Generating sets for powers of finite algebras and the complexity of quantified constraints

Barnaby Martin

Algorithms and Complexity Group, Durham University, UK

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Let us study the growth rate of generating sets for direct powers of an algebra $\mathbb{A}$.

For $\mathbb{A}$ we have a function $f_\mathbb{A}: \mathbb{N} \to \mathbb{N}$, giving the cardinality of the minimal generating sets of the sequence
- $\mathbb{A}$, $\mathbb{A}^2$, $\mathbb{A}^3$, ... as
- $f(1)$, $f(2)$, $f(3)$, ...

We say $\mathbb{A}$ has the XGP with:

(PGP) polynomial, when $f_\mathbb{A} = O(i^c)$, for some $c$; and

(EGP) exponential, when exists $b$ so that $f_\mathbb{A} = \Omega(b^i)$. 

Theorem (Wiegold 1987)

Let $B$ be a finite semigroup. If $B$ is a monoid then $B$ has the (linear) PGP. Otherwise, $B$ has the EGP.

Proof of PGP.
If $B$ is a monoid with identity $1$ and $|B| = n$, then

\[
\begin{align*}
(B, 1, \ldots, 1, 1) \\
(1, B, \ldots, 1, 1) \\
& \quad \vdots \\
(1, 1, \ldots, B, 1) \\
(1, 1, \ldots, 1, B)
\end{align*}
\]

is a generating set for $B^m$ of size $mn$. 
Theorem (Wiegold 1987)

Let $\mathbb{B}$ be a finite semigroup. If $\mathbb{B}$ is a monoid then $\mathbb{B}$ has the (linear) PGP. Otherwise, $\mathbb{B}$ has the EGP.

Proof of EGP.
Otherwise, without an identity, $\mathbb{B}$ and $\mathbb{B}^m$ have the properties that

$$|x \cdot B| \leq n - 1, \text{ for each } x \in B.$$  
$$|z \cdot B^m| \leq (n - 1)^m, \text{ for each } z \in B^m.$$  

Thus, a subset of $B^m$ of size $r$ can generate no more $r + r(n - 1)^m$ elements in $\mathbb{B}^m$. Thus, a generating set must be of size

$$\geq \left( \frac{2n}{2n-1} \right)^m.$$
Constraint Satisfaction Problems

The constraint satisfaction problem (CSP) is a popular formalism in Artificial Intelligence in which one is given

- a triple \((V, D, C)\) of variables, domain, constraints

and in which one asks for an assignment of the variables to the domain that satisfies the constraints.

A popular parameterisation involves fixing \(D\) and restricting

- the constraint language \(C\).

This can be formulated combinatorially as CSP(\(C\)) with

- Input: a structure \(A\).
- Question: does \(A\) have a homomorphism to \(C\)?

or logically as CSP(\(C\)) with

- Input: a sentence \(\phi\) of \(\{\exists, \land, =\}\)-FO.
- Question: does \(C \models \phi\)?
Example

CSP($\mathcal{K}_3$), or CSP($\{r, g, b\}; \neq$), is \textit{Graph 3-colourability}.

Combinatorially, one looks for a homomorphism from $\mathcal{C}_5$ to $\mathcal{K}_3$. Logically, one asks if $\mathcal{K}_3 \models \Phi$.

$$\Phi := \exists v_1, v_2, v_3, v_4, v_5 \quad E(v_1, v_2) \wedge E(v_2, v_1) \wedge E(v_2, v_3) \wedge E(v_3, v_2) \wedge E(v_3, v_4) \wedge E(v_4, v_3) \wedge E(v_4, v_5) \wedge E(v_5, v_4) \wedge E(v_5, v_1) \wedge E(v_1, v_5).$$
Quantified Constraint Satisfaction

The *quantified constraint satisfaction problem* QCSP(\(\mathcal{B}\)) has

- Input: a sentence \(\phi\) of \(\{\forall, \exists, \land, =\}\)-FO.
- Question: does \(\mathcal{B} \models \phi\)?

It is the CSP with \(\forall\) returned.
“The QCSP might be thought of as the dissolute younger brother of its better-studied restriction, the CSP. . . . CSPs are ubiquitous in CS . . . , while QCSPs can not nearly claim to be so important in applications.”

<table>
<thead>
<tr>
<th>useful QCSPs</th>
<th>classified?</th>
</tr>
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<tbody>
<tr>
<td>relativised $(\forall x \in X, \exists y \in Y)$</td>
<td>√</td>
</tr>
<tr>
<td>Boolean (QBF or QSAT)</td>
<td>√</td>
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“...what is left of the true non-Boolean QCSP is a problem I believe to be mostly of interest to theorists.”
### Complexity of Model Checking

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<tr>
<th>Fragment</th>
<th>Dual</th>
<th>Classification?</th>
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<tbody>
<tr>
<td>({\exists, \lor}) ({\forall, \land})</td>
<td>({\forall, \land}) ({\forall, \land, \neq}) ({\forall, \land, =})</td>
<td>Logspace</td>
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<tr>
<td>({\exists, \lor, =}) ({\forall, \land, \neq}) ({\forall, \land, =})</td>
<td>Logspace if there is some element (a) s.t. all relations are (a)-valid, and NP-complete otherwise</td>
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<tr>
<td>(\forall, \exists, \land, \lor)</td>
<td>Tetrachotomy: P, NP-complete, co-NP-complete or Pspace-complete</td>
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<tr>
<td>({\forall, \exists, \land, \lor, =}) ({\forall, \exists, \land, \lor, \neq}) ({\neg, \exists, \forall, \land, \lor, =})</td>
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First-order structures

Relational structures:

\[ B := (B; R_1, R_2, \ldots) \]

Functional structures:

\[ B := (D; f_1, f_2, \ldots) \]

functional structures = algebras.

What is the interplay between relational and functional structures?

Model Theory = Logic + Universal Algebra

All our structures are finite-domain.
Let $R$ be an $m$-ary relation on $\mathbb{B}$. We say that a $k$-ary operation $f : B^k \rightarrow B$ preserves $R$ (or $R$ is invariant) under $f$ if:

$$
\begin{align*}
\quad & f, \quad f, \quad \ldots, \quad f \\
\quad & (x_{11}, \quad x_{12}, \quad \ldots, \quad x_{1m}) \in R \\
\quad & (x_{21}, \quad x_{22}, \quad \ldots, \quad x_{2m}) \in R \\
\vdots & \quad \quad \quad \vdots \\
\quad & (x_{k1}, \quad x_{k2}, \quad \ldots, \quad x_{km}) \in R \\
\quad & (y_1, \quad y_2, \quad \ldots, \quad y_m) \in R
\end{align*}
$$

where each $y_i = f(x_{1i}, x_{2i}, \ldots, x_{ki})$.

- operations that preserve each of the relations of $\mathbb{B}$ are $\text{Pol}(\mathbb{B})$.
- relations invariant under each operation of $\mathbb{B}$ are $\text{Inv}(\mathbb{B})$. 
Let $\mathbb{B}$ and $\mathbb{B}$ be over the same finite domain $B$.

\begin{align*}
\text{Inv}(\text{Pol}(\mathbb{B})) &= \langle \mathbb{B} \rangle \{\exists, \wedge, =\} \\
\text{Inv}(\text{surPol}(\mathbb{B})) &= \langle \mathbb{B} \rangle \{\forall, \exists, \wedge, =\}
\end{align*}

**Idempotent** operations are **surjective**! The **algebraic** definition for $\text{QCSP}(\mathbb{B})$ has

- **Input**: a sentence $\phi$ of $\{\forall, \exists, \wedge\}$-FO with some relations $\mathbb{B} \in \text{Inv}(\mathbb{B})$.
- **Question**: does $\mathbb{B} \models \phi$?

What if $\text{Inv}(\mathbb{B})$ is **infinite**?
Each relation $R$ can be given as a list of tuples, but this is far too lengthy! How about a Boolean formula $\phi$ in atoms

- $v = v'$ and $v = c$,

where $c$ is a domain element. The problem is that recognising, e.g., non-emptiness of the relation can be NP-hard! Following others, e.g. [Bodirsky & Dalmau 2006] we will ask for

- $\phi$ in DNF,

However, our main result will be a separation NP versus co-NP-hard, so this is not a big deal!
Infinite languages on a finite domain

Example 1.

\[
\{ (1, 2), (2, 1), (x \neq y \lor x = 1) \\
2, 3), (3, 2), \\
(1, 3), (3, 1), \\
(1, 1) \}
\]

Example 2.

\[
\{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (x \neq y \lor y \neq z) \\
1, 1, 0), (1, 0, 1), (1, 1, 0), \}
\]
Call an algebra $B$ $k$-PGP-switchable if $B^m$ is generated from the set of $m$-tuples of the form

- $(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, \ldots, x_{k'}, \ldots, x_{k'})$ for some $k' \leq k$.

Switchability were originally introduced in connection with the QCSP by Hubie Chen!

**Theorem (Chen 2008)**

*If $\mathbb{A}$ is switchable then $QCSP(\mathbb{A})$ is in NP.*

**Theorem (LICS 2015)**

$\mathbb{A}$ is PGP-switchable iff it is switchable.
A number of handsome people worked on the PGP-EGP dichotomy conjecture.

**Conjecture**

*Let $B$ be a finite idempotent algebra, then either $B$ has PGP or it has EGP.*

In 2015, Dmitriy Zhuk solved it.

**Theorem (Zhuk 2015)**

*Let $B$ be a finite algebra, then either $B$ is PGP-switchable or it has EGP.*

In order to prove this result, Zhuk assumes $B$ is not PGP-switchable and finds the existence of a certain class of relations in $\text{Inv}(B)$. 
Church of Switchability

B. M.

switchability


Results

Henceforth, let $A$ be an idempotent algebra on a finite domain $A$.

Conjecture (Chen Conjecture 2012)

Let $B$ be a finite relational structure expanded with all constants.
If $\text{Pol}(B)$ has PGP, then $\text{QCSP}(B)$ is in NP; otherwise $\text{QCSP}(B)$ is Pspace-complete.

Theorem (Revised Chen Conjecture)

If $A$ satisfies PGP, then $\text{QCSP}($Inv$(A))$ is in NP. Otherwise, if $A$ satisfies EGP, then $\text{QCSP}($Inv$(A))$ is co-NP-hard.

Conjecture (Alternative Chen Conjecture)

If $A$ satisfies PGP, then for every finite reduct $B \subseteq \text{Inv}(A)$, $\text{QCSP}(B)$ is in NP. Otherwise, there exists a finite reduct $B \subseteq \text{Inv}(A)$ so that $\text{QCSP}(B)$ is co-NP-hard.
Results

Henceforth, let $A$ be an idempotent algebra on a finite domain $A$.

Conjecture (Chen Conjecture 2012)

Let $B$ be a finite relational structure expanded with all constants. If $Pol(B)$ has PGP, then $QCSP(B)$ is in NP; otherwise $QCSP(B)$ is $P$space-complete.

Theorem (Revised Chen Conjecture)

Either $QCSP(Inv(A))$ is co-$NP$-hard or $QCSP(Inv(A))$ has the same complexity as $CSP(Inv(A))$.

Conjecture (Alternative Chen Conjecture False)

If $A$ satisfies PGP, then for every finite reduct $B \subseteq Inv(A)$, $QCSP(B)$ is in NP. Otherwise, there exists a finite reduct $B \subseteq Inv(A)$ so that $QCSP(B)$ is co-$NP$-hard.
Tractability

We know from Zhuk 2015 that

\[ \text{PGP} \rightarrow \text{PGP-switchability} \]

and from [LICS 2015]

\[ \text{PGP-switchability} \rightarrow \text{switchability} \]

whereupon Chen 2008 gives

\[ \text{switchability} \rightarrow \text{QCSP tractability}. \]
Henceforth, $\alpha, \beta$ be strict subsets of $A$ so that $\alpha \cup \beta = A$.

**Theorem (Zhuk 2015)**

*Algebra $\mathbb{A}$ (idempotent) has EGP iff exists such $\alpha, \beta$ with*

$$
\sigma_k(x_1, y_1, \ldots, x_k, y_k) := \rho(x_1, y_1) \lor \ldots \lor \rho(x_k, y_k),
$$

*where $\rho(x, y) = (\alpha \times \alpha) \cup (\beta \times \beta)$, is in $\text{Inv}(\mathbb{A})$, for each $k \in \mathbb{N}$.*

We prefer the relation $\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k)$ defined by

$$
\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k) := \rho'(x_1, y_1, z_1) \lor \ldots \lor \rho'(x_k, y_k, z_k),
$$

*where $\rho'(x, y, z) = (\alpha \times \alpha \times \alpha) \cup (\beta \times \beta \times \beta)$.*

**Corollary**

*Algebra $\mathbb{A}$ (idempotent) has EGP iff exists such $\alpha, \beta$ with $\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k)$ in $\text{Inv}(\mathbb{A})$, for each $k \in \mathbb{N}$.*
Theorem

If $\text{Inv}(A)$ satisfies EGP, then QCSP($\text{Inv}(A)$) is co-NP-hard.

Proof.

Reduce from the complement of (monotone) 3-not-all-equal-sat.

$$\exists x^1_1, x^2_1, x^3_1, \ldots, \ldots, x^1_m, x^2_m, x^3_m \text{NAE}(x^1_1, x^2_1, x^3_1) \land \ldots \land \text{NAE}(x^1_m, x^2_m, x^3_m)$$

becomes

$$\forall x^1_1, x^2_1, x^3_1, \ldots, \ldots, x^1_m, x^2_m, x^3_m \rho'(x^1_1, x^2_1, x^3_1) \lor \ldots \lor \rho'(x^1_m, x^2_m, x^3_m)$$

where we note that $\tau_m(x_1, y_1, z_1 \ldots, x_m, y_m, z_m) :=$

$$\rho'(x_1, y_1, z_1) \lor \ldots \lor \rho'(x_m, y_m, z_m)$$

has a DNF representation that is polynomially-sized in $m$. 
Recall, $\alpha, \beta$ be strict subsets of $A$ so that $\alpha \cup \beta = A$. Now ask further that $\alpha \cap \beta \neq \emptyset$.

Corollary

$\text{QCSP}(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$ is co-NP-hard.

In fact,

Proposition

$\text{QCSP}(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$ is in co-NP.

Proof.

Roughly speaking, evaluate all existential variables to something in $\alpha \cap \beta$. But $(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$ is not finitely related. □

Proposition

For every finite reduct $\mathcal{B}$ of $(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$, $\text{QCSP}(\mathcal{B})$ is in NL.
The conventional definition for $\text{QCSP}(\mathcal{B})$, where $(\mathcal{B})$ is a finite constraint language, is

- Input: a sentence $\phi$ of $\{\forall, \exists, \land\}$-FO.
- Question: does $\mathcal{B} \models \phi$?

Conjecture (Chen Conjecture + CSP Dichotomy)

Let $\mathcal{B}$ be a finite relational structure expanded with all constants. Either $\text{QCSP}(\mathcal{B})$ is in $P$, is NP-complete or is Pspace-complete.
Example $R_{\delta,3}$.

\[\{(1, -, -), (2, -, -), (0, 0, 0), (0, 1, 1), (0, 2, 2), (x \neq 0 \lor y = z)\}\]

Example $R_{\text{and},2}$.

\[\{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1), (2, -, -), (-, 2, -)\}\]

- QCSP($\{0, 1, 2\}; 0, 1, 2, R_{\text{and},2}, R_{\delta}$) is co-NP-complete.
Death of the Chen Conjecture II

Example $R_{\delta,2}$.

\[
\{ (0, 0), (1, 0), (2, 0), \\
(1, 2), (2, 2) \}
\]

Example $R_{\text{and},2}$.

\[
\{ (0, 0), (1, 0), (2, 0), \\
(1, 2), (2, 2) \}
\]

- $\text{Pol}(\{0, 1, 2\}; 0, 1, 2, R_{\text{and},2}, R_{\delta,3})$ has EGP.
- $\text{QCSP}(\{0, 1, 2\}; 0, 1, 2, R_{\text{and},2}, R_{\delta,3})$ is in P.
QCSP Monsters

There are finite \( \mathcal{B} \) so that QCSP(\( \mathcal{B} \)) ranges over

- in P.
- NP-complete.
- Pspace-complete.
- co-NP-complete.
- DP-complete.
- \( \Theta^P_2 \)-complete.
- ... 

Theorem (Zhuk & M. 2019)

Let \( \mathcal{B} \) be a finite 3-element relational structure expanded with all constants. Either QCSP(\( \mathcal{B} \)) is in P, is NP-complete, is co-NP-complete or is Pspace-complete.
Future of the Chen Conjecture

The conservative case is a natural large class on which the Chen Conjecture holds.

**Theorem (Zhuk & M. 2019)**

Let $\mathcal{B}$ be a finite relational structure expanded with all unary relation. Either $\text{QCSP}(\mathcal{B})$ is in $P$, is NP-complete, or is $P$space-complete.

Can PGP and EGP be sensibly modified to make the Chen Conjecture “true”? 