QS101: Introduction to Quantitative Methods in Social Science

Week 7: Probability Theory

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Why Probability Theory?

- Systematic treatment of uncertainty
- Humans often think in probabilistic terms (even if not gambling)
- Precursor to statistical inference (week 10 and beyond)

What we do, moves towards objective probability, which is defined as a limiting relative frequency: the long-run behaviour of a nondeterministic outcome or just an observed proportion in a population.

Counting Rules and Permutations

Minor Complexities of Counting

- Two features of counting:
  - Does the order matter?
  - Are we counting events more than once?

How many Ways to Order?

Factorial Function

$$n(n-1)(n-2)\ldots(2)(1) = n!$$

Four Scenarios of Counting

- Ordered, with Replacement
- Ordered, without replacement
- Unordered, without replacement
- Unordered, with replacement

Ordered, with Replacement

- We have n objects
- We want to pick $k < n$ from them
- We replace on each iteration
- So we always have n choices

$$n \times n \times \cdots n = n^k$$

Ordered, without Replacement

- We have n objects
- We want to pick $k < n$ from them
- We do not replace on each iteraction
- So we have n choices on the first draw, $n-1$ on the second draw, $n-2$ on the third draw, and so on

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+2) \times (n-k+1) = \frac{n!}{(n-k)!}$$

Note: This is wrong in the Gill book!

Unordered, without Replacement

- Extremely common sampling procedure
- Think of this like the last case, but we cannot see the order of picking
- Imagine we have balls in an urn: red, blue, white
- Now, it does not matter, whether we have *red, blue, white* or *white, blue, red*, and so on
- Recalling the factorial function, we now have $k!$ fewer choices than with ordered counting

$$\frac{n!}{(n-k)!k!} = \binom{n}{k}$$

Unordered, with Replacement

- Terribly unintuitive
- Think of this one as to be adjusted upwards to reflect the increased number of choices

$$\frac{(n+k-1)!}{(n-1)!k!} = \binom{n+k-1}{k}$$

Does it matter?

- 10 students
- 3 come to see me in advice and feedback hours
  - Ordered, with replacement: 1,000
  - Ordered, without replacement: 720
  - Unordered, without replacement: 120
  - Unordered, with replacement: 220

Sets and Operations of Sets

Sets and Operations of Sets

Introduction to Sets

What's a Set?

- A set is a bounded collection, defined by its contents (or lack thereof) and is denoted with curly braces.
- So, a set of even, positive integers less than 10 is

$$\{2, 4, 6, 8\}$$

Terminology

- "Thingies" contained within a set are called *elements*
- An *event* is any collection of possible outcomes of an experiment (any subset, or full set of possibilities, including the full set itself)
- Events are usually labelled with capital Roman letters, such as A, B, T, H, . . .
- Example: The event that an even number occurs when we throw a die, is

$$A = \{2, 4, 6\}$$

Characteristics of Sets

- Countability
  - Countable: One-to-one correspondence to a positive integer, such as $S = \{1, 2, 3, 4, 5, 6\}$ for rolling a die
  - Uncountable: $\mathcal{S} = [0 : 2\pi]$, for spinning a pointer on a circle, and looking at the angle in radians
- Finiteness (finite, or infinite)
- Cardinality: number of elements in a set, usually given by $n(A)$

Note: $\mathcal{S}$ denotes the sample space of a given experiment, such as rolling a die.

The Empty Set

- Does not contain any elements
- Useful later on
- Is denoted as ϕ (Greek letter phi)

Sets and Operations of Sets

Operations on Sets

Subsets

- Set A is a subset of B, if every element of A is also an element of B

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- $A = B \Leftrightarrow A \subset B$ and $B \subset A$

Union of Sets

- A union of Sets A and B, $A \cup B$, is the new set that contains all of the elements that belong to either A OR B
- Formally

$$A \cup B = \{X : X \in A \text{ or } X \in B\}$$

Example

- We throw a single die, and define the following sample space and sets:

$$\begin{aligned} \mathcal{S} &= \{1, 2, 3, 4, 5, 6\} \\ A &= \{2, 4, 6\} \\ B &= \{4, 5, 6\} \\ C &= \{1\} \end{aligned} \tag{1}$$

Then

$$A \cup B = \{2, 4, 5, 6\}$$

Intersection of Sets

- An intersection of Sets A and B, $A \cap B$, is the new set that contains all of the elements that belong to either A *AND* B
- Formally

$$A \cap B = \{X : X \in A \text{ and } X \in B\}$$

Note that if $A \cap B = \phi$, then the two sets A and B are called *disjoint*.

Example

- We throw a single die, and define the following sample space and sets:

$$\begin{aligned} \mathcal{S} &= \{1,2,3,4,5,6\} \\ A &= \{2,4,6\} \\ B &= \{4,5,6\} \\ C &= \{1\} \end{aligned} \tag{2}$$

Then

$$A \cap B = \{4,6\}$$

Complement of Sets

- A complement of a given set is the set that contains all elements not in the original set
- Formally

$$A^C = \{X : X \notin A\}$$

Example

- We throw a single die, and define the following sample space and sets:

$$\begin{aligned} \mathcal{S} &= \{1, 2, 3, 4, 5, 6\} \\ A &= \{2, 4, 6\} \\ B &= \{4, 5, 6\} \\ C &= \{1\} \end{aligned} \tag{3}$$

Then

$$A^C = \{1, 3, 5\}$$

Difference Operator

- Defines which portion of a given set is NOT a member of the other
- The difference of A relative to B is the set of elements X whereby

$$A \setminus B = \{X : X \in A \text{ and } X \notin B$$

or

$$A \setminus B = A \cap B^C$$

Example

- We throw a single die, and define the following sample space and sets:

$$\begin{aligned} \mathcal{S} &= \{1, 2, 3, 4, 5, 6\} \\ A &= \{2, 4, 6\} \\ B &= \{4, 5, 6\} \\ C &= \{1\} \end{aligned} \tag{4}$$

Then

$$A \setminus B = \{2\}$$
$$B \setminus A = \{5\}$$

The Probability Function & Calculations

The Probability Function

- Is a mapping of an event (or events) onto a metric bounded by zero (it cannot happen) and one (it will happen with absolute certainty).
- Allows us to discuss various degrees of likelihood of occurrence in a systematic and practical way

Calculations with Probabilities

- Good news: Rules are straightforward
- For A and B in $\mathcal{S}$

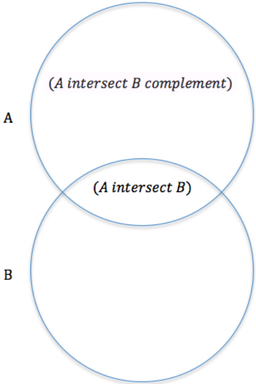
$\rightarrowtail$	Probability of Unions	$p(A \cup B)$ $= p(A) + p(B) - p(A \cap B)$
$\rightarrowtail$	Probability of Intersections	$p(A \cap B)$ $= p(A) + p(B) - p(A \cup B)$ (also denoted $p(A, B)$)
$\rightarrowtail$	Probability of Complements	$p(A^C) = 1 - p(A)$, $p(A) = 1 - p(A^C)$
$\rightarrowtail$	Probability of Null Set	$p(\phi) = 0$
$\rightarrowtail$	Probability of the Sample Space	$p(\mathcal{S}) = 1$

The Theorem of Total Probability

- Given any events A and B,

$$p(A) = p(A \cap B) + p(A \cap B^C)$$

- Probability of an event A can be decomposed into two parts
  - One that intersects with another set B
  - One that intersects with the complement of B
- If there is no intersection or if B is a subset of A, then one of the two parts has probability zero.



Odds

- Imagine a sample space with two outcomes: success and failure
- Actually quite common in social sciences: wars, marriages, memberships, crimes, etc.
- Let $p(S)$ be the probability of success
- Then, the define the probability of failure as $q = p(F) = 1 - p$
- Then, the odds of success are

$$odds(S) = \frac{p}{q}$$

Note: Odds are positive, but unbounded.