all other choices.

$t(n)$ is a threshold for a property $\mathcal{A}$ if, for every $p(n)$,

$$
\lim_{n \to \infty} \mathbb{P}(G(n, p) \in \mathcal{A}) = \begin{cases} 
0 & \text{if } p(n) = o(t(n)), \\
1 & \text{if } p(n) = \omega(t(n)).
\end{cases}
$$

**Theorem (Bollobás – Thomason, 1987)**

Every non-trivial monotone property $\mathcal{A}$ has a threshold.

$\mathcal{A}$ is monotone if it is closed under addition of edges (containing $H$ as subgraph vs. containing $H$ as induced subgraph)
Thresholds in random graphs

There has been a lot of work in this area.

Problem

Determine the threshold at which $G(n, p)$ contains a.a.s. a given spanning subgraph $H$. 

- Hamilton cycle (Pósa | Koršunov, 1976)
  $$t(n) = n - 1 \log n$$

- Perfect $K_r$-tiling (Johansson, Kahn and Vu, 2008)
  $$t(n) = n - \frac{2}{r} \left( \frac{1}{\log n} \right)^2 \frac{1}{r^2 - r}$$

- Conjectured the thresholds for perfect $H$-tiling for every $H$; resolved the case when $H$ is a strictly balanced graph;
- Gerke and McDowell (2015) gave a proof when $H$ is a non vertex-balanced graph.

The problem is still open for some graphs $H$. 
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Randomly perturbed graphs

Problem
Starting from a (dense) graph, determine how many random edges need to be added to ensure that the resulting graph a.a.s. contains a given spanning subgraph $H$. 

• The dense graph ‘helps’ $G(n, p)$ to have the spanning structure. (small $\alpha$)
• Random edges ‘help’ the dense graph to have the spanning structure. (small $p$)
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Definition (Bohman, Frieze and Martin, 2003)

Let $\alpha, p \in [0, 1]$, $n \in \mathbb{N}$ and $G_\alpha$ be a graph on $n$ vertices with minimum degree at least $\alpha n$. We call $G_\alpha \cup G(n, p)$ a randomly perturbed graph.
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Randomly perturbed graphs: Hamiltonicity

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Given $\alpha$, determine the threshold $t(n)$ at which $G_\alpha \cup G(n, p)$ contains a.a.s. a given spanning subgraph $H$:

0-s: If $p(n) = o(t(n))$, then, for all $n$, there is an $n$-vertex $G_\alpha$ such that $G_\alpha \cup G(n, p)$ a.a.s. does not contain a perfect $H$-tiling.

1-s: If $p(n) = \omega(t(n))$, then, for all $n$-vertex $G_\alpha$, we have that $G_\alpha \cup G(n, p)$ a.a.s. contains a perfect $H$-tiling.
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Given \( \alpha \), determine the threshold \( t(n) \) at which \( G_{\alpha} \cup G(n, p) \) contains a.a.s. a given spanning subgraph \( H \):

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Theorem (Bohman, Frieze and Martin, 2003)
For every \( \alpha > 0 \), there is a \( C = C(\alpha) \) such that with \( p \geq C/n \), a.a.s. \( G_{\alpha} \cup G(n, p) \) is Hamiltonian.

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One sees a decrease in the probability threshold (by a logarithmic factor).
Randomly perturbed graphs: Perfect tilings

Theorem (Balogh, Treglown and Wagner, 2019)

Let $r \geq 2$. For every $\alpha > 0$, there is a $C = C(\alpha, r)$ such that with $p \geq Cn^{-2/r}$, a.a.s. $G_\alpha \cup G(n, p)$ contains a perfect $K_r$-tiling.

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They give bounds for perfect $H$-tilings, for every graph $H$ (when $\alpha > 0$).

- When $H = K_r$, their result is optimal for $0 < \alpha < 1/r$.
- What more can be said if $\alpha \geq 1/r$?
Theorem (Han, Morris and Treglown, 2020+)

Let $2 \leq k \leq r$ and $1 - \frac{k}{r} < \alpha < 1 - \frac{k-1}{r}$.

There is a $C = C(\alpha, r)$ such that with $p \geq Cn^{-2/k}$, a.a.s. $G_\alpha \cup G(n, p)$ contains a perfect $K_r$-tiling. Moreover, this is indeed the threshold.

- The threshold exhibits a 'jumping' behaviour.
Randomly perturbed graphs: Perfect tilings (ctd.)

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Example for perfect \( K_3 \)-tiling:

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Question

What about the boundary cases: \( \alpha = 1/r, 2/r, \ldots, (r - 2)/r \)?

- For perfect \( K_3 \)-tiling, the only left case is \( \alpha = 1/3 \).
Perfect $K_3$-tiling: The boundary case

$$\alpha = 0 \quad 0 < \alpha < 1/3 \quad \alpha = 1/3 \quad 1/3 < \alpha < 2/3 \quad 2/3 \leq \alpha$$

$$n^{-2/3} (\log n)^{1/3} \quad n^{-2/3} \quad ? \quad n^{-1} \quad 0$$
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For $\alpha = 1/3$, $\omega(1/n)$ is not enough and $\omega(\log n/n)$ is needed:

$K_{n/3, 2n/3}$
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Theorem (Böttcher, Parczyk, Sgueglia and S., 2020+)

There exists $C > 0$ such that for $p \geq C \log n/n$ the following holds. $G_{1/3} \cup G(n, p)$ a.a.s. contains a triangle factor.
Perfect $K_3$-tiling: Boundary case (a bit more)

**Theorem (Böttcher, Parczyk, Sgueglia and S., 2020+)**

For any $1/12 > \beta > 0$ there exist $\gamma > 0$ and $C > 0$ such that for $4\beta \leq \alpha \leq 1/3$ and $p \geq C/n$ the following holds. If $G$

- has minimum degree at least $(\alpha - \gamma) n$ and
- is not '$\beta$-close to the extremal graph',

then a.a.s. $G \cup G(n, p)$ contains $\min\{n/3, \alpha n\}$ disjoint triangles.
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$\min\{n/3, \delta(G)\}$ disjoint triangles.

- $\omega(1/n)$ is enough unless the deterministic graph is close to the (unique) extremal graph.
A glimpse into the proof: Extremal case

Embedding Lemma
For all $d \in (0, 1)$ there are $\varepsilon > 0$ and $C$ so that: Let $U, V, W$ be of size $n$. If

- $(V, U)$ and $(V, W)$ are $(\varepsilon, d)$-super-regular pairs,
- $G(U, W, p)$ is a random bipartite graph with $p \geq C \log n/n$,

then a.a.s. there is a triangle factor in $G[U \cup V \cup W]$. 

A glimpse into the proof: Non-extremal case

Stability tool concerning matchings

If $\delta(G) \geq (\frac{1}{3} - \gamma)n$ and $G$ is not $\beta$-close to the extremal graph, then its reduced graph $R$ has a matching with $(\frac{1}{3} + 4\gamma)v(R)$ edges.
Larger Cliques

Boundary cases for larger cliques

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However, we know that if the deterministic graph is not extremal, then $n - 2/3$ is the right threshold.

Is this true in general?

A similar behaviour happens in the case of $K_r$-tiling for all $\alpha = 1 - k/r$ with $2 < k < r$ and $r \geq 4$. 
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$K_4$-tiling at $n/4$: More complicated than expected.

Let $\log^3 n \leq m \leq n^{1/7}$ and $p = n^{-2/3} (\log n)^{1/3}$.
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Let $\log^3 n \leq m \leq n^{1/7}$ and $p = n^{-2/3}(\log n)^{1/3}$.

- Construct $G$: take $A \cup B$ with $|A| = n/4 - m$ and $|B| = 3n/4 + m$, $A$ is an independent set, $G[B]$ is given by $|B|/(2m)$ disjoint copies of $K_{m,m}$ and $G[A, B]$ is complete.
- From the construction: $\delta(G) \geq n/4$.
- If $G \cup G(n, p)$ contains a $K_4$-factor, since $A$ only contains $n/4 - m$ vertices, at least $m$ copies of $K_4$ must lie within $B$. 
\(K_4\)-tiling at \(n/4\): More complicated than expected.

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One can take small $\varepsilon > 0$, $n^{7\varepsilon} \leq m \leq n^{1/7}$ and $p = n^{-2/3+\varepsilon}$.
2-universality

\[ \mathcal{F}(n, \Delta) := \{ F : |V(F)| = n \text{ and } \Delta(F) \leq \Delta \} \]

A graph is 2-universal if it contains every element of \( \mathcal{F}(n, 2) \) as a subgraph.
Ongoing work: Universality

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The hardest graph to embed is the \( K_3 \)-factor:

In progress (Böttcher, Parczyk, Sgueglia and S.)

When \( \alpha = 1/3 \), the threshold for 2-universality is \( \log n/n \).

• We know \( \omega(1/n) \) suffices if \( G \) is not close to the extremal graph.
More open problems

- Balogh, Treglown and Wagner gave optimal thresholds for perfect \(H\) tiling in \(G_\alpha \cup G(n, p)\) for \(\alpha < 1/|H|\). The problem is still wide open for larger values of \(\alpha\).
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In progress (Böttcher, Parczyk, Sgueglia and S.)

When $\alpha = 1/3$, $\omega(\log n/n)$ is the correct threshold for 2-universality.

- Find the correct threshold for 3-universality in $G_\alpha \cup G(n, p)$.

This is much harder because $\mathcal{F}(n, 3)$ contains expanders while $\mathcal{F}(n, 2)$ only unions of cycles and paths.
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Thank you!