SCIENCE OF MUSIC, SESSION 3, SCALES AND TUNING, 29^{th} JANUARY 2016

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1. The problem

We have seen that there have been many different attempts at coming up with a satisfactory musical scale: the Pythagorean approach, Just Intonation, and Tempering, for example. We have seen such a variety because of one simple, unavoidable fact, which we establish below: any "complete" set of musical notes with perfectly in tune octaves and fifths needs infinitely many notes. The different scales are ways of dodging or ignoring that problem.

Pythagorean tuning uses the rules of deriving octaves and fifths to generate a twelve-note chromatic scale with a discrepancy between G-sharp and A-flat, if one starts on D. What happens if we just accept the discrepancy, admit that they are two different notes, and carry on?

(Notation: we identify a note with the frequency of its fundamental sound wave. Thus a note is a number to which we may apply the usual rules of arithmetic.)

Theorem 1. Suppose we want to construct a scale, or set of playable notes, obeying the following two rules:

- for any playable note x, successive octaves above and below x are playable; and
- for any playable note x, successive fifths above and below x are playable.

Then any non-empty set of playable notes contains infinitely many playable notes *in any one octave.*

Proof. Suppose x is playable. Going up one octave corresponds to doubling the frequency, going down one octave corresponds to halving the frequency. Iterating this we see that we need every multiple of x which is a positive or negative whole number power of 2 to be playable. Similarly, since going up one fifth corresponds to multiplying the frequency by $3/2$, and going down one fifth corresponds to dividing the frequency by $3/2$, iterating this shows that we need every multiple of x which is a positive or negative whole number power of $3/2$ to be playable. In summary, the octaves and fifths criteria imply that

- $2^n x$ is playable for all $n = 0, \pm 1, \pm 2, \ldots$; and
- $(3/2)^m x$ is playable for all $m = 0, \pm 1, \pm 2, \ldots$.

We shall show that this implies that between any two distinct playable notes, there exists another playable note not equal to the first two. Iterating this we find infinitely many playable notes, between any given pair of playable notes (in particular within any given octave).

Let y be another playable note, strictly larger than x say, so $x < y$. Since then $x/y < 1$ *and since* 2 *and* 3 *are prime*, we may choose $n, m = 0, \pm 1, \pm 2, \ldots$ such that

$$\frac{x}{y} < \frac{2^n}{3^m} < 1.$$

(For the curious or sceptical, this is proved formally in section 3 below.)

So then

$$x < \frac{2^n}{3^m} y < y, \text{ where } \frac{2^n}{3^m} y = 2^{n-m} \left(\frac{3}{2}\right)^{-m} y,$$

and therefore this is playable, since this corresponds to taking the playable note y, and from it taking fifths $-m$ times and octaves $n - m$ times. (Positive numbers correspond to going "up" a certain number of octaves or fifths, negative numbers correspond to going "down" a certain number of octaves or fifths.) Note carefully that since the resulting number lies between x and y, although the numbers of octaves and fifths taken might in general be very large, we can choose an order of doing so so that we never involve notes more than an octave above or below y. Thus this result does *not* rely on the range in terms of very high or low notes of our scale being very great. □

This argument would not work if there existed whole numbers $n, m \geq 2$ such that $2^n = 3^m$. In this case, the set of all notes derived by octaves and fifths from one starting note within a given range would indeed be finite.

So suppose indeed that $2^n = 3^m$. Let p, q be positive or negative whole numbers. We shall see what we can obtain by taking p octaves and q fifths from a starting note. Since the starting note itself will just remain throughout as a constant factor, we assume it to be 1, so can ignore it.

There exist positive or negative whole numbers k, l and whole numbers r, s such that $0 \leq r \leq n - 1$ and $0 \leq s \leq m - 1$ such that

$$p - q = kn + r \text{ and } q = lm + s.$$

Then, since $2^n/3^m = 1$, and so any power of $2^n/3^m$ equals 1, we see that

$$2^p \left(\frac{3}{2}\right)^q = \frac{2^{p-q}}{3^{-q}} = \frac{2^{kn+r}}{3^{-(lm+s)}} = \left(\frac{2^n}{3^m}\right)^{-l} \cdot \frac{2^{(k+l)n+r}}{3^{-s}} = 2^{(k+l)n} \cdot \left(2^r 3^s\right).$$

Since $0 \leq r \leq n - 1$ and $0 \leq s \leq m - 1$, there are only finitely many numbers $(2^r 3^s)$. The factor $2^{(k+l)n}$, corresponding to taking $(k + l)n$ octaves from each $2^r 3^s$, can take infinitely many values, but if we restrict to a plausible situation of having only a bounded range to the frequencies of playable notes, then only finitely many are permitted. Thus we have finitely many octaves of finitely many notes, i.e. only finitely many playable notes.

The point is that this situation does not, or rather cannot, occur. There are no two whole numbers $n, m \geq 2$ such that $2^n = 3^m$. Such a situation would contradict one of the cornerstones of number theory.

Theorem 2 (The Fundamental Theorem of Arithmetic)**.** Every whole number can be expressed in a *unique* way as a product of prime numbers.

Since 2 and 3 are indeed prime numbers (i.e. each has no whole divisors other than 1 and itself), if there were such a pair $n, m \geq 2$ with $2^n = 3^m$, then this common value would be a number written in two different ways as a product of prime numbers. This would contradict the uniqueness assertion in the Fundamental Theorem of Arithmetic.

2. The solution

But, of course, pianos don't have infinitely many keys. What do we do to get around this problem? The customary approach is to preserve the octave, but then divide that into k evenly spaced intervals, such that the ratio of each note to its preceding note is $2^{1/k}$, since if we go up k notes, we must reach the octave, i.e. double the frequency: $(2^{1/k})^k = 2$. And having decided that, we just have to hope that for some choice of k, going l notes, say, up the scale to a multiple of $(2^{1/k})^l = 2^{l/k}$ of the starting note gives us in fact a multiple of 3/2 of the starting note, i.e. we have managed to preserve the fifth as well. *But this is impossible*: if $2^{l/k} = 3/2$, then we would have that

$$2^{k+l} = 3^k,$$

which once more would contradict the Fundamental Theorem of Arithmetic.

The point is that we are lucky in that there are certain choices for which the discrepancy is tolerably small. Standard western classical music chooses $k = 12$, thus there are twelve (what we call) semitones in an octave and, pleasingly, seven semitones very nearly make a fifth:

$$(2^{1/12})^7 = 2^{7/12} = 1.498\cdots \approx 3/2.$$

Happily the fourth comes out rather well too:

$$(2^{1/12})^5 = 2^{5/12} = 1.335\cdots \approx 4/3.$$

This is why we choose to divide the octave into twelve semitones: it gives good approximations to other harmonious intervals. But there are other choices: we could divide into 29, where 17 steps approximates the fifth, or 53, where 31 steps approximates the fifth. But already we face quite an engineering feat to build such a piano and quite a musical challenge to play such a thing. See Malcolm Savage, "Beautiful Music: An Amazing Mathematical Fluke?", *Mathematics Today*, April 2014, for some discussion of this, in particular remarks about how, in some sense, we are "lucky" that such a good solution exists, requiring comparatively few notes in the scale.

The basic point I want to bring to this session as a mathematician is that the difficulties inherent in finding a satisfactory scale and building satisfactory instruments are *not physical*. The limitations are not just that we haven't found the right answer or been unable to build it. It is a *logically* impossible to find a scale, which allows one to modulate and preserve both octaves and fifths perfectly in tune, with only finitely many notes per octave.

3. The auxiliary result

Here for those who wish it is a proof of the technical fact about powers of 2 and 3 we used in the proof of theorem 1. Since I suspect this will be of no interest to those unable to follow such a presentation, I will adopt a more rigorous mathematical approach and notation. Nonetheless I have attempted to choose an argument with a minimum of specific mathematical prerequisites.

Lemma 3. Let $\varepsilon > 0$. Then there exist $n, m \in \mathbb{Z}$ such that $2^n/3^m \in (1-\varepsilon, 1)$.

Proof. Taking logarithms we are required to prove that there exist n, m such that

$$\log(1-\varepsilon) < \log(2^n/3^m) = n\log 2 - m\log 3 < \log 1 = 0.$$

This amounts to proving that for a given $\delta > 0$, there exist $n, m \in \mathbb{Z}$ such that

$$|n\log 2 - m\log 3| < \delta.$$

Let such a $\delta > 0$ be given. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N} \cup \{0\}$ and $r_n \in [0, \log 3)$ such that

$$n\log 2 = k_n \log 3 + r_n.$$

Note that since 2 and 3 are prime, no two r_n are equal. For suppose otherwise, that $r_n = r_m$ for $n \neq m$. Then by assumption

$$n\log 2 - k_n\log 3 = r_n = r_m = m\log 2 - k_m\log 3,$$

whence

$$(n-m)\log 2 = (k_n - k_m)\log 3,$$

which implies that $2^{n-m} = 3^{k_n - k_m}$, which would contradict the Fundamental Theorem of Arithmetic. So in particular, choosing $N \in \mathbb{N}$ such that $N > (\log 3)/\delta$, the first $N+1$ terms r_n are all distinct. Considering $[0, \log 3)$ to be divided into N partitions of length $(\log 3)/N$, we see (by the pigeonhole principle) that at least two of the r_n must lie in the same partition, r_{n_1} and r_{n_2} say. That is,

$$|r_{n_1} - r_{n_2}| \leq (\log 3)/N < \delta.$$

But then by definition of the r_n, we have that

$$\begin{aligned} |(n_1 - n_2)\log 2 - (k_{n_1} - k_{n_2})\log 3| &= |(n_1\log 2 - k_{n_1}\log 3) - (n_2\log 2 - k_{n_2}\log 3)| \\ &= |r_{n_1} - r_{n_2}| \\ &< \delta. \end{aligned}$$

So setting $n := n_1 - n_2$ and $m := k_{n_1} - k_{n_2}$ gives n, m as required. □