Center for Scientific Computing, University of Warwick

Optical lattices and Bose gases

MIR@W-Day and CSC One-Day Workshop

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Anisotropic Generalised BEC with Two Critical Densities

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- 1.Perfect Bose-gas.
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- 3.Exponential BEAM and CIGAR Traps.
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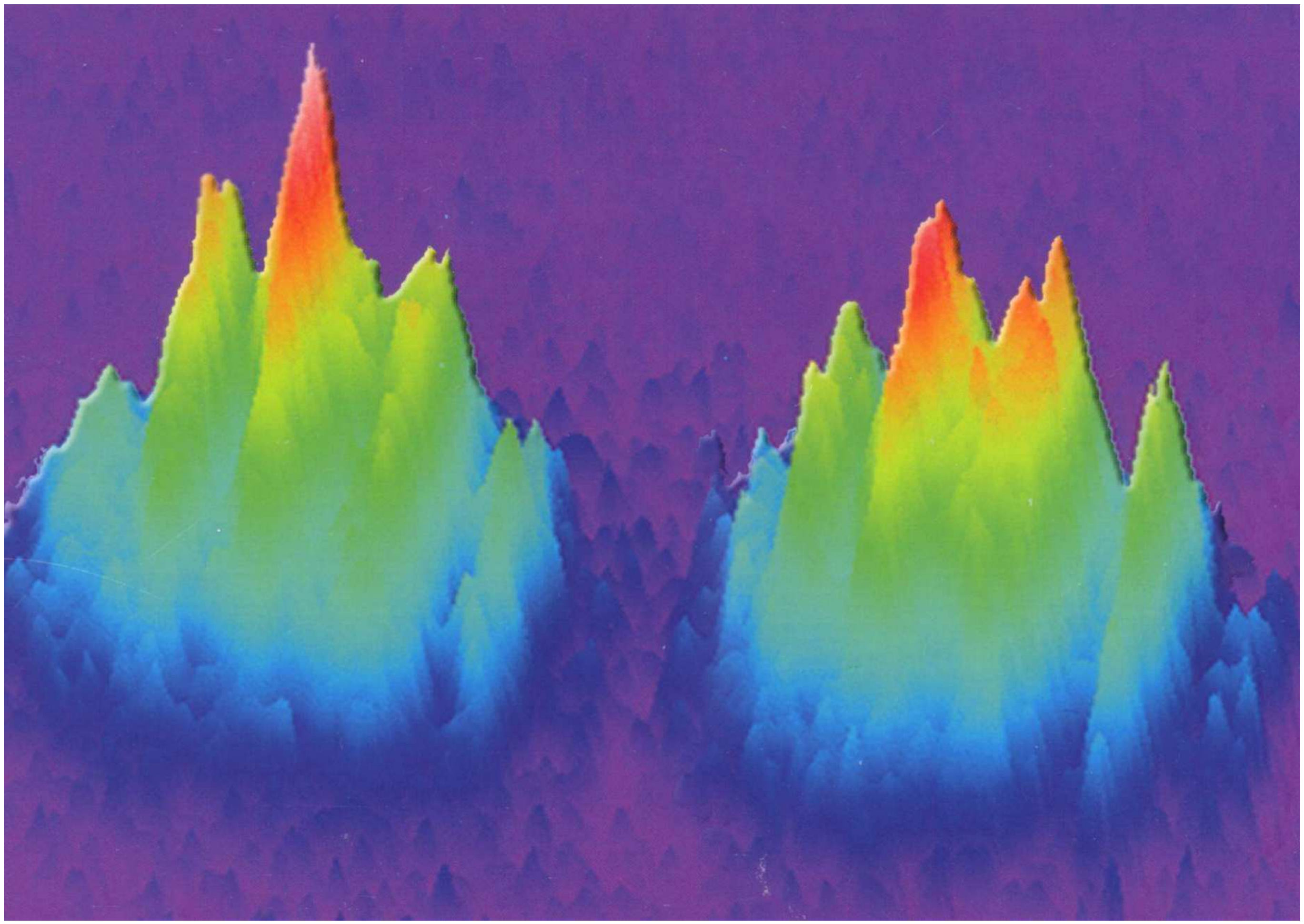
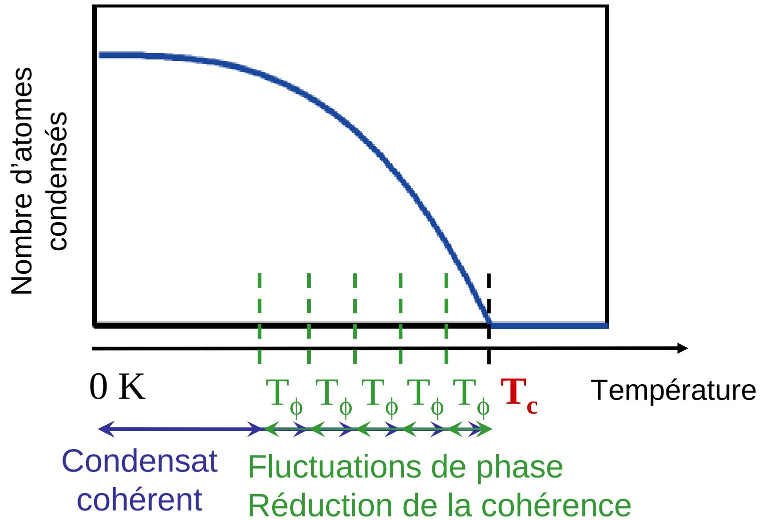
- 5.Anisotropy and Localisation.
- 6.Coherence Length and Anisotropy.

M.Beau and V.A.Z arXiv:1002.1242 (February 2010)

0. Experimental Data

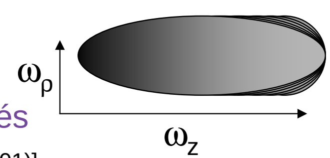


Température de phase T_ϕ (caractérise la cohérence)



$$T_\phi = \frac{15}{32}\frac{(\hbar\omega_z)^2 N}{\mu}$$

T_ϕ petite :
- condensat long
- peu d'atomes condensés



D. Petrov *et al.* [PRL **87**, 050404 (2001)]

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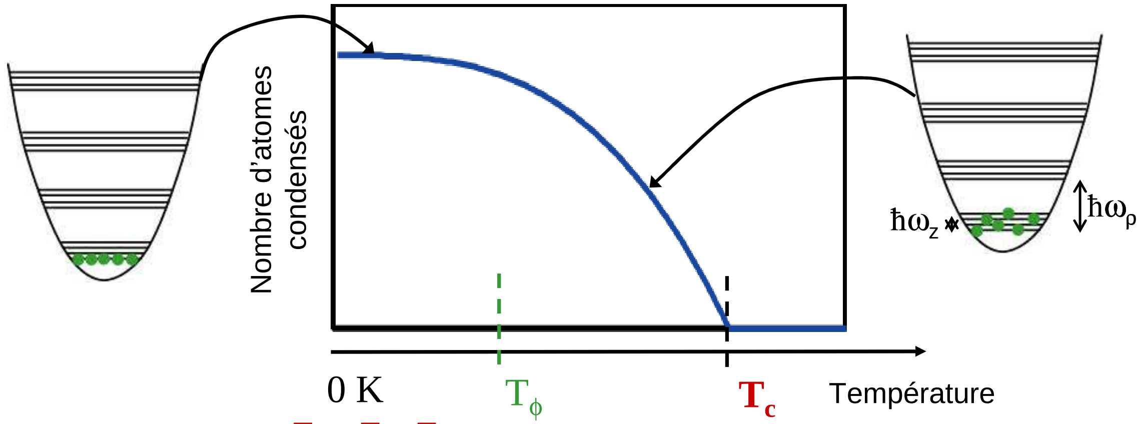
Origine des fluctuations de phase

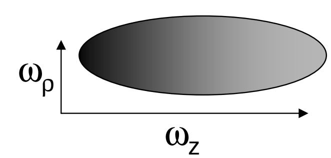


$T_\phi < T < T_c$:

Distribution aléatoire sur plusieurs niveaux d'énergie très proches

$\Rightarrow$ Fluctuations de phase suivant l'axe long du condensat

Amplitude des fluctuations de phase : $\frac{T}{T_\varphi}$



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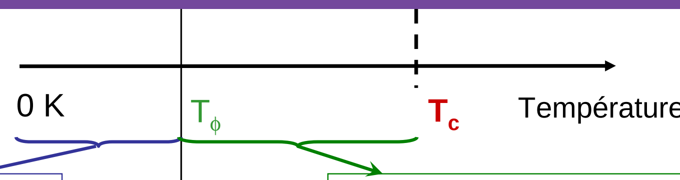
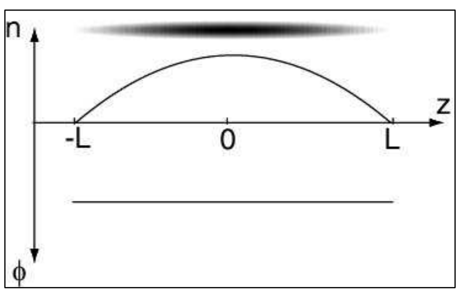
Densité et phase du quasi-condensat

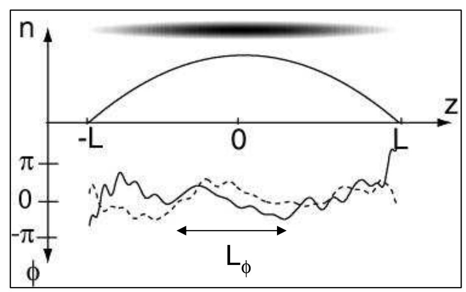


- **Condensat :**

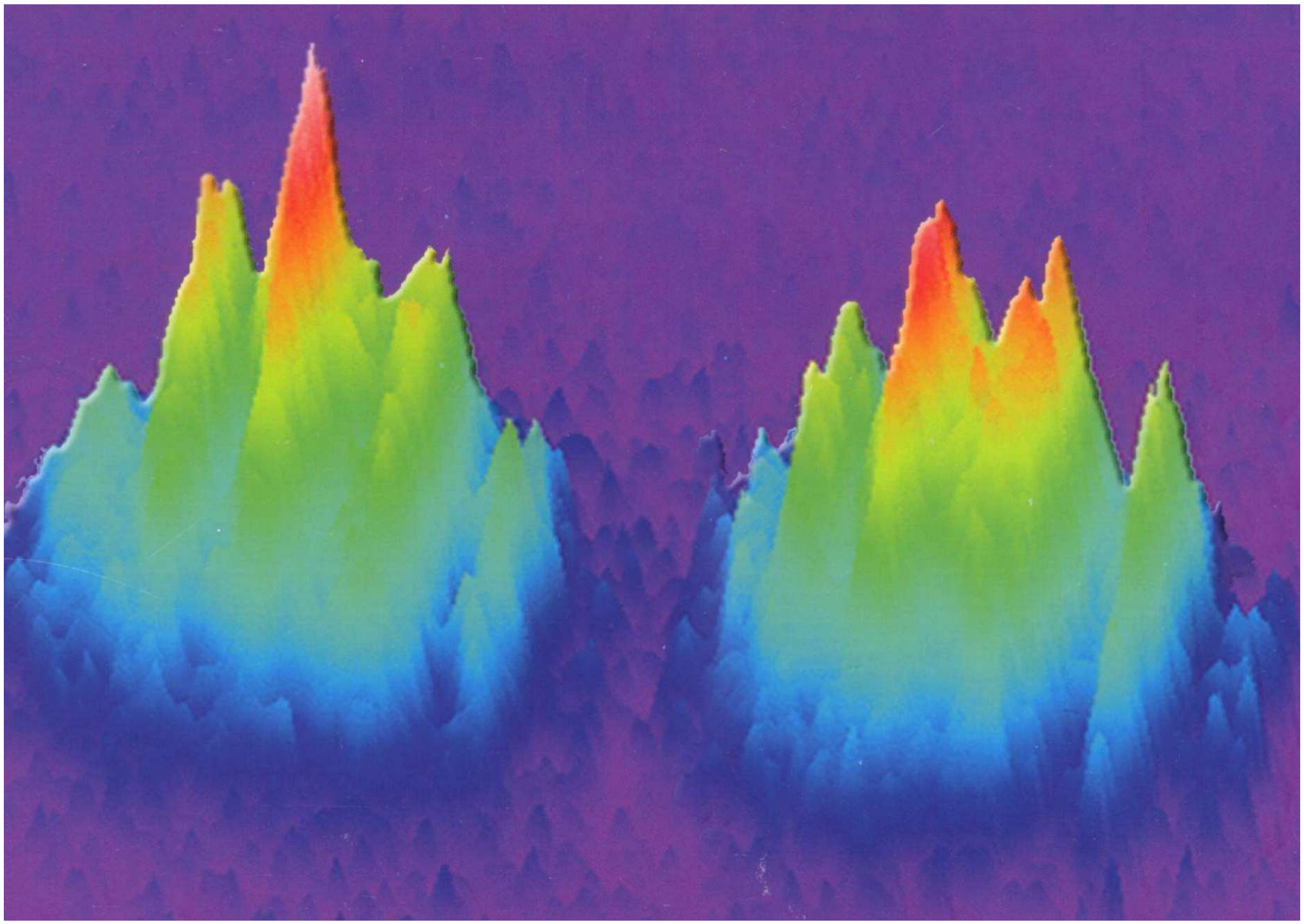


Phase φ est uniforme
$L_c = L$

- **Quasi-condensat :**

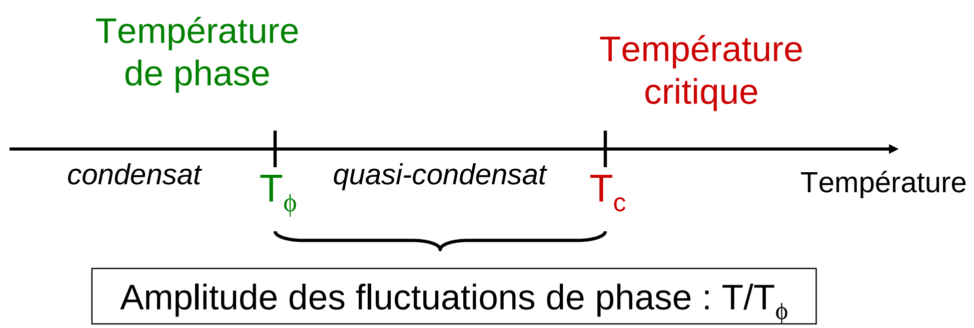


Phase fluctue suivant l'axe long du piège
$L_c < L$



Résumé

- Deux températures pour la caractérisation de la condensation :



1. Perfect Bose-gas

- For $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^3$ and $T_\Lambda^{(N=1)} = \left(-\hbar^2\Delta/(2m)\right)_D$ the spectrum:

$$\left\{\varepsilon_s = \frac{\hbar^2}{2m}\sum_{j=1}^{3}(\pi s_j/L_j)^2\right\}_{s_j\in\mathbb{N}}$$

- Eigenfunctions: $\{\phi_{s,\Lambda}(x) = \prod_{j=1}^3 \sqrt{2/L_j}\,\sin(\pi s_j x_j/L_j)\}_{s_j\in\mathbb{N}}$, $s := (s_1, s_2, s_3) \in \mathbb{N}^3$
- In (T, V, μ), $V = L_1L_2L_3$ the Gibbs mean occupation number of $\phi_{s,\Lambda}$ is $N_s(\beta,\mu) = (e^{\beta(\varepsilon_s-\mu)} - 1)^{-1}$, $\mu < \inf_s\ \varepsilon_s$.
- Particle density $\rho_\Lambda(\beta,\mu) = \sum_{s\in\mathbb{N}^3} N_s(\beta,\mu)/V =: N_\Lambda(\beta,\mu)/V$
- The **first** critical density: $\rho_c(\beta) := \sup_{\mu\le 0}\lim_\Lambda \rho_\Lambda(\beta,\mu) = \zeta(3/2)/\lambda_\beta^3$, $\lambda_\beta := \hbar\sqrt{2\pi\beta/m}$, de Broglie thermal length.

2. Exponential SLAB and the Second Critical Point

2.1 Let $\Lambda = Le^{\alpha L} \times Le^{\alpha L} \times L$. For any fixed s_1, s_2 and $\mu \le 0$

$$\lim_{L\to\infty} \sum_{s\neq(s_1,s_2,1)} \frac{N_s(\beta,\mu)}{V_L} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{d^3k}{e^{\beta(\hbar^2k^2/2m-\mu)}-1} \;.$$

2.2 Let $\mu_L(\beta,\rho) := \varepsilon_{(1,1,1)} - \Delta_L(\beta,\rho)$, where $\Delta_L(\beta,\rho) \geq 0$ is a unique solution of the equation:

$$\rho = \sum_{s=(s_1,s_2,1)} \frac{N_s(\beta,\mu)}{V_L} + \sum_{s\neq(s_1,s_2,1)} \frac{N_s(\beta,\mu)}{V_L}. \qquad (1)$$

2.3 Since: $\lim_{L\to\infty}\sum_{s\neq(s_1,s_2,1)} N_s(\beta,\mu=0)/V_L = \rho_c(\beta)$, for $\rho > \rho_c(\beta)$ the limit of the first sum is

$$\lim_{L\to\infty} \sum_{s=(s_1,s_2,1)} \frac{N_s(\beta,\mu)}{V_L} =$$

$$\lim_{L\to\infty} \frac{1}{L}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{d^2k}{e^{\beta(\hbar^2k^2/2m+\Delta_L(\beta,\rho))}-1} =$$

$$\lim_{L\to\infty} -\frac{1}{\lambda_\beta^2 L}\ln[\beta\Delta_L(\beta,\rho)] = \rho-\rho_c(\beta).$$

This implies the asymptotics:

$$\Delta_L(\beta,\rho) = \frac{1}{\beta}\, e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L} + \dots\ .$$

2.4 Remark 2.1. Since $L_{j=1,2} = Le^{\alpha L}$ and

$$\varepsilon_{(s_1,s_2,1)} - \mu_L(\beta,\rho) = \frac{\hbar^2}{2m}\sum_{j=1}^{2}[(\pi s_j/L_j)^2 - 1] + \Delta_L(\beta,\rho)$$

the representation of the first sum by the **integral** is valid only when $\lambda_\beta^2(\rho - \rho_c(\beta)) \leq 2\alpha$, i.e.

$$L^{-2}e^{-2\alpha L} < \beta^{-1}\ e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L} + \dots\ .$$

2.5 Definition 2.2. The **second** critical density:

$$\rho_m(\beta) := \rho_c(\beta) + 2\alpha/\lambda_\beta^2 > \rho_c(\beta)\ .$$

2.6 Remark 2.3. For $\rho > \rho_m(\beta)$ the convergence $\Delta_L(\beta,\rho) \to 0$ **must** be **faster** than $e^{-2\alpha L}$.

2.7 To keep the difference $\rho - \rho_m(\beta) > 0$ one **must** return back to the finite volume **sum representation** to take into account the input of the ground state occupation density.

Theorem 2.4. The asymptotics of $\Delta_L(\beta, \rho > \rho_m(\beta))$ is

$$\Delta_L(\beta,\rho) = [\beta(\rho - \rho_m(\beta))V_L]^{-1} + \ldots < L^{-2}e^{-2\alpha L} \ .$$

2.8 Since $V_L = L^3 e^{2\alpha L}$, the first sum without the ground-state:

$$\lim_{L\to\infty} \sum_{s=(s_1>1,s_2>1,1)} \frac{N_s(\beta,\mu)}{V_L} = \lim_{L\to\infty} \frac{1}{\lambda_\beta^2 L} \ln[\beta\Delta_L(\beta,\rho)]^{-1} =$$

$$2\alpha/\lambda_\beta^2 = \rho_m(\beta) - \rho_c(\beta).$$

2.9 The ground-state term gives the **macroscopic** occupation:

$$\rho - \rho_m(\beta) = \lim_{L\to\infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_{(1,1,1)} - \mu_L(\beta,\rho))} - 1} \ .$$

2.10 Corollary 2.5 Since for $\rho_c(\beta) < \rho < \rho_m(\beta)$

$$\varepsilon_s - \mu_L(\beta,\rho) = \Delta_L(\beta,\rho) + \varepsilon_s - \varepsilon_{(1,1,1)} = \mathcal{O}(\beta^{-1}\ e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L})\ ,$$

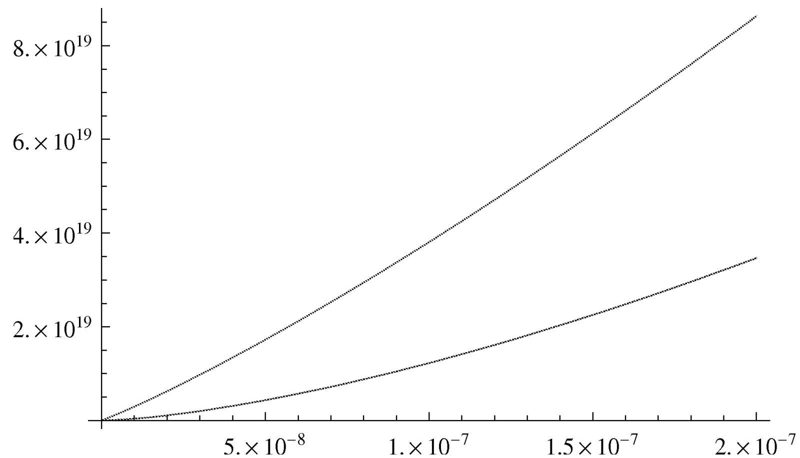
one gets the *type* III van den Berg-Lewis-Pulé generalised condensation (vdBLP-GC): when **none** of the single-particle states are *macroscopically* occupied:

$$\rho_s(\beta,\rho) := \lim_{L\to\infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))}-1} = 0\ .$$

The asymptotics $\Delta_L(\beta,\rho > \rho_m(\beta)) = [\beta(\rho-\rho_m(\beta))V_L]^{-1}$ implies

$$\lim_{L\to\infty} \rho_{s\neq(1,1,1)}(\beta,\rho) := \lim_{L\to\infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))}-1} = 0\ ,$$

and $\lim_{L\to\infty} \rho_{(1,1,1)}(\beta,\rho) = \rho - \rho_m(\beta) > 0$, the *type* I vdBLP-GC.



2.11 For $\rho > \rho_m(\beta)$ there is a *coexistence* of the *saturated* **type III** vdBLP-GC, with the **constant** density $\rho_m(\beta) - \rho_c(\beta)$, and the standard BEC (the **type I** vdBLP-GC) in the the ground state with the density $\rho - \rho_m(\beta)$.

3. Exponential BEAM and CIGAR Traps

3.1 Remark 3.1 It is curious to note that neither **Casimir shaped boxes** $\Lambda = L^{\alpha_1} \times L^{\alpha_2} \times L^{\alpha_3}$, nor the **van den Berg boxes** $\Lambda = Le^{\alpha L} \times L \times L$, with **one-dimensional anisotropy** do not produce the *second* critical density $\rho_m(\beta) \neq \rho_c(\beta)$.

3.2 Remark 3.2 (BEAM) For beams with two critical densities we consider the Hamiltonian: $T_\Lambda^{(N=1)} = -\hbar^2\Delta/(2m) + m\omega_1^2 x_1^2/2$, with **harmonic trap** in direction x_1 and Dirichlet boundary conditions in directions x_2, x_3. Then the spectrum:

$$\left\{\epsilon_s := \hbar\omega_1(s_1 + 1/2) + \frac{\hbar^2}{2m}\sum_{j=2}^{3}(\pi s_j/L_j)^2\right\}_{s\in\mathbb{N}},$$

$s = (s_1, s_2, s_3) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2$, the ground-state energy: $\epsilon_{(0,1,1)}$.

3.3 For $\mu_L(\beta,\varrho) := \epsilon_{(0,1,1)} - \Delta_L(\beta,\varrho)$, the $\Delta_L(\beta,\varrho) \geq 0$ is a unique solution of the equation:

$$\varrho := \sum_{s=(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3} + \sum_{s\neq(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3},$$

$N_s(\beta,\mu) = (e^{\beta(\epsilon_s-\mu)} - 1)^{-1}$, $\omega_1 := \hbar/(mL_1^2)$ and $L_2 = L_3 = L$.

3.4 Similar to SLAB, for any $s_1 \geq 0$ and $\mu \leq 0$

$$\varrho(\beta,\mu) := \lim_{L_1,L\to\infty} \sum_{s\neq(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3} =$$

$$\frac{1}{(2\pi)^2}\int_0^\infty dp \int_{\mathbb{R}^2} \frac{d^2k}{e^{\beta(\hbar p+\hbar^2k^2/2m-\mu)} - 1} .$$

The **first** critical density is *finite*: $\varrho_c(\beta) := \sup_{\mu\leq 0} \varrho(\beta,\mu) = \varrho(\beta,\mu=0) < \infty$.

3.5 For $\varrho > \varrho_c(\beta)$ the limit $L \to \infty$ of the first sum in **3.3**

$$\lim_{L_1, L \to \infty} \sum_{s=(s_1,1,1)} \omega_1 \frac{N_s(\beta, \mu_L)}{L_2 L_3} =$$
$$\lim_{L\to\infty} \frac{1}{L^2} \int_0^\infty \frac{dp}{e^{\beta(\hbar p + \Delta_L(\beta,\varrho))} - 1} =$$
$$\lim_{L\to\infty} \frac{1}{\hbar\beta L^2} \ln[\beta \Delta_L(\beta, \varrho)]^{-1} = \varrho - \varrho_c(\beta).$$

This gives the asymptotics : $\Delta_L(\beta, \varrho) = \beta^{-1}\, e^{-\hbar\beta(\varrho - \varrho_c(\beta)) L^2} + \ldots$.
3.6 Let $L_1 := Le^{\gamma L^2}$, $\gamma > 0$. Then, similar to SLAB, the representation of the limit in **3.5** by the integral is valid for $\hbar\beta(\varrho - \varrho_c(\beta)) \leq 2\gamma$ and we reach to necessity of the **second** critical density $\varrho_m(\beta) := \varrho_c(\beta) + 2\gamma/(\hbar\beta)$.
3.7 The rest of scenario is identical to the case of the SLAB.

3.8 **Remark 3.3 (CIGAR)** A "cigar"-type geometry is ensured by the anisotropic harmonic trap:

$$T_{\Lambda}^{(N=1)} = -\hbar^2\Delta/(2m) + \sum_{1\leq j\leq 3} m\omega_j^2 x_j^2/2 \ .$$

with $\omega_1 = \hbar/(mL_1^2),\ \omega_2 = \omega_3 = \hbar/(mL^2)$. Here $L_1, L_2 = L_3 = L$ are the *characteristic* sizes of the trap in three directions and the spectrum $\eta_s = \sum_{1\leq j\leq 3} \hbar\omega_j(s_j + 1/2)$.

3.9 For $\mu_L(\beta, n) := \eta_{(0,0,0)} - \Delta_L(\beta, n)$ and factor $\kappa > 0$:

$$\lim_{L_1, L\to\infty} \sum_{s=(s_1,0,0)} \kappa^3\omega_1\omega_2\omega_3 N_s(\beta, \mu_L) =$$

$$\lim_{L\to\infty} \frac{\kappa^3\hbar}{\beta(mL^2)^2} \ln[\beta\Delta_L(\beta, n)]^{-1} = n - n_c(\beta).$$

3.10 Again the **first** critical density $n_c(\beta) := n(\beta, \mu = 0)$ is finite:

$$n(\beta,\mu) := \lim_{L_1,L\to\infty} \sum_{s\neq(s_1,0,0)} \kappa^3\omega_1\omega_2\omega_2 N_s(\beta,\mu) =$$

$$\int_{\mathbb{R}^3{}_+} \frac{\kappa^3 d\omega_1 d\omega_2 d\omega_3}{e^{\beta[(\omega_1+\omega_2+\omega_3)-\mu]}-1} ,$$

and asymptotics:

$$\Delta_L(\beta, n > n_c(\beta)) = \beta^{-1}\ e^{-\beta(n-n_c(\beta))m^2 L^4/(\hbar\kappa^3)} + \ldots .$$

3.11 If $L_1 := Le^{\widehat{\gamma}L^4}$, $\widehat{\gamma} > 0$, then the **second** critical density:

$$n_m(\beta) := n_c(\beta) + (\widehat{\gamma}\hbar\kappa^3)/(\beta m^2) .$$

is defined by the standard argument of the energy level spacing.

3.12 Bose-condensation (CIGAR) For $n_c(\beta) < n < n_m(\beta)$ we obtain the *type* III vdBLP-GC, when *none* of the single-particle states are *macroscopically* occupied:

$$n_s(\beta, \rho) := \lim_{L \to \infty} \frac{\kappa^3 \omega_1 \omega_2 \omega_3}{e^{\beta(\eta_s - \mu_L(\beta,n))} - 1} = 0 \ .$$

Although for $n_m(\beta) < n$ there is a coexistence of the *type* III vdBLP-GC, with the saturated constant density $n_m(\beta) - n_c(\beta)$, and the standard BEC (*type* I vdBLP-GC) in the ground-state:

$$n - n_m(\beta) = \lim_{L \to \infty} \frac{\kappa^3 \omega_1 \omega_2 \omega_3}{e^{\beta(\eta_{(0,0,0)} - \mu_L(\beta,n))} - 1} > 0 \ .$$

4. Temperature Dependence of the Bose-Condensate

4.1 The *first* critical temperatures: $T_c(\rho)$, $\widetilde{T}_c(\rho)$ or $\widehat{T}_c(\rho)$ are well-known. For a given density ρ they verify the identities:

$$\rho = \rho_c(\beta_c(\rho)) \ , \ \varrho = \varrho_c(\widetilde{\beta}_c(\varrho)) \ , \ n = n_c(\widehat{\beta}_c(n)) \ ,$$

respectively for slabs, squared beams or "cigars".

4.2 Since $\rho_c(\beta) =: T^{3/2} I_{sl}$, $\varrho_c(\beta) =: T^2 I_{bl}$, $n_c(\beta) =: T^3 I_{cg}$, the expressions for the **second** critical densities one gets relations between the *first* and the *second* critical temperatures:

$$\begin{aligned} T_m^{3/2}(\rho) + \tau^{1/2}\, T_m(\rho) &= T_c^{3/2}(\rho) \quad \text{(slab)} \ , \\ \widetilde{T}_m^2(\varrho) + \widetilde{\tau}\, \widetilde{T}_m(\varrho) &= \widetilde{T}_c^2(\varrho) \quad \text{(beam)} \ , \\ \widehat{T}_m^3(n) + \widehat{\tau}^2\, \widehat{T}_m(n) &= \widehat{T}_c^3(n) \quad \text{(cigar)} \ . \end{aligned}$$

$\tau = [\alpha m k_B/(\pi\hbar^2 I_{sl})]^2$, $\widetilde{\tau} = 2\gamma k_B/(\hbar I_{bl})$, $\widehat{\tau} = [(\widehat{\gamma}\hbar\kappa^3 k_B)/(m^2 I_{cg})]^{1/2}$ are "effective" temperatures related to the corresponding geometrical shapes.

4.3 Since the **total** condensate density is $\rho - \rho_c(\beta) := \rho_0(\beta) = \rho_{0c}(\beta) + \rho_{0m}(\beta)$, where $\rho_{0m}(\beta) := (\rho - \rho_m(\beta))\ \theta(\rho - \rho_m(\beta))$, the *second* critical temperature modifies the usual law for the condensate fractions temperature dependence.

4.4 For the *type* III vdBLP-GC, $\rho_{0c}(\beta)$, in the SLAB geometry:

$$\frac{\rho_{0c}(\beta)}{\rho} = \begin{cases} 1 - (T/T_c)^{3/2}\ , & T_m \leq T \leq T_c\ , \\ \sqrt{\tau}\ T/T_c^{3/2} & , \quad T \leq T_m\ . \end{cases}$$

For the BEC (*type* I vdBLP-GC) in the ground state $\rho_{0m}(\beta)$:

$$\frac{\rho_{0m}(\beta)}{\rho} = \begin{cases} 0\ , & T_m \leq T \leq T_c, \\ 1 - (T/T_c)^{3/2}(1 + \sqrt{\tau/T}), & T \leq T_m, \end{cases}$$

The **total** condensate density $\rho_0(\beta) := \rho_{0c}(\beta) + \rho_{0m}(\beta)$ results from *coexistence* of both of them: this gives the standard PBG expression $\rho_0(\beta)/\rho = 1 - (T/T_c)^{3/2}$.

4.5 For the "cigars" geometry the *type* III vdBLP-GC $r_{0c}(\beta)$:

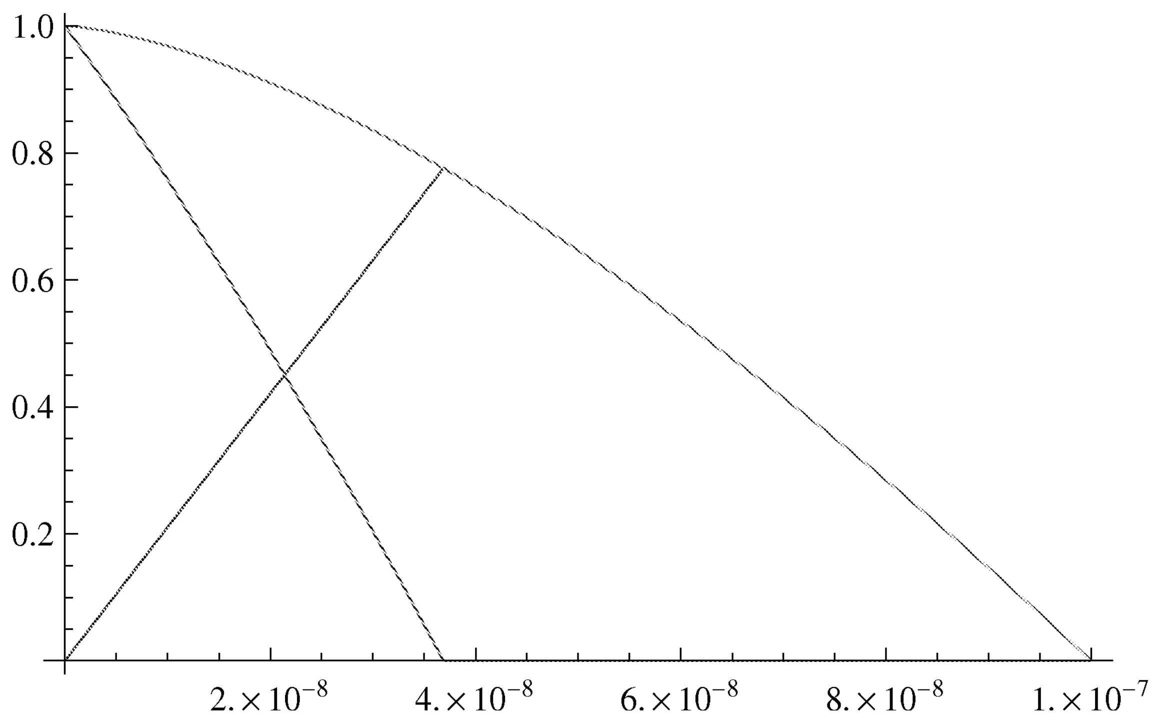
$$\frac{n_{0c}(\beta)}{n} = \begin{cases} 1-(T/\widehat{T}_c)^3 \ , & \widehat{T}_m \leq T \leq \widehat{T}_c \ , \\ \widehat{\tau}^2 \ T/\widehat{T}_c^3 \ , & T \leq \widehat{T}_m \ . \end{cases}$$

The ground state conventional BEC is

$$\frac{n_{0m}(\beta)}{n} = \begin{cases} 0 \ , & \widehat{T}_m \leq T \leq \widehat{T}_c, \\ 1-(T/\widehat{T}_c)^3(1+\widehat{\tau}^2/T^2), & T \leq \widehat{T}_m, \end{cases}$$

and again for the two *coexisting* condensates one gets a standard expression:

$$n - n_c(\beta) := n_0(\beta) = n_{0c}(\beta) + n_{0m}(\beta) = (1-(T/T_c)^{3/2})n \ .$$


1.0
0.8
0.6
0.4
0.2
2.×10^-8
4.×10^-8
6.×10^-8
8.×10^-8
1.×10^-7

5. Anisotropy and Localisation

5.1 Global Scaled Particle Density :

$$\xi_L(u) := \sum_s \frac{|\phi_{s,\Lambda}(L_1 u_1, L_2 u_2, L_3 u_3)|^2}{e^{\beta(\varepsilon_s - \mu)} - 1} ,$$

with the scaled distances $\{u_j = x_j/L_j \in [0,1]\}_{j=1,2,3}$.

5.2 For a given ρ in the slab geometry

$$\xi^{slab}_{\rho,L}(u) := \sum_s \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j} [\sin(\pi s_j u_j)]^2 .$$

Since $2[\sin(\pi s_j u_j)]^2 = 1 - \cos\{(2\pi s_j/L_j) u_j L_j\}$ and $\lim_{L\to\infty} \mu_L(\beta, \rho < \rho_c(\beta)) < 0$, by the Riemann-Lebesgue lemma we obtain that $\lim_{L\to\infty} \xi^{slab}_{\rho,\Lambda}(u) = \rho$ for any $u \in (0,1)^3$.

5.3 If $\rho > \rho_c(\beta)$, then for any $u \in (0,1)^3$:

$$\lim_{L\to\infty} \sum_{s=(s_1,s_2,1)} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j}[\sin(\pi s_j u_j)]^2$$

$$= \lim_{L\to\infty} \frac{2[\sin(\pi u_3)]^2}{(2\pi)^2 L} \int_{\mathbb{R}^2} \frac{\prod_{j=1}^{2}(1-\cos(2k_j u_j L_j) d^2k}{e^{\beta(\hbar^2 k^2/2m + \Delta_L(\beta,\rho))} - 1}$$

$$= (\rho - \rho_c(\beta))\ 2[\sin(\pi u_3)]^2 \ ,$$

$$\lim_{L\to\infty} \sum_{s\neq(s_1,s_2,1)} \frac{1}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j}[\sin(\pi s_j u_j)]^2$$

$$= \rho_c(\beta))$$

$$\Rightarrow \quad \xi_\rho^{slab}(u) = (\rho - \rho_c(\beta))\ 2[\sin(\pi u_3)]^2 + \rho_c(\beta) \ ,$$

which manifests a *space anisotropy* of the type III vdBLP-GC for $\rho_c(\beta) < \rho < \rho_m(\beta)$ in direction u_3.

5.4 For $\rho > \rho_m(\beta)$ one has to use representations and asymptotics from **2**. Then

$$\xi_\rho^{slab}(u) = (\rho - \rho_m(\beta)) \prod_{j=1}^{3} 2[\sin(\pi u_j)]^2 + (\rho_m(\beta) - \rho_c(\beta))\, 2[\sin(\pi u_3)]^2 + \rho_c(\beta) \ .$$

So, the anisotropy of the space particle distribution is still only in direction u_3 due to the type III vdBLP-GC ("quasi-condensate") $(\rho_m(\beta) - \rho_c(\beta))$. The input of the standard type I vdBLP-GC (one mode BEC) $(\rho - \rho_m(\beta))$ is **isotropic**.

6. Coherence Length and Anisotropy

6.1 ODLRO kernel:

$$K(x,y) := \lim_{L\to\infty} K_\Lambda(x,y) = \lim_{L\to\infty} \sum_s \frac{\overline{\phi}_{s,\Lambda}(x)\phi_{s,\Lambda}(y)}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \ .$$

Let us center the box Λ at the origin of coordinates: $x_j = \tilde{x}_j + L_j/2$ and $y_j = \tilde{y}_j + L_j/2$. Then the ODLRO kernel gets the form:

$$K_\Lambda(\tilde{x},\tilde{y}) = \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \ R_l^{(2)} \ R_l^{(1)} \ .$$

6.2 Here after the shift of coordinates and using additive form of the spectrum we put

$$R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) = \sum_{s=(s_1,s_2)} e^{-l\beta\varepsilon_{s_1,s_2}} \, \overline{\phi}_{s_1,s_2,\Lambda}(\tilde{x}_1, \tilde{x}_2) \; \phi_{s_1,s_2,\Lambda}(\tilde{y}_1, \tilde{y}_2)$$

$$R_s^{(1)}(\tilde{x}_3, \tilde{y}_3) = \sum_{s=(s_3)} e^{-l\beta\varepsilon_{s_3}} \sqrt{\frac{2}{L_3}} \sin(\frac{\pi s_3}{L_3}(\tilde{x}_3 + \frac{L_3}{2})) \times \sqrt{\frac{2}{L_3}} \sin(\frac{\pi s_3}{L_3}(\tilde{y}_3 + \frac{L_3}{2})) \ .$$

6.3 By the Weyl theorem one gets for the first two directions:

$$\lim_{L\to\infty} R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) = \frac{1}{l\lambda_\beta^2} \, e^{-\pi\|\tilde{x}^{(2)}-\tilde{y}^{(2)}\|^2/l\lambda_\beta^2} \ .$$

6.4 For exponentially anisotropic box and for $\rho_c(\beta) < \rho < \rho_m(\beta)$ we must split the sum over $s = (s_1, s_2, s_3)$ in **6.1** into two parts: sum over $s = (s_1, s_2, 1)$ and the rest. For the first sum by **6.3** we obtain:

$$\lim_{L\to\infty} \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s=(s_1,s_2,1)} e^{-l\beta\varepsilon_{s_1,s_2,1}} \times$$
$$\times \overline{\phi}_{s_1,s_2,1\Lambda}(\tilde{x})\ \phi_{s_1,s_2,1\Lambda}(\tilde{y}) =$$
$$\lim_{L\to\infty} \sum_{l=1}^{\infty} e^{-l\beta\Delta_L(\beta,\rho)} \frac{1}{l\lambda_\beta^2}\ e^{-\pi\|\tilde{x}^{(2)}-\tilde{y}^{(2)}\|^2/l\lambda_\beta^2} \times$$
$$\times \frac{2}{L} \sin(\frac{\pi}{L}(\tilde{x}_3 + \frac{L}{2})) \sin(\frac{\pi}{L}(\tilde{y}_3 + \frac{L}{2}))\ .$$

6.5 For the second part we apply the Weyl theorem for 3 component function:

$$\lim_{L\to\infty}\sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s\neq(s_1,s_2,1)} e^{-l\beta\varepsilon_s} \times$$
$$\times\overline{\phi}_{s,\Lambda}(\tilde{x})\ \phi_{s,\Lambda}(\tilde{y}) = \sum_{l=1}^{\infty}\frac{1}{l\lambda_\beta^3}\ e^{-\pi\|\tilde{x}-\tilde{y}\|^2/l\lambda_\beta^2}\ .$$

6.6 Since $\Delta_L(\beta,\rho_c(\beta) < \rho < \rho_m(\beta)) \to 0, L \to \infty$, the change $l \to l\ \Delta_L(\beta,\rho)$ in **6.4** gives the integral Darboux-Riemann sum, where $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2$ is scaled as $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2\ \Delta_L(\beta,\rho)$.
6.7 Definition 6.1 The *coherence length* L_{ch} in direction perpendicular to x_3 is $L_{ch}(\beta,\rho)/L := \Delta_L^{-1/2}(\beta,\rho)$. A similar argument is valid for $\rho > \rho_m(\beta)$ with obvious modifications due to BEC for $s = (1,1,1)$ and adapted asymptotics for $\Delta_L(\beta,\rho)$.

6.7 To compare $L_{ch}(\beta,\rho)$ with the *scale* $L_{1,2} = Le^{\alpha L}$, we define the critical exponent $\gamma(T,\rho)$ such that

$$\lim_{L\to\infty}(L_{ch}(\beta,\rho)/L)(L_1/L)^{-\gamma(T,\rho)} = 1$$

Then

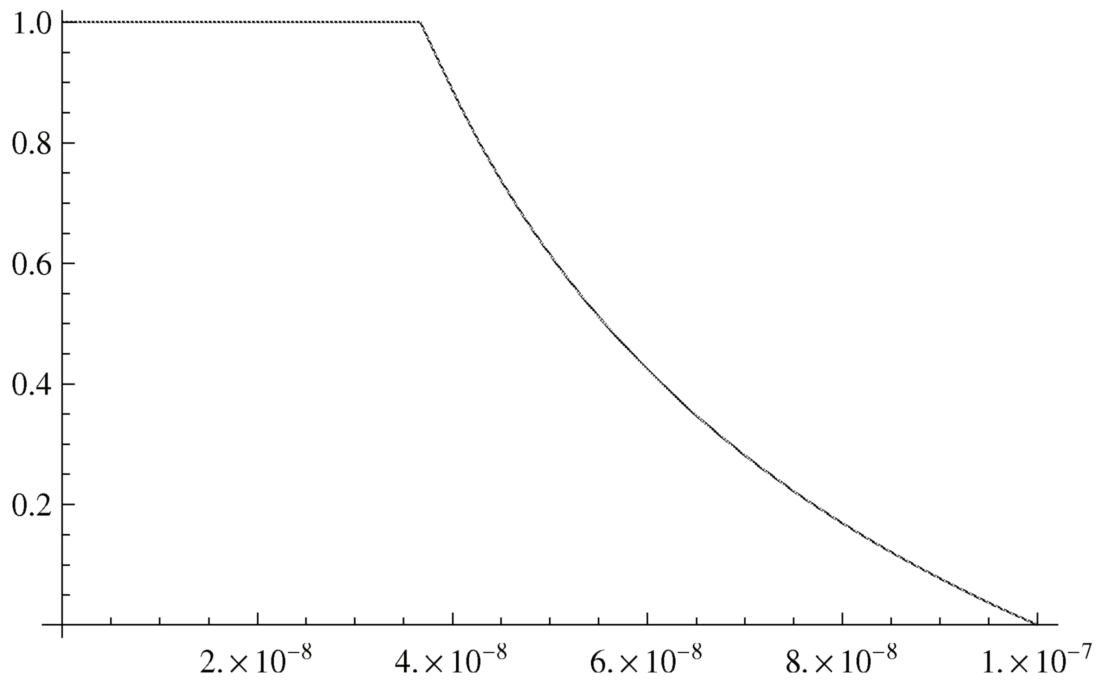
$$\begin{aligned}\gamma(T,\rho) &= \lambda_\beta^2\,(\rho-\rho_c(\beta))/2\alpha\ ,\ \ \rho_c(\beta)<\rho<\rho_m(\beta)\\ &= \lambda_\beta^2\,(\rho_m(\beta)-\rho_c(\beta))/2\alpha\ ,\ \ \rho_m(\beta)\le\rho\ .\end{aligned}$$

For a fixed density, taking into account temperature dependence of condensates we find the temperature dependence of the exponent $\gamma(T) := \gamma(T,\rho)$, see Fig:

$$\begin{aligned}\gamma(T) &= \sqrt{T/\tau}\ \{(T_c/T)^{3/2}-1\}\ ,\ T_m<T<T_c\ ,\\ &= 1,\ T\le T_m\ .\end{aligned} \tag{2}$$

6.8 Notice that in the both cases the ODLRO kernel is anisotropic due to the type III condensation in the states $s = (s_1, s_2, 1)$, whereas the other states give a symmetric part of correlations, which includes a constant term $\rho_c(\beta)$.

6.9 Numerically, for $L_1 = L_2 = 100\mu m$, $L_3 = 1\mu m$ and $T_m < T = 0.75T_c$ the coherence length of the condensate is equal to $2.8\mu m \ll 100\mu m$. This decreasing of the *coherence length* for $T_c < T < T_m$ is experimentally observed (2003).


1.0
0.8
0.6
0.4
0.2
2. × 10−8
4. × 10−8
6. × 10−8
8. × 10−8
1. × 10−7

Thank you for your attention !