PREFACE

1, 2, 3, . . . The "counting numbers" are part of us. We know them forward and backward. Babies as young as five months old, psychologists claim,[1] are sensitive to the difference between 1 + 1 and 2 − 1. We sing numbers, counting up the days of Christmas and counting down to the poignant monotheism of "One is one and all alone and evermore shall be so."

Our ancestors have added to this repertoire and reckoned with zero and the negative numbers, which were sometimes referred to as fictions (*fictae*) before they gained familiarity.

All these together constitute what we call the *whole numbers*,

$$\ldots, -2, -1, 0, +1, +2, \ldots$$

More formally, they are called *integers*, from the Latin adjective meaning "whole, untouched, unharmed."

"Whole," "untouched"; their very name hints that integers *can be* touched, or fractionated. Indeed they can be, and when they are, we get the larger array of numbers that are *fractions*, ratios of whole numbers.

Fractions, as their notation vividly displays, also stand for proportions (think of $\frac{1}{2} = \frac{2}{4}$ as "one is to two as two is to four") and for actions (think of $\frac{1}{2}$ as "halving," ready to cut in half anything that follows it).

To bring fractions into line, we express them as decimals ($\frac{1}{2}$ = 0.5000000 . . .). The modern world gives us much experience with decimals to a high degree of accuracy—to "many decimal places"; mathematics, as always, goes all the way, happy to deal with numbers with complete accuracy—to *infinitely* many decimal places. Numbers represented by infinitely many decimal places, whether they are fractions or not, are called *real numbers.*

But the telltale adjective *real* suggests two things: that these numbers are somehow real to us and that, in contrast, there are *unreal* numbers in the offing. These are the *imaginary numbers.*

The *imaginary* numbers are well named, for there is some imaginative work to do to make them as much a part of us as the real numbers we use all the time to measure for bookshelves.

This book began as a letter to my friend Michel Chaouli. The two of us had been musing about

whether or not one could "feel" the workings of the imagination in its various labors.[2] Michel had also mentioned that he wanted to "imagine imaginary numbers." That very (rainy) evening, I tried to work up an explanation of the idea of these numbers, still in the mood of our conversation.

The text of my letter was the welcome excuse for continued conversation with a number of friends, many of whom were humanists interested in understanding what it means to imagine the square root of negative numbers. The successive revisions and expansions of my letter were shaped by their questions, comments, critiques, and insights. This book, then, is written for people who have no training in mathematics and who may not have actively thought about mathematics since high school, or even during it, but who may wish to experience an act of mathematical imagining and to consider how such an experience compares with the imaginative work involved in reading and understanding a phrase in a poem. Of course, poetry and mathematics are far apart. All the more glorious, then, for people at home in the imaginative life of poetry to use their insight and sensibility to witness the imagination at work in mathematics.

Although no particular mathematical knowledge is necessary, pencil and paper are good to have at hand, to make a few calculations (multiplying small numbers, mostly). The operation of multiplication itself is something we will be looking at. There are three standard ways of denoting the act of multiplication: by a *cross* ×, by a *centered dot* ·, or, when there is no ambiguity, by *simple juxtaposition* of the objects to be multiplied. Which notation we use reflects where we wish to direct our attention: the equation

$$2 \times 3 = 6$$

emphasizes the *act* of multiplying 2 times 3, whereas

$$2 \cdot 3 = 6$$

focuses on the *result*, 6, of that operation. Nevertheless, despite this difference in nuance, both equations, $2 \times 3 = 6$ and $2 \cdot 3 = 6$, are saying the same thing. When we deal with an unknown quantity X, here are three equivalent ways of denoting 5 times that unknown quantity:

$$5 \times X = 5 \cdot X = 5X.$$

Again, we write $5 \times X$ if we want to emphasize the act of multiplying and $5 \cdot X$ or $5X$ if we want to emphasize the result; and that last variant notation, juxtaposition, is used for visual conciseness.

This book has footnotes and endnotes. Some of the endnotes are side comments requiring more mathematical background than is assumed in the text.

PART I

1

THE IMAGINATION AND SQUARE ROOTS

1. Picture this.

Picture Rodin's *Thinker*, crouched in mental effort. He has his supporting right elbow propped *not* on his right thigh, as you or I would have placed our right elbow, but rather on his left thigh,[1] which bolts him into an awkward striving, his muscles tense with thought. But does he, can we, really *feel* our imaginative faculty at work, striving toward, and then finally achieving, an act of the imagination?

Consider the range of our imaginative experiences. Consider, for example, how immediate is the experience of imagining what we read. Elaine Scarry has remarked that there is *no* "felt experience" corresponding to this imaginative act.[2] We experience, of course, the *effect* of what we are reading. Scarry claims that if we read a phrase like

> the yellow of the tulip[3]

we form, perhaps, the image of it in our mind's eye and experience whatever emotional effect that image produces within us. But, says Scarry, we have no *felt experience* of coming to form that image. We will return to this idea later.

Perhaps one should contrast reading with trying to think something up for ourselves. Rainer Maria Rilke's comment on the working of our imagination,

> We are the bees of the invisible[4]

paints our imaginative quests as not entirely *unfelt experiences* (following Scarry), but not *contortions* (following Rodin) either. Our gathering of the honey of the imaginative world is not immediate; it takes work. But though it requires traveling some distance, merging with something not of our species, communicating by dance to our fellow creatures what we've done and where we've been, and, finally, bringing back that single glistening drop, it is an activity we do without contortion. It is who we bees are.

Thinking about the *imagination imagining* is made difficult by the general swiftness and efficacy of that faculty. The imagination is a fleet genie at your service. You want an elephant? Why, there it is:

(picture your elephant here)

You read "the yellow of the tulip."

And, again, there it is: a calligraphed swath of yellow on your mental movie screen.

More telling, though, are the other moments of thought, when our genie is not so surefooted. Moments composed half of bewilderment and half of expectation; moments, for example, when some new image, or viewpoint, is about to reveal itself to us. But it resists emerging. We are forced to angle for it.

At those times, it is as if the waters of the imagination are roiling; you have cast your fishing line from a somewhat shaky boat, and you feel a tug on that line, but have no clear sense what you have hooked onto. Bluefish, old boot, or some underwater species never before seen? But you definitely feel the tug.

I want to think about the inner articulations of our imaginative life by "re"-experiencing a particular example of such a tug. The example I propose to consider occurs in the history of mathematics. It might be described as a *moment of restless anticipation in the face of a slowly emerging act of imagining. Moment*, though, is not the right word here, for the period, rather, stretches over three centuries. And *anticipation* carries too progressivist and perhaps too personal a tone, for this "act" doesn't take place fully in any single mind. There are many "bees of the invisible" in the original story.

If we are successful, we will be reenacting, for ourselves, the imagining of a concept that, for the original thinkers, had never been seen or thought before, and that seemed to lie athwart things seen or thought before.* Of course, thinking about things never thought before is the daily activity of thought, certainly in art or science. The cellist Yo-Yo Ma has suggested that the job of the artist is to go to the edge and report back.[5] Here is how Rilke expressed a similar sentiment: "Works of art are indeed always products of having-been-in-danger, or having-gone-to-the-very-end in an experience, to where one can go no further."[6]

In contrast to the instantly imaginable "yellow of the tulip," the *square root of negative quantities* was a concept in common use in mathematics for over three hundred years before a satisfactory geometric under-

*This sentence echoes the caption of an old cartoon in which a child is pursued by a demon of his imagination and cries out, "It looks very much like something I have never seen before!"

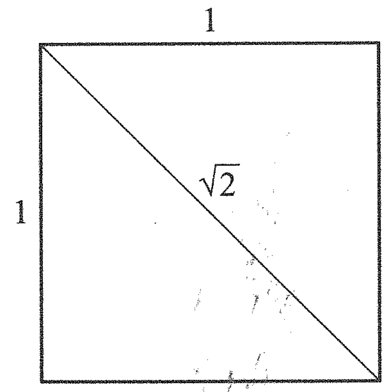
standing of it was discovered. If you deal *exclusively with positive quantities*, you have less of a challenge in coming to grips with square roots: the *square root of a positive number* is just a quantity whose square is that number.

Any positive number has only one (positive) square root. The square root of 4, for example, is 2. What is the square root of 2? We know, at the very least, that its square is 2. Using the equation that asserts this,

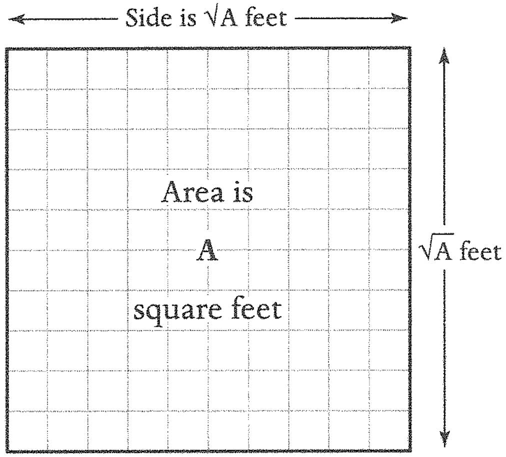
$$(\sqrt{2})^2 = \sqrt{2} \cdot \sqrt{2} = 2,$$

try your hand at estimating $\sqrt{2}$. Is it smaller than $\frac{3}{2}$? Do you see why $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$?

Square roots are often encountered geometrically, as lengths of lines. We will see shortly, for example, that $\sqrt{2}$ is the length of the diagonal of a square whose sides have length 1.



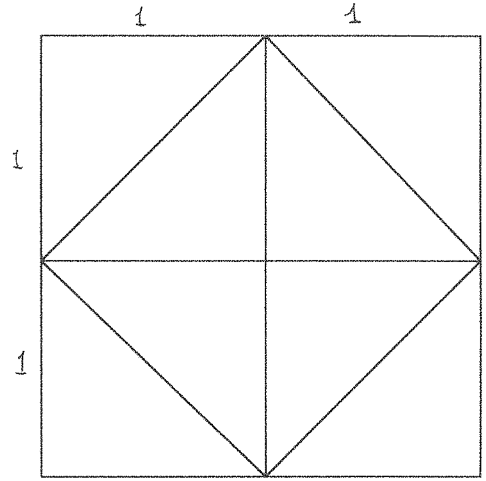
Also, if we have a square figure whose area is known to be *A* square feet, then the length of each of its sides, as in the diagram below, is $\sqrt{A}$ feet.



The square root as "side"

Suppose that each box in this diagram has an area equal to 1 square foot. There are a hundred boxes, so A = 100, and the dimensions of the large square are $\sqrt{A}$ by $\sqrt{A}$—that is, 10 by 10.

In Plato's *Meno*,[7] Socrates asks Meno's young slave to construct a square whose area is twice the area of a given square. Here is the diagram that Socrates finally draws to help his interlocutor answer the question:



The profile of this diagram is a 2 × 2 square (whose area is therefore 4) built out of four 1 × 1 squares (each of area 1). But in its midst, we can pick out a catercorner square (standing, as it seems, on one of its corners). By rearranging the triangular pieces that make up the diagram, can you see, as Socrates' young friend in the *Meno* did, that the catercorner square has area 2, and therefore each of its sides has length $\sqrt{2}$?

The sides of the catercorner square play a double role: they are also the diagonals of the small (1 × 1) squares. So, as promised a few paragraphs earlier, we see $\sqrt{2}$ as the length of the diagonal drawn in a square whose sides are of length 1.

The early mathematicians thought of the square root as a "side"; the sixteenth-century Italians would at times simply refer to the square root of a number as its *lato*, its "side." Thus, at first glance, negative numbers don't have square roots, for (as I discuss later) the square of any numerical quantity (positive or negative) is positive. In fact, a second and third glance will tend to confirm the suspicion that negative numbers are not *entitled* to have square roots.

If we think of square roots in the geometric manner, as we have just done, to ask for the square root of a negative quantity is like asking: "What is the length of the side of a square whose area is *less than zero*?" This has more the ring of a Zen koan than of a question amenable to a quantitative answer.[8] Nevertheless, these seemingly nonexistent square roots were, early on, seen to be *useful*. But the first users of square roots of negative numbers were queasy about the practice of invoking such airy objects. These strange square roots were called *imaginary numbers*, meaning they were difficult to place among *real* mathematical objects.

And then, an astonishingly satisfying image of these square roots emerged. A way of *imagining* these otherwise unpicturable "numbers" was found independently, and almost simultaneously, by two, or possibly three (or more), people.* What a dramatic act: to find a

*A friend suggested that since none of those directly involved in the publication of this discovery had any other significant mathematical contribution to their credit (with the exception of Legendre, who, as we shall see, plays a curious role)—that is, since all these individuals are peripheral to the intense mathematical progress of the time (the end of the eighteenth century)—it is possible that the "pictorial image" they came

home in our imagination for such an otherwise troublesome concept!

This "way of imagining" has become our common intellectual property. It and the numbers it helped us imagine have found thorough and ubiquitous use, not only by mathematicians but by every engineer who works with the calculus, by every physicist.

The aim of this book is not to give a historical account.* Rather, it is to re-create, in ourselves, the shift of mathematical thought that makes it possible to imagine these numbers.

Poetry, to be sure, has "shifts of thought" at its core, the "turn" of the poem, in both its viewpoint and its typography, being celebrated in the word *verse*. Poetry demands our paying attention to these turns. For people who pay such attention while reading poetry but who have never done anything similar with mathematics, I hope the style of presentation I have adopted—which passes back and forth between reflections on the imaginative work of thinking about poetry and thinking about mathematics—will be helpful.

In proceeding with our mathematical theme, we

up with was, in fact, "in the air," was in the "public domain," at least to the extent that the "public" included Euler and his colleagues. In any event, plucking such coins out of the air is a pretty good trick, which, with luck, we too will do in subsequent pages.

*See the annotated bibliography at the end of the book for a list of sources that provide a systematic logical or historical account of the concept of number.

want first to feel the uncomfortableness of the early mathematicians who encounter imaginary numbers; then to sense the possibility that some shift, some new viewpoint in thinking about number, may help to tame the concept of imaginary number; then to be conscious of the emergence of this viewpoint within ourselves. Finally, we will see that our new attitude toward number unifies otherwise disparate intuitions and helps us interpret an amazing formula that perplexed sixteenth-century mathematicians.

As for prerequisites, the less mathematics you know, the better prepared you are for the task ahead. To follow the mathematics presented here, you will only need to have the skill to perform certain simple multiplications and substitutions when the text requests this, and to allow with equanimity the occasional appearance of simple algebraic equations of the type encountered in the first weeks of high school algebra.* If you can do, or follow, the sample exercises in this endnote,[9] you are ready for the math in this book.

Let us start by considering that imaginative construct, the faculty of imagination itself.

*A comment by one of the readers of an early draft of this book led me to revise it substantially. My manuscript, the reader said, reminded him of the time he paged through the *Kama Sutra*. The *Kama Sutra* promised that a wonderful world was his if only he had (but he hadn't) sufficient flexibility and skill. The present version of this book requires neither.

2. Imagination.

A certain mathematical article opens with the invitation:

> Imagine . . . an infinite completely symmetrical array of points.[10]

In the prologue to Shakespeare's *Henry V*, the Chorus asks that you, the audience, let the actors,

> *. . . ciphers to this great accompt,*
> *On your imaginary forces work.*

Paul Scott's *The Raj Quartet* begins with a request of the reader:

> IMAGINE, then, a flat landscape, dark for the moment, but even so conveying to a girl running in the still deeper shadow cast by the wall of the Bibighar Gardens an idea of immensity.[11]

What a problematic instruction: to be told to *imagine*! What are we doing, and do we have the language to say what we are doing, when we fulfill that instruction?

Our English word *imagination* has a direct antecedent in Latin, but the earlier Latin word, which connoted "object of the imagination" (at least as a side meaning), is *visio*, whose standard meaning is "sight." For a discussion about this (and for a comprehensive history and commentary on what has been said about the imagination), see Eva Brann's majestic *The World of the Imagination—Sum and Substance.*[12] Here is Quintilian explaining the Greek origin of the Latin term *visio*:

> What the Greeks call "phantasies" we rightly term "sights" through which the images of absent things are so represented in the mind that we seem to discern and have them present.[13]

Quintilian's definition of *sights* as meaning "objects of the imagination" is a serviceable definition, as far as it goes. It includes things we have seen before but which happen to be absent. Its reach, however, does not encompass the unicorns and sphynxes that tinkers and joiners of the imagination have thrown together for us.

One might try to extend Quintilian's definition, following the lead of Jeremy Bentham, by claiming that the imagination is a faculty by which "a number of abstracted ideas are compounded into one image."[14] Bentham's definition goes a bit further than Quintilian's, but not much, for surely there are objects of thought that cannot be parsed in terms of the algebra of simple, previously known images.

And Bentham's definition, which has the imaginative faculty playing the menial role of editing table for videotapes of the mind's eye, would hardly satisfy

Wordsworth, who would prefer a loftier function of the imagination: the function of connecting mere fact with "that infinity without which there is no poetry."[15] For Wordsworth, the imaginative faculty is the transcendental alchemist that turns, for example, the "mere" gold band of a wedding ring into a symbol of eternal unity.

Quintilian, Bentham, Wordsworth, et al., notwithstanding, there are those who simply shrug off "imagination" as an " 'onomatoid,' that is, a namelike word which in fact designates nothing because it signifies too broadly."[16] Is it *one* thing, deserving of the pronoun *it*? Coleridge makes a distinction in *Biographia Literaria* between what he calls the *imagination* and its less daring sibling *fancy*, which "is indeed no other than a mode of memory emancipated from the order of time and space."[17] In some circles, the concept of the imaginative faculty (or, at least, the idea that you can say anything about it) raises philosophical suspicion; in other circles, its very mention raises religious fears. For example, a recent review of high school history textbooks reports that, to satisfy the religious right, the word *imagine* is largely banished from textbooks. An editor at McGraw-Hill is quoted as saying, "We were told to try to avoid using the word 'imagine' because people in Texas felt it was too close to the word 'magic' and therefore might be considered anti-Christian."[18]

Nevertheless, there are certain experiences of the intellect that cannot be discussed at all without grappling with the issue of the imagination.

3. Imagining what we read.

When we look at a page of writing, our mind's eye sees something quite different than the white page, the black ink. John Ashbery, in his prose poem "Whatever It Is, Wherever You Are," writes of reading:

> [T]he yellow of the tulip, for instance—will flash for a moment in such a way that after it has been withdrawn we can be sure there was no imagining, no auto-suggestion here, but at the same time it becomes as useless as all subtracted memories.

He muses about the *inventors of writing*:

> To what purpose did they cross-hatch so effectively, so that the luminous surface that was underneath is transformed into another, also luminous but so shifting and so alive with suggestiveness that it is like quicksand, to take a step there would be to fall through the fragile net of uncertainties into the bog of certainty . . .

and suggests that the images conjured by reading flash onto our mental screen and convey "certainty without heat or light."[19] For Scarry, the "vivacity" of the yellow

flash of the tulip compels conviction, and the suddenness of its appearance in our mind precludes our having any "felt experience of image-making." She says that

> the imagination consists exclusively of its objects, that it is only knowable through its objects, that it is remarkable among intentional states for not being easily separable into the double structure of state and object.[20]

To give a sense of what she means by the double structure of "state and object," Scarry offers the comparison of the acts of "imagining a flower" and "fearing an earthquake." She points out that "fear of an earthquake" has two parts to it: the contemplated object of the fear, and the inner experiencing of this fear. In contrast, suggests Scarry, "imagining a flower" has as object the imagined flower, but comes along with no further, separable, inner experience of the exercise of the faculty of the imagination.

This may not be surprising, for even with the (happily) true and not imagined sensory perception of *the smell of coffee in the morning*, one has the inner experience of the smell of that coffee without any palpable separate experience of the exercise of one's sense of smell.

Scarry herself, I should emphasize, does not alight for too long upon these ideas. For those who have not read her prose-poem essay, I wouldn't want to ruin its dramatic momentum by revealing how it evolves from this theme, or to reveal its further surprises. But can we, as a modest test of Scarry's claim, catch glimpses of our *imagination at work*? With this in mind, let us turn to the initial setting of our mathematical story.

4. Mathematical problems and square roots. As already hinted, square roots show up as answers to even some of the simplest of geometric questions. And if your appetite for mathematical problems grows, you find, as did the sixteenth-century Italian algebraists, more complicated numerical quantities like

$$\sqrt{\sqrt{52}+2}$$

(this one happens to be roughly 3.03) appearing routinely as *solutions*.[21]* Reading these Italian mathematicians, you can only have admiration for the tongue-twisting lengths to which they went to indulge their tastes for mathematical puzzles, which were often allowed to masquerade as practical(?!) problems:

> A certain king sent 128,000 *aurei* to the proconsul who was leading his army so that he might hire 7000

*The general notation for square root, cube root, fourth root, and so forth, is $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. In the case of square root, however, the 2 can be omitted (i.e., the signs $\sqrt{\ }$ and $\sqrt[2]{\ }$ both mean square root).

> foot soldiers and 7000 horsemen. The ratio of the stipend was such that 100 *aurei* would hire 18 more foot soldiers than it would mounted men. A certain tribune of soldiers came to the proconsul with 1700 foot men and 200 horsemen and asked for his share of the pay . . .[22]

If the martial setting of this algebra word problem is not to your liking, you can turn your talents to trying to solve an earlier one, posed in the twelfth-century text *Vija-Gan'ita* of Bháskara, Problem 132 (see Colebrooke's *Algebra*):

> The square-root of half the number of a swarm of bees is gone to a shrub of jasmin; and so are eight-ninths of the whole swarm; a female is buzzing to one remaining male; that is humming within a lotus, in which he is confined, having been allured to it by its fragrance at night. Say, lovely woman, the number of bees.[23]

Ineluctably, however, as the sixteenth-century Italian mathematicians allowed particular tactics of solution to particular problems to give way to more general methods applied to more general problems, in their calculations they found themselves nudged more and more urgently, by the momentum of their ideas,[24] to make use of quantities like $\sqrt{-1}$. Especially puzzling is that some of these calculations succeed in giving perfectly comprehensible answers to perfectly comprehensible questions, but only by dealing along the way with somewhat incomprehensible quantities like $\sqrt{-1}$. This can be unsettling; rather like discovering that there is an efficacious way of getting from Brooklyn to Boston, but that somewhere in mid-journey one has to descend to the Underworld.

Here is a concrete example of the type of ordinary-sounding problem that might move a sixteenth-century mathematician to use quantities like $\sqrt{-1}$ to effect its (theoretical, but not practical) solution.

> Suppose that someone has given you the following information about an aquarium tank. The tank holds a volume of 25 cubic feet, and is 1 foot taller than it is wide, and 1 foot longer than it is tall. Find the (precise) dimensions (length, width, height) of the tank.

I said, parenthetically, that quantities like $\sqrt{-1}$ are used to establish a theoretical, not a practical, solution to the problem. To figure out an approximate answer, good enough for any practical considerations about the care and feeding of the fish in the aquarium, there are easier, rougher methods, and even trial and error will do quite well (the aquarium is about $24\frac{1}{2}$ inches wide). The aim here would be to find an exact solution to the problem and, in the course of this, to understand the solution's conceptual structure. You might respond, "What can

you possibly mean by the *conceptual structure* of an answer to this problem, which is, after all, a mere number?" Wait.

It was not that such puzzling answers to problems had never been explicitly encountered before then. Nicolas Chuquet, in his 1484 manuscript *Le Triparty*, attempting to find that number whose triple is 4 plus its square, discovers that his method comes up with the "answers" (which I give in modern notation)

$$3/2 + \sqrt{-1.75} \text{ and } 3/2 - \sqrt{-1.75}.$$

And Chuquet concludes that there is no number whose triple is 4 plus its square, because the above answers are, as he puts it, "impossible."[25] This is a perfectly valid conclusion, given that Chuquet was seeking "ordinary number" solutions to his problem. To get a sense, though, of why Chuquet might have been led to think of such expressions—deemed by him impossible—as candidate solutions to the problem, you might try to square $3/2 + \sqrt{-1.75}$ (i.e., multiply this expression by itself using "laws of ordinary arithmetic" plus the fact that the square of $\sqrt{-1.75}$ is $-1.75 = -7/4$), adding 4 to the result, and seeing whether you get 3 times $3/2 + \sqrt{-1.75}$ as the answer.[26]

In contrast to the way in which $\sqrt{-1.75}$ entered as a possible but discarded solution to Chuquet's problem, the novel element in the early Italian involvement with things like $\sqrt{-1.75}$ is that the Italian mathematicians were working on problems having perfectly possible ("ordinary numerical," i.e., real-number) answers, but their methods, at times, involved dealing with numbers like $\sqrt{-1.75}$ along the way.

5. What is a mathematical problem?

Problems are different from questions. We sometimes ask questions in full expectation that the answer will be easily given. "Do you want some more pie?" But we pose (throw out) problems for solution only if we expect that something of a mental stretch is required to come up with the answer.

One can classify categories of straightforward question-asking, as Aristotle does in the *Metaphysics*: "What?" "By what means?" "How?" "Why?" But *problems* are a different story. They seem not to submit easily to any simple categorization. Their posing may take ingenuity:

> *How hadde this cherl ymaginacioun*
> *To shewe swich a probleme to the frere?*

asks the lord in "The Summoner's Tale" in Chaucer's *Canterbury Tales.*[27] *Problems* are the mainstay of the schoolroom, and the melancholy plight of students is that they are bent over their desks working out problems set by others, not by themselves.

All the best mathematical problems are *come-ons*: there is a gentle irony behind them. The problem-setter usually presents to you a very precise task. *Solve this!* An equation, perhaps: just solve it. But if the problem is really good, a solution of it is nothing more than a letter of introduction to a level of interaction with the material that you hadn't achieved before. Solving the problem gets you to a deeper level of question-asking. The problem itself is an invitation, a goad, to extend your imagination. This is true of good school problems but is also true of some—perhaps all—of the famous and venerable mathematical problems. For example, there is the *Poincaré conjecture*, one of the great yet unachieved goals of three-dimensional geometry.[28] The Poincaré conjecture is a precise claim about the characterization of three-dimensional space, and mathematicians would keenly like to know: "Is it true?" "Is it not true?" But the impetus behind the problem is far greater than determining whether it is true or false. Work on the problem presents a possible way of extending our three-dimensional geometric intuition. Now, you might say that we all know three-dimensional space: we get into and out of our sweaters, we tie things together with knots, we dance, we explore caves and mountains. The Poincaré conjecture tells us—plus ultra*—that there is

*Before the discovery of America, *Ne plus ultra* was the motto of the royal arms of Spain, the western limit of the known world. *Beyond us*, proclaimed the motto, *there is no more*. After the discovery, however, more to be imagined, there are yet ways in which our three-dimensional intuition might be refined, and it challenges us to do so.

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when Charles V inherited the throne of Aragon and Castille, he simply deleted the *Ne* from the motto: *There is [even] more.*

2

SQUARE ROOTS AND THE IMAGINATION

6. What is a square root?

Thus far we have discussed, for example, the square root of 2 ($\sqrt{2}$), the number whose square is equal to 2, and have seen that $\sqrt{2}$ is also the length of the diagonal of a square whose sides are of length equal to 1. We can give the square root of 2 to any degree of accuracy we wish. Do you want it to ninety-nine decimal places? Here it is:

1.41421356237309504880168872420969807856967187537694807317667973799907324784 62107038850387534327641572...

It was known to the Pythagoreans that $\sqrt{2}$ *cannot* be expressed as a fraction, that is, as a ratio of whole num-