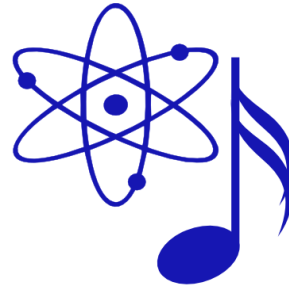**Session 3**
Scales and Tuning
Paul Harrison

Science of Music

Today's session

- Harmonics and the harmonic series
- Pythagorean Tuning
- Quarter comma meantone
- Just intonation
- Equal temperament
- Richard Gratwick on Pure Maths and Tuning

Scales and Tuning: Introduction

- Almost all tuning traditions based on the octave
- Scales typically divide octave into several intervals
- How many and which intervals is a matter of cultural convention, informed by the behaviour of instruments
- The perfect fifth is also a common, though not universal interval
- Popularity of octave and perfect fifth is no accident of culture

What is an Octave?

- At least 3 definitions:
  - the "same" note at a different pitch (music)
  - frequency ratio equal to exactly 2 (maths)
  - Interval between the first and second harmonics of an oscillator (ie. an instrument), (physics)

- They are all equivalent!

⚛♪ Why is Octave so significant?

- It is a "natural" interval (I claim)
  - arises naturally in oscillating physical systems
  - basis for its musicality, (our ear is a natural physical system).
- Consider a string stretched between two fixings (like in a stringed instrument)
  - Excite it - it will oscillate and emit a sound at a recognisable pitch
  - This note (of the open string) is called the fundamental or first harmonic of the string's vibration.

⚛♪ Demo/Video

- Hopefully, we have a demo which works
- If not, this video gives an idea:
  - https://www.youtube.com/watch?v=BSIw5SgUirg
- And here's one we made:
  - https://www.youtube.com/watchv=VZxybLZFcnI&feature=youtu.be

Do The Math!

- String pitch corresponds to frequency given by Newton's Laws in terms of its physical properties:

$$frequency = \frac{1}{2}\frac{1}{Length}\sqrt{\frac{Tension}{Density}}$$

- Tighten/loosen string => increase/decrease frequency/pitch of note (tuning instrument)
- Shorten string (by "stopping" it against neck)
  => increase frequency/pitch of note
  - ie. playing the instrument - physically different origin of change in pitch.
  - Can get ANY note (unfretted instrument)

Back to Octaves (and fifths)!

- Can also get different notes from the string in a much more subtle way, using harmonics.
- And this time, we cannot get any note we want!**
- 1st harmonic is "fundametal": frequency = "f"
- 2nd harmonic: frequency 2xf = octave above 1st
- 3rd "   freq. 3xf = + perf 5th above 2nd harm
- 4th "   freq. 4xf = + perf 4th above 3rd "
- 5th "   freq. 5xf = + maj 3rd above 4rd "
- 6th "   freq. 6xf = + min 3rd above 5th "

Called the Harmonic series (goes to infinity).
These special pitches I call "natural" for the string.

Compare Fretted Notes

- Fretted notes have nothing to do with harmonics.
- By contrast, there is little natural about them - man-made by the manufacturer (given by position of frets on the fingerboard).

Harmonic Series

- Equations of physics dictate (further) that arbitrary oscillation of a string consists of an "admixture" of all the different harmonics.
- Where you strike, pluck or bow the string affects the proportions of each harmonic (which, in turn, determines the timbre of the sound).
- Almost identical considerations affect wind instruments - the same set of integer relations affect the harmonics of sound waves in pipes.
- Thus, harmonics naturally sound together and are consonant.

Harmonic Series (Contd.)

- Such natural intervals as members of the harmonic series played together (as a chord) are perceived as consonant, for several reasons:
  - since the notes naturally occur together when generated on any instrument.
  - since they generate no beats (or rather, their beats also belong to the harmonic series)
- Other intervals will produce audible "beats" when played together, and these are harsh on the ear and dissonant.

Last word on the Science

- Have covered the basic science behind harmonics, their naturalness and consonance.
- Also the frequencies to which they correspond, ie. integer multiples of the fundamental frequency, f.
- Remark: such infinite sequences of discrete frequencies arise naturally whenever we have standing waves existing physically on a well-defined interval, because
  - physically, we have to fit an exact number of waves between the fixed ends of the "oscillator" (instrument).
  - mathematically, since such integer sequences arise naturally in the solutions of second order partial differential equations according to periodic boundary conditions.
- They are also found in eg. the physics of atoms, and for similar reasons.
- But that is another story!

⚛♫ Pythagorean Tuning
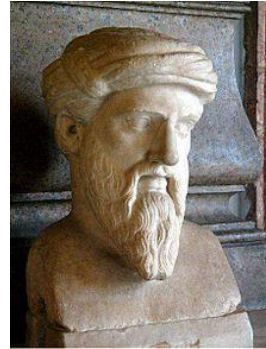
- Recognition of importance of natural "modes" (ie harmonics/overtones) on a string or pipe drove emergence of scales and tuning of instruments in earliest times.
  - Esp. importance as "natural" of the octave and the perfect fifth.
- No accident that strings on members of violin family are tuned to relative pitches of perfect fifths.
  - (cf. guitar, it is largely perfect 4ths, the octave-complement of the fifth).
- If I want to tune the next string to a convenient note (especially on unfretted instrument) what better than the perfect fifth, which the string will naturally generate for you (once you find the node).

⚛♫ Pythagoras Motivation

- Octave (1/2 the length of string) sounds same for both the harmonic _and_ when stopping the string onto the fret-board - the only place where this happens.
- 1/3 of the length gives an octave plus a perfect fifth (in harmonics) - 3f in frequency. But it also gives exactly a perfect fifth on the stopped string (ie. an octave lower - 3f/2 in frequency). The only place that _this_ happens.
- So, it must have been "obvious" to the early music theorists, like Pythagoras, that these natural intervals were of huge fundamental importance.
- Harmonics must have seemed almost magical, since the instrument itself "knows" them - it insists on them!

Pythagoras Details

- Perfect Fifth thus had practical, as well as aesthetic (and potentially mystical) value.
- Pythagoras set out to explore all the notes you can reach by taking multiple intervals of a perfect 5th, starting at an initial fundamental pitch, or the octave above it.
  - E.g. drop a p. 5th and then go up an octave => p. 4th above fundamental. Frequency ratio here is factor 2 (octave) divided by 3/2 (goes down a 5th) i.e. 2x2/3=4/3=1.333 relative to the fundamental.
  - Two p. fifths above start is factor (3/2)^2=9/4 in frequency (> Octave), so then drop factor 2 (octave) to bring it back in range => frequency ratio of 9/8=1.125. This is a "major tone".
  - Etc. (Nb. Also known as the "Circle of fifths").
- Starting on a D, can quickly generate a pentatonic or 7-note C major scale (C, D, E, F, G, A, B).

Note	Interval from D	Formula	Frequency ratio	Size (cents)	ET-dif (cents)
A♭	diminished fifth	$\left(\frac{2}{3}\right)^6 \times 2^4$	$\frac{1024}{729}$	588.27	-11.73
E♭	minor second	$\left(\frac{2}{3}\right)^5 \times 2^3$	$\frac{256}{243}$	90.22	−9.78
B♭	minor sixth	$\left(\frac{2}{3}\right)^4 \times 2^3$	$\frac{128}{81}$	792.18	−7.82
F	minor third	$\left(\frac{2}{3}\right)^3 \times 2^2$	$\frac{32}{27}$	294.13	−5.87
C	minor seventh	$\left(\frac{2}{3}\right)^2 \times 2^2$	$\frac{16}{9}$	996.09	−3.91
G	perfect fourth	$\frac{2}{3} \times 2$	$\frac{4}{3}$	498.04	-1.96
D	unison	$\frac{1}{1}$	$\frac{1}{1}$	0 .00	0.00
A	perfect fifth	$\frac{3}{2}$	$\frac{3}{2}$	701.96	1.96
E	major second	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$	$\frac{9}{8}$	203.91	3.91
B	major sixth	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$	$\frac{27}{16}$	905.87	5.87
F♯	major third	$\left(\frac{3}{2}\right)^4 \times \left(\frac{1}{2}\right)^2$	$\frac{81}{64}$	407.82	7.82
C♯	major seventh	$\left(\frac{3}{2}\right)^5 \times \left(\frac{1}{2}\right)^2$	$\frac{243}{128}$	1109.78	9.78
G♯	augmented fourth	$\left(\frac{3}{2}\right)^6 \times \left(\frac{1}{2}\right)^3$	$\frac{729}{512}$	611.73	11.73

Pythagorean "chromatic" scale (notes ordered by construction).

Final column gives deviation from standard (modern) pitch in "cents".

100 cents=1 semitone

⚛♪ Features of Pythagoras' Major Scale

- Has 5 major tones of interval 9/8=1.125
- Has 2 semitones of $\dfrac{2^8}{3^5} = 1.0535$ (not half!)
- If we start on a different note, C say, all the generated notes will be different from those in D
  - So, can only play in one key on a tuned instrument.
- Does not close octave after complete cycle:
  - Pitches of Ab and G# are not equal!
  - Differ by ~1/4 of semitone.

⚛♪ The Wolf Fifth and the Pythagorean Comma

- Require the octave to be a factor 2 in frequency
- The interval from G# to Eb above the 2nd D is then reduced by the same 1/4 of a semitone.
- This flat fifth is called a wolf fifth and is badly out of tune.
- Cannot play a piece combining these notes.
- The 1/4 of a semitone (= 23.46 cents) is called the Pythagorean comma.

Problems with Pythagoras Tuning

- Already described some issues with Pythagoras' tuning.
- Construction was based only on octaves and fifths.
- Other important intervals from the harmonic series were not correctly reproduced.
  - In particular, the "major third" does not correspond well to the one in the harmonic series - is dissonant.
- Later tuning schemes were developed to try to address these problems.

Just Intonation

- All the intervals of Harmonic series are described by small-integer freq. ratios.
- JI is defined as any tuning system which uses only small integer ratios.

- Claudius Ptolemy (~100 AD) considered it a mistake not to get the major third right.
- 5th harmonic = 5xf is 2 octaves plus major third above fundamental.
- Projecting back into 1st octave gives ratio of 5/4=1.25 for 3rd (cf. 81/64 =1.266 in Pythagoras).
- Difference = "Syntonic Comma".

Ptolemy's Intense Diatonic Scale

- Ptolemy suggested to generate the major third as a ratio 5/4 and the perfect fifth as the ratio 3/2.
- This created what we recognise as a major triad
- Then proceeded as Pythagoras to generate scale.

	Name	C	D	E	F	G	A	B	C
	Solfege	Do	Re	Mi	Fa	Sol	La	Ti	Do
Note	Ratio	1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
	Harmonic	24 (help·info)	27 (help·info)	30 (help·info)	32 (help·info)	36 (help·info)	40 (help·info)	45 (help·info)	48 (help·info)
	Cents	0	204	386	498	702	884	1088	1200
	Name	T		t	s	T	t	T	s
Step	Ratio	9/8		10/9	16/15	9/8	10/9	9/8	16/15
	Cents	204		182	112	204	182	204	112

Tempering

- Ptolemy's scale was lost to the Western world for centuries and Pythagoras' was used until the 15th century, with some improvements.
- While Pythagorean tradition was used, it was gradually appreciated that something wasn't quite right.
- To improve the wolf fifth, and to make the pythagorean third more consonant, some of the notes in the scale were adjusted in pitch slightly.
- Many of these "temperings" were presumably arrived at by trial-and-error.

Quarter Comma Meantone Temperament (16thc & 17thc)

- A modification to Pythagorean tuning
  - an attempt to make the major third "just" while sacrificing slightly the just-ness of the fifth.
- 12 intervals were created in the traditional way, by stacking p.5ths
  - But p.5th was redefined to force 4 stacked fifths to make an octave plus a major third (ie. 5xf) exactly.
  - Thus freq. ratio of fifth defined by:

$$r \times r \times r \times r := r^4 = 5$$
$$\implies r = \sqrt[4]{5} = 1.495$$

Note	Interval from D	Formula	Freq. ratio	Size (cents)
A♭	diminished fifth	$x^{-6} \cdot 2^4 = \dfrac{16\sqrt{5}}{25}$	1.4311	620.5
E♭	minor second	$x^{-5} \cdot 2^3 = \dfrac{8\sqrt{5}x}{25}$	1.0700	117.1
B♭	minor sixth	$x^{-4} \cdot 2^3 = \dfrac{8}{5}$	1.6000	813.7
F	minor third	$x^{-3} \cdot 2^2 = \dfrac{4x}{5}$	1.1963	310.3
C	minor seventh	$x^{-2} \cdot 2^2 = \dfrac{4\sqrt{5}}{5}$	1.7889	1006.8
G	perfect fourth	$x^{-1} \cdot 2^1 = \dfrac{2\sqrt{5}x}{5}$	1.3375	503.4
D	unison	$x^0 \cdot 2^0 = 1$	1.0000	0.0
A	perfect fifth	$x^1 \cdot 2^0 = x$	1.4953	696.6
E	major second	$x^2 \cdot 2^{-1} = \dfrac{\sqrt{5}}{2}$	1.1180	193.2
B	major sixth	$x^3 \cdot 2^{-1} = \dfrac{\sqrt{5}x}{2}$	1.6719	889.7
F♯	major third	$x^4 \cdot 2^{-2} = \dfrac{5}{4}$	1.2500	386.3
C♯	major seventh	$x^5 \cdot 2^{-2} = \dfrac{5x}{4}$	1.8692	1082.9
G♯	augmented fourth	$x^6 \cdot 2^{-3} = \dfrac{5\sqrt{5}}{8}$	1.3975	579.5

Quarter Comma Meantone 13-note scale

Pitch/freq ratio difference between this fifth and p. 5th is 0.3% or ~5 cents. This is ~ ¼ a Syntonic Co-mma (inaudible to most).

3rd and 5th are good, but still have wolf fifth (sharp and worse than in Pythagoras).

⚛♫ Just Intonation Rediscovered

- JI rediscovered in the late 16th c. and described by Gioseffo Zarlino
- Following intervals were defined (all members of the harmonic series or intervals between them):

    - Octave = 2/1
    - Perf 5th = 3/2
    - Perf 4th = 4/3 (octave complement of Perf 5th)
    - Maj 3rd = 5/4
    - Min 3rd = 6/5 (Per 5th complement of Maj 3rd)

Result was essentially the Ptolemy's Intense Diatonic Scale, which had not survived in the West, but was preserved in the arabic tradition in the meantime, and rediscovered in the crusades

⚛♫ The Main Problem

- Although the intervals in JI are the most consonant and harmonious, this tuning, together with most of the preceding ones all suffer the problem that they do not transpose between keys.
    - An instrument justly tuned in one key will sound great in that key, but the intervals are all wrong for other keys, in which it cannot therefore be played.
- Thus, equal temperament.

Equal Temperament

- In some ways, the simplest, and easiest tuning system to understand.
- Designed to smooth out all the imperfections associated with the other systems
  - distributes them evenly among all the notes
  - imperfections in any one note are inaudible to the general human ear.
- Designed to be transposable between keys, one of its most important properties.
- It seems to have first been written down (and perhaps invented) independently in China by Zhu Zaiyu in 1584 and by Simon Stevin (Holland) in 1585.

Equal Temperament

- Its defining properties are:
  - To respect the just octave (ie. freq. ratio = 2)
  - To divide the just octave into a fixed number of equal frequency-ratio steps
  - There should be 12 such steps per just octave
- The 2nd property suffices to make the tuning transposable between keys, which, apart from the first, is its most important feature.
- The 3rd property is based on the traditional western scales, but since they are based on harmonic series, it means that it works remarkably well.

ET Frequency Ratio?

- The defining properties of ET suffice to determine the ratio, "r" as follows.
- Since multiplying by r takes us from one pitch to the next, and r is the same in each of 12 steps:

$$r^{12} = 2$$
$$\implies r = \sqrt[12]{2} = 1.0595$$

- Which gives the ratio for the basic (universal) semitone.
- Is "default" modern tuning – used on keyboards and fretted instruments etc.

Compare ET with JI Intervals

Name	Exact value in 12-TET	Decimal value in 12-TET	Cents	Just intonation interval	Cents in just intonation	Error
Unison (C)	$2^{0/12} = 1$	1.000000	0	$\frac{1}{1} = 1.0000000$	0.00	0
Minor second (C#/Db)	$2^{1/12} = \sqrt[12]{2}$	1.059463	100	$\frac{16}{15} = 1.06666...$	111.73	−11.73
Major second (D)	$2^{2/12} = \sqrt[6]{2}$	1.122462	200	$\frac{9}{8} = 1.1250000$	203.91	−3.91
Minor third (D#/Eb)	$2^{3/12} = \sqrt[4]{2}$	1.189207	300	$\frac{6}{5} = 1.2000000$	315.64	−15.64
Major third (E)	$2^{4/12} = \sqrt[3]{2}$	1.259921	400	$\frac{5}{4} = 1.2500000$	386.31	+13.69
Perfect fourth (F)	$2^{5/12} = \sqrt[12]{32}$	1.334840	500	$\frac{4}{3} = 1.33333...$	498.04	+1.96
Tritone (F#/Gb)	$2^{6/12} = \sqrt{2}$	1.414214	600	$\frac{7}{5} = 1.4000000$	582.51	+17.49
Perfect fifth (G)	$2^{7/12} = \sqrt[12]{128}$	1.498307	700	$\frac{3}{2} = 1.5000000$	701.96	−1.96
Minor sixth (G#/Ab)	$2^{8/12} = \sqrt[3]{4}$	1.587401	800	$\frac{8}{5} = 1.6000000$	813.69	−13.69
Major sixth (A)	$2^{9/12} = \sqrt[4]{8}$	1.681793	900	$\frac{5}{3} = 1.66666...$	884.36	+15.64
Minor seventh (A#/Bb)	$2^{10/12} = \sqrt[6]{32}$	1.781797	1000	$\frac{16}{9} = 1.77777...$	996.09	+3.91
Major seventh (B)	$2^{11/12} = \sqrt[12]{2048}$	1.887749	1100	$\frac{15}{8} = 1.8750000$	1088.27	+11.73
Octave (C)	$2^{12/12} = 2$	2.000000	1200	$\frac{2}{1} = 2.0000000$	1200.00	0

Some intervals are remarkably close to JI, particularly the fourth and fifth

Pictorial Comparison
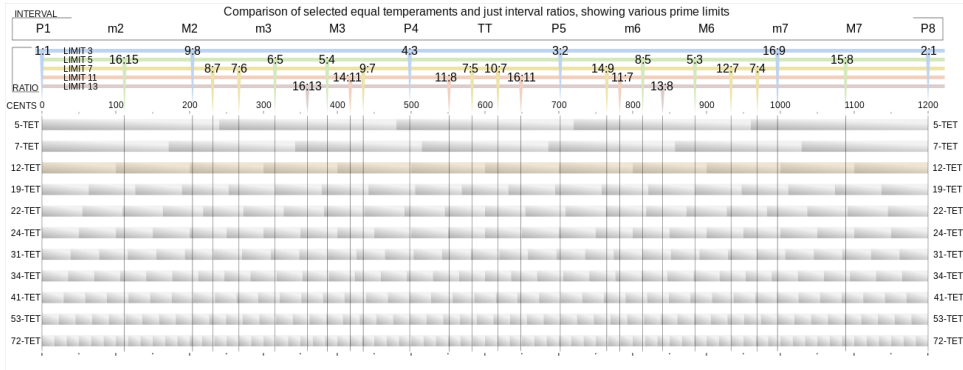
"Equal Temper w limits" by Just plain Bill at en.wikipedia. Licensed under CC BY-SA 3.0 via Commons – https://commons.wikimedia.org/wiki/File:Equal_Temper_w_limits.svg#/media/File:Equal_Temper_w_limits.svg

Some Links

- The Wiki page on JI has some great comparison sound clips:
- https://en.wikipedia.org/wiki/Just_intonation
- Pachelbel in 3 tunings:
  - https://www.youtube.com/watch?v=d2I1zNw2w-c
- Piano tuner
  - https://www.youtube.com/watch?v=TBt6APk21tU