Discreet Games, Light Affine Logic and PTIME Computation

A. S. Murawski* C.-H. L. Ong**

Oxford University Computing Laboratory
Wolfson Building, Parks Rd, Oxford OX1 3QD, UK
{andrzej,Luke.Ong}@comlab.ox.ac.uk

Abstract. This paper introduces a model of IMLAL, the intuitionistic multiplicative $(\otimes, \multimap, \S, !)$-fragment of Light Affine Logic, based on games and *discreet strategies*. We define a generalized notion of *threads*, so that a play of a game (of depth k) may be regarded as a number of interwoven threads (of depths ranging from 1 to k). To constrain the way threads communicate with each other, we organize them into *networks* at each depth (up to k), in accord with a protocol:

- A network comprises an O-thread (which can only be created by O) and finitely many P-threads (which can only be created by P).
- A network whose O-thread arises from a $!$-game can have at most one P-thread which must also arise from a $!$-game.
- No thread can belong to more than one network.
- Only O can switch between networks, and only P can switch between threads within the same network.

Strategies that comply with the protocol are called *discreet*, and they give rise to a fully complete model of IMLAL. Since IMLAL has a polytime cut-elimination procedure, the model gives a basis for a denotational-semantic characterization of PTIME.

Keywords: Game Semantics, Linear Logic, Complexity, PTIME.

1 Introduction

Light Linear Logic (LLL) [4] has a polytime cut-elimination procedure and can encode all polytime numeric functions. In Girard's words, it is an "intrinsically polytime system" whose proofs may be regarded as (representations of) polytime algorithms. An intuitionistic affine variant of the Logic has recently been introduced by Asperti [3]. The system, called IMLAL2 (Second-order Intuitionistic Multiplicative Light Affine Logic), is arguably simpler than LLL, and yet gives the same characterization of PTIME.

Our goal is to give a denotational characterization of PTIME by constructing a good game model of *proofs* (not just provability) of such *light logics* as LLL

* On leave from Nicholas Copernicus University, Toruń, Poland.
** Webpage: `http://www.comlab.ox.ac.uk/oucl/work/luke.ong.html`

or IMLAL. This seems a non-trivial task: the only model of a light logic known to us is one of provability, and is based on a fibred system of phase spaces [7]. The main result therein is a strong completeness theorem[1]: if a formula is valid (i.e. every valuation in any fibred phase space validates it) then it is provable. For modelling proofs, an appropriate criterion to aim for is *full completeness* [1], which is best formulated in terms of a categorical model of the logic, in which formulas are denoted by objects and proofs by maps. We say that the model $\mathbb{C}$ is *fully complete* just in case the unique functor from the relevant free category (typically the classifying category of the logic or type theory) to $\mathbb{C}$ is full. In this paper, we take the first step towards a denotational-semantic characterization of PTIME by presenting a fully complete model for the quantifier-free IMLAL.

One way to approach our model is to start from the AJM games in [2], and consider total, history-free strategies that are $\approx$-reflexive ($\approx$ is a partial equivalence relation that relates strategies which have "equivalent responses at equivalent positions"). Thus a play of a shriek game $!A$ may be viewed as a number of interwoven threads of A. The whisper game $\S A$ of light logic is a degenerate form of $!A$; a play of $\S A$ consists of only one thread of A. We introduce a generalized notion of threads at each depth i, called *i-threads*. To constrain the way threads communicate with each other, we organize threads into *networks* at each depth, in accord with a protocol. P-strategies that comply with the protocol are called *discreet* (to underline the property that P only "communicates" within a network). We can show from first principles that such strategies compose, and they give rise to a fully complete model of IMLAL. We refer the reader to the protocol (as presented in the framed box) in Section 4, and illustrate it with an example.

Example 1. Consider the two-player game of a valid sequent in Figure 1 and the twelve-move play which switches between the subgames as indicated therein (we use P and O to indicate which player has made the move). The play has two networks. Threads of the first network are contained in dashed shapes in Figure 1; they are the O-thread[2] $m_1 m_{12}$, and two P-threads, namely, $m_2 m_3 m_6 m_{11}$ and $m_4 m_5$. Threads of the second network are contained in dotted shapes in the Figure; they are the O-thread $m_7 m_{10}$ and the P-thread $m_8 m_9$. Protocol (p2) says that whenever O starts a new (O-)thread (e.g. with m_7 but not m_3), a new network is created. Protocol (p5) requires that only O may switch from one network to another existing network (e.g. from m_{10} to m_{11}), but only P may switch from one thread to another existing thread within the same network (e.g. from m_5 to m_6).

Now by replacing the formula on the right of the turnstile by $\S(B \multimap A)$, we get an invalid sequent. Consider the play that moves between the subgames, following exactly the same pattern of the preceding play. Note that the corresponding m_8 violates protocol (p5) as it is a case of P attempting to switch back to the first network.

[1] The result is for a slight variant of LLL that has a $\S$-operator which is not self-dual.

[2] For the purpose here, an O-thread is just a thread beginning with an O-move.

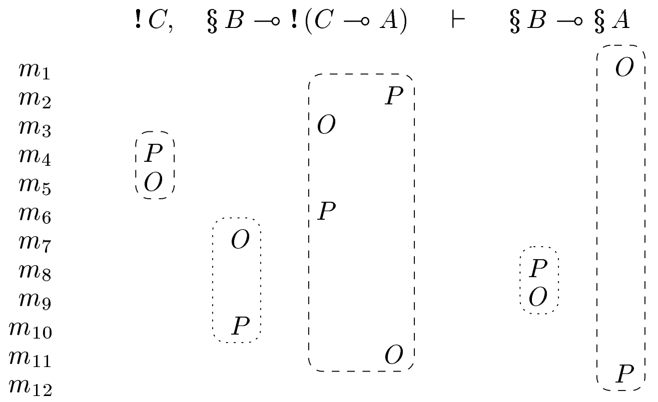

Fig. 1. Networks of (threads of) a play

To the best of our knowledge, the construction presented here is the first model of a system of light logic proofs. Our main contribution is a semantic analysis of the two crucial connectives $!$ and $\S$. We could have worked on a model for LLL instead but decided on the intuitionistic affine system because it is a simpler (but perfectly adequate) setting to explain our ideas. We believe we can extend discreet games to model multiplicative LLL without difficulty: games for the classical system (i.e. extended by involutive negation) can be obtained by admitting positions that begin with P-moves; weakening can be invalidated by either introducing *fairness* [5] at the level of positions or *exhaustion* [8] at the level of strategies.

2 IMLAL

Intuitionistic Multiplicative Light Affine Logic (IMLAL) formulas are generated from atoms $a, b, c, \cdots$ by the connectives $\otimes$, $\multimap$, $\S$ (read "whisper") and $!$ (read "shriek"). Affine here means that the weakening rule is valid. We let Γ, Δ range over finite sequences of IMLAL formulas. The valid IMLAL sequents are defined by the rules in Figure 2. The two main results are:

Theorem 1 (Girard, Asperti).

1. *Cut elimination in a proof p of $\Gamma \vdash A$ can be done in time proportional to $|p|^{2^d}$ where d is the depth of $\bigotimes \Gamma \multimap A$ and $|p|$ is the size of p.*
2. *All PTIME numeric functions can be encoded in IMLAL2, the second-order extension of IMLAL.* $\square$

Unfortunately we do not have the space to say anything more about the Theorem other than to direct the reader to the main references [4, 3, 9] for proofs and examples. The following properties of the modalities $!$ and $\S$, which are the essence of light logics, are worth emphasizing:

$$
\begin{array}{llll}
\textbf{(var)} & a \vdash a & \textbf{(exch)} & \dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \\[2ex]
\textbf{(wk)} & \dfrac{\Gamma \vdash B}{\Gamma, A \vdash B} & \textbf{(contr)} & \dfrac{!A, !A, \Gamma \vdash C}{!A, \Gamma \vdash C} \\[2ex]
(\otimes\textbf{-l}) & \dfrac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C} & (\otimes\textbf{-r}) & \dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \\[2ex]
(\multimap\textbf{-l}) & \dfrac{\Gamma \vdash A \qquad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C} & (\multimap\textbf{-r}) & \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \\[2ex]
(!_0) & \dfrac{\vdash A}{\vdash !A} & (!) & \dfrac{A \vdash B}{!A \vdash !B} \\[2ex]
(\S_0) & \dfrac{\vdash A}{\vdash \S A} & \textbf{(cut)} & \dfrac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \\[2ex]
(\S) & \dfrac{A_1, \cdots, A_k, B_1, \cdots, B_l \vdash C \quad k + l > 0}{!A_1, \cdots, !A_k, \S B_1, \cdots, \S B_l \vdash \S C} & &
\end{array}
$$

Fig. 2. The rules defining valid IMLAL sequents

1. $!$ is *not* a comonad: $!A \nvdash A$ and $!A \nvdash !!A$, but we have *duplication* $!A \vdash !A \otimes !A$
2. $\S$ is a degenerate or neutral form of $!$ i.e. we have $!A \vdash \S A$
3. $\S A \otimes \S B \vdash \S(A \otimes B)$ but $!A \otimes !B \nvdash !(A \otimes B)$.

3 Games and strategies

We consider two-player games between P (Proponent) and O (Opponent). Every play is started by O (this paper is concerned only with the intuitionistic fragment), and thereafter it alternates between P and O. Formally a ***game*** G is a three-tuple $\langle M_G, \lambda_G, P_G \rangle$ where

- M_G is a set of moves
- $\lambda_G : M_G \longrightarrow \{ O, P \}$ partitions moves into those that O can make or *O-moves*, and those that P can make or *P-moves* (we will write M_G^O, M_G^P for the set of O-moves and P-moves of G respectively)
- P_G is a prefix-closed set of finite alternating sequences of moves from M_G, each beginning with an O-move; we call elements of P_G *positions* or *plays*.

For example $\langle \varnothing, \varnothing, \{ \epsilon \} \rangle$ (where ϵ is the empty sequence) is a game, which we call the *empty game*. We interpret atomic formulas a as single-move games G_a, which we also call *atomic*, defined as $G_a = \langle \{ a \}, \{ (a, O) \}, \{ \epsilon, a \} \rangle$. In a game context, a is called a ***token***. In the following we shall abuse notation and often write G_a simply as a when it is clear from the context what we mean (e.g. we abbreviate $G_a \multimap G_a$ to $a \multimap a$).

We construct new games from old using standard game constructions. We write $s \upharpoonright A$ to mean the subsequence of s consisting only of moves from A, and define $\overline{P} = O$ and $\overline{O} = P$. For a game G, we write $M_G^{\circledast}$ to mean the set of finite alternating sequences of moves from M_G. The first two, ***tensor games*** $A \otimes B$ and ***linear function space games*** $A \multimap B$, are standard. For $\circledcirc = \otimes$ and $\multimap$, we have

$$M_{A \circledcirc B} = M_A + M_B$$
$$P_{A \circledcirc B} = \{\, s \in M_{A \circledcirc B}^{\circledast} \mid s \upharpoonright A \in P_A,\ s \upharpoonright B \in P_B \,\}$$

where $\lambda_{A \otimes B}$ is defined to be the canonical map $[\lambda_A, \lambda_B] : M_A + M_B \longrightarrow \{\, P, O \,\}$, and $\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$. Note that it is a consequence of the definition that every $s \in P_{A \otimes B}$ satisfies the *O-Switching Condition*: for each pair of consecutive moves mm' in s, if m and m' are from different components (i.e. one is from A the other from B), then m' is an O-move. Similarly it follows that every $s \in P_{A \multimap B}$ satisfies the *P-Switching Condition* i.e. only P can switch component.

The next two constructions, which we call ***box constructions***, are related. The idea is that a play of a shriek game $!\,A$ consists of a number of interwoven "threads"[3] (or plays) of A, each is tagged explicitly by a number. The whisper game $\S\,A$ is a degenerate form of $!\,A$ in the sense that a play consists of just one thread, which is tagged by $\star$.

Shriek games $!\,A$	***Whisper games*** $\S\,A$
$M_{!\,A} = M_A \times \mathbb{N}$	$M_{\S\,A} = M_A \times \{\star\}$
$\lambda_{!\,A}(m, i) = \lambda_A(m)$	$\lambda_{\S\,A}(m, \star) = \lambda_A(m)$
$P_{!\,A} = \{s \in M_{!\,A}^{\circledast} \mid \forall i \in \mathbb{N}. s \upharpoonright i \in P_A\}$	$P_{\S\,A} = \{s \in M_{\S\,A}^{\circledast} \mid \pi_1^*(s) \in P_A\}$

where $s \upharpoonright i$ is the sequence of A-moves obtained first by projecting s onto the subsequence consisting of pairs whose second component is i, and then by taking the respective first projection of each pair; and where $\pi_1^*(s)$ is the sequence of A-moves obtained from s by taking the respective first projection of each pair.

A *deterministic P-strategy*, or simply ***strategy***, for a game G is a non-empty, prefix-closed subset σ of P_G satisfying: (i) for any even-length s, if $s \in \sigma$ and $sm \in P_G$ then $sm \in \sigma$, and (ii) (*determinacy*) if even-length sm and sm' are both in σ, then $m = m'$. We say that σ is *history-free* if there is a partial function $f : M_G^O \rightharpoonup M_G^P$ such that for any odd-length $sm \in \sigma$, we have $smm' \in \sigma$ if and only if $f(m)$ is defined and equal to m'; we write $\sigma = \sigma_f$ just in case f is the least such function. Further it is said to be ***injective history-free*** if the least such f is injective. If for every odd-length $s \in \sigma$, there is some m such that $sm \in \sigma$, we say that σ is ***total***. For any games A_1, A_2 and A_3 we define $\mathcal{L}(A_1, A_2, A_3)$ to be the set of finite sequences s of moves from $M_{A_1} + M_{A_2} + M_{A_3}$ such that for any pair of consecutive moves mm' in s, if $m \in M_{A_i}$ and $m' \in M_{A_j}$ then $|i - j| \leqslant 1$.

[3] In Section 4 we give a formal definition of a generalized notion of threads.

(We call $\mathcal{L}(A_1, A_2, A_3)$ the set of ***interaction sequences*** over (A_1, A_2, A_3).) Take strategies σ and τ for games $A \multimap B$ and $B \multimap C$ respectively. We define the composite $\sigma\ ;\ \tau$ of σ and τ as:

$$\sigma\ ;\ \tau = \{\, s \upharpoonright (A, C) : s \in \mathcal{L}(A, B, C) \wedge s \upharpoonright (A, B) \in \sigma \wedge s \upharpoonright (B, C) \in \tau \,\}.$$

This is the standard notion of composition of strategies. Games and strategies (maps from A to B are strategies of $A \multimap B$) form a symmetric monoidal closed category; the category whose maps are injective history-free strategies forms a subcategory. All this is standard (see e.g. [1, 8]).

Important Notation. The moves of a tensor game are just the disjoint union of the moves of the respective component games, similarly for function space games. For technical convenience, we fix representations of moves specific to the two constructions

$$\begin{array}{rcl} M_{A \otimes B} & = & M_A \times \{\, l \,\} \cup M_B \times \{\, r \,\} \\ M_{A \multimap B} & = & M_A \times \{\, L \,\} \cup M_B \times \{\, R \,\} \end{array}$$

We call ***free games*** those that are constructed from atomic games. *We emphasize that from now on, by games we shall mean free games.* A move of a free game has the form

$$m = ((\cdots((a, i_n), i_{n-1})\cdots), i_1)$$

where a is a token and each i_j ranges over $\mathbb{N} \cup \{\, \star, l, r, L, R \,\}$. For convenience, we shall write the move as a pair $(a, i_n \cdots i_1)$, and call the second component its *occurrence*. Let $i_{r_d} \cdots i_{r_1}$ be the subsequence of $i_n \cdots i_1$ consisting only of numbers and $\star$. We define the ***depth*** of the move m to be d, the *index at depth* j, or ***j-index***, to be the *sequence* $i_{r_j} i_{r_j - 1} \cdots i_3 i_2 i_1$, and the ***token*** to be a. For example, the 1-index, 2-index and 3-index of the move $(a, 2r\star Ll33R)$ are $3R, 33R$ and $\star Ll33R$ respectively. We say that a subgame A of a game G occurs at depth i if A is in the scope of i box constructors. A game is said to have ***depth*** k just in case k is the maximum depth of all its subgames. (It is straightforward to show that k is equal to the maximum depth of its moves.) We say that a strategy σ for a free game is ***token-reflecting*** if for any even-length $smm' \in \sigma$, m and m' have the same token.

Example 2. Consider the game $\mathbf{!}\,(\mathbf{!}\, a \otimes \mathbf{!}\, b) \multimap \mathbf{!}\,\S\, a$ (which, *qua* formula, is provable); $(a, \star 3R)(a, 2l5L)$ and $(a, \star 3R)(b, 1r4L)$ are positions of the game. Observe that the occurrence of a move may be read as a path by which its token can be reached from outside-in.

It is straightforward to show that IMLAL proofs are denoted by token-reflecting, injective history-free, total strategies. Indeed these properties are enough to characterize denotations of proofs of the $(\otimes, \multimap)$-fragment of Intuitionistic Multiplicative Affine Logic, which we abbreviate to IMAL. (Valid IMAL sequents are those defined by the first eight rules in Figure 2.)

Theorem 2 (Full Completeness). *For any free game given by an IMAL-sequent $\Gamma \vdash A$, and for any* ***IMAL-winning*** *(i.e. token-reflecting, injective history-free, total) strategy σ for the game, there is a derivation of $\Gamma \vdash A$ whose denotation is σ.* □

(A proof of the Theorem can be extracted from [8].)

However, to characterize IMLAL proofs, among other conditions, we require strategies to act uniformly on threads, as the next example illustrates.

Example 3. Consider the game $!\,a \otimes !\,a \multimap !\,a$; the strategy defined by the function f: for $n \in \mathbb{N}$

$$f \;:\; \begin{cases} (a,(2n)R) \mapsto (a,nlL) \\ (a,(2n+1)R) \mapsto (a,nrL) \end{cases}$$

is not the denotation of any proof of the corresponding sequent (there are in fact only two proofs).

Thread uniformity and $\approx$-reflexivity. In order to cut the strategies down to size, we use a partial equivalence relation $\approx$ over strategies introduced in [2]. First we define a $\approx$-***game*** G to be a four-tuple $\langle M_G, \lambda_G, P_G, \approx_G \rangle$ where $\langle M_G, \lambda_G, P_G \rangle$ is a game, and $\approx_G$ is an equivalence relation on P_G satisfying:

- $sa \approx_G tb \Rightarrow s \approx_G t$
- $s \approx_G t \Rightarrow \lambda_G^*(s) = \lambda_G^*(t)$
- $s \approx_G t \wedge sa \in P_G \Rightarrow \exists b \in M_G . tb \in P_G \wedge sa \approx_G tb.$

We extend the four game constructions introduced earlier to their respective $\approx$-game constructions. For $\circledcirc = \otimes$ or $\multimap$, we have $s \approx_{A \circledcirc B} t$ just in case $s \upharpoonright A \approx_A t \upharpoonright A$, $s \upharpoonright B \approx_B t \upharpoonright B$ and $\pi_2(s_i) = \pi_2(t_i)$, for each i, where s_i denotes the i-th move of s. For whisper games $\S\, A$, we have $s \approx_{\S A} t$ if $\pi_1^*(s) \approx_A \pi_1^*(t)$. For shriek games $!\,A$, we have $s \approx_{!A} t$ if

$$\exists \alpha \in S(\mathbb{N}).\forall i \in \mathbb{N}. s \upharpoonright i \approx_A t \upharpoonright \alpha(i) \;\wedge\; \alpha^*(\pi_2^*(s)) = \pi_2^*(t)$$

where $S(\mathbb{N})$ is the collection of permutations of $\mathbb{N}$. The equivalence relation $\approx_G$ extends to a *partial* equivalence relation over strategies of G: $\sigma \approx_G \tau$ holds just in case for all $s \in \sigma$, $t \in \tau$, $sa \in P_G$, $tc \in P_G$, if $sa \approx_G tc$ then the following bisimulation-style properties hold:

- $sab \in \sigma \Rightarrow \exists d \in M_G . tcd \in \tau \wedge sab \approx_G tcd$
- $tcd \in \tau \Rightarrow \exists b \in M_G . sab \in \sigma \wedge sab \approx_G tcd.$

The strategies that we are interested in are those σ that are $\approx$-***reflexive*** i.e. $\sigma \approx \sigma$; we write the partial equivalence class of σ as $[\sigma]$. The intuition is that such strategies "behave equivalently at equivalent positions". For instance two threads of a shriek game $!\,A$ that have equivalent underlying A-positions, but tagged by different numbers, are equivalent. Thus we reject the strategy σ_f in Example 3: its response at even-numbered threads is different from that at equivalent but odd-numbered threads); see Example 6 for another illustration of the effect of $\approx$.

It turns out that $\approx$ is preserved by composition, so that games, with maps from A to B given by $\approx_{A\multimap B}$ partial equivalence classes of injective history-free strategies of $A \multimap B$, form a symmetric monoidal closed category (see [2]). From now on, by games we shall mean $\approx$*-reflexive free games.*

4 Network protocol and discreet strategies

In this section we introduce the network protocol and consider the strategies that comply with it. First we need to formalize a generalized notion of *threads* of a game. Fix a free game G of depth k. By an i*-index* of G, we mean the i-index of some move of G (so $i \leqslant k$). *Threads of depth* i, or simply i***-threads***, of G are named by i-indices of G. A thread named by θ is the set of moves of G whose i-index is θ. (We start counting from outside-in, so that the outermost thread is at depth 1.) An i-thread named by θ is said to be a ***P-thread*** if there are an odd number of occurrences of L in θ, otherwise it is said to be an ***O-thread***; if the leftmost symbol of the index θ is a number we say the i-thread is of $\mathbf{!}$***-type***, otherwise the thread is of $\S$***-type***.

Remark 1. (i) For any game G, we write $T^P_{G,i}$ (respectively $T^O_{G,i}$) for the set of P-threads (respectively O-threads) of G at depth i. We shall omit the subscript i whenever we can get away with it. For any A and B it is easy to see that

$$\begin{array}{lll} T^O_{A\multimap B} \cong T^P_A + T^O_B & \quad T^O_{A\otimes B} \cong T^O_A + T^O_B & \quad T_{\mathbf{!}\,A} \cong T_A \times \mathbb{N} \\ T^P_{A\multimap B} \cong T^O_A + T^P_B & \quad T^P_{A\otimes B} \cong T^P_A + T^P_B & \quad T_{\S\,A} \cong T_A \end{array}$$

(ii) We shall often analyse a play s by considering its subsequence consisting of moves belonging to a given i-thread. We call that subsequence the i***-thread of the play*** s. (Thus i*-thread of a game* is a set, and i*-thread of a play* is a sequence, of moves.)

(iii) It is straightforward to prove, by induction on the structure of the game, that any O-thread (respectively P-thread) of a play must begin with an O-move (respectively P-move).

Example 4. Consider the game $\mathbf{!\,!}\,(a \multimap a) \multimap \mathbf{!\,!}\,(a \multimap a)$ and the play

$$(a, R41R)\ (a, R32L)\ (a, L32L)\ (a, R92L)\ (a, L92L)\ (a, L41R)$$

which we refer to as $m_1 \cdots m_6$ in the picture (where only the respective numeric subsequences of the six occurrences are shown)

	$\mathbf{!\,!}\,(a$	$\multimap$	$a)$	$\multimap$	$\mathbf{!\,!}\,(a$	$\multimap$	$a)$
m_1							41
m_2			32				
m_3	32						
m_4			92				
m_5	92						
m_6					41		

The two 1-threads of the play are $m_1 m_6$ (named by $1R$) and $m_2 m_3 m_4 m_5$ (named by $2L$); and the three 2-threads are $m_1 m_6$ (named by $41R$), $m_2 m_3$ (named by $32L$) and $m_4 m_5$ (named by $92L$).

A network protocol. To model IMLAL, we need to constrain the way threads interact or communicate with each other. A key innovation of this paper is to organize threads into *networks* in accord with a protocol. We are interested in plays that obey the following network protocol at *every* depth (up to the depth of G):

(p1) A ***network*** (at depth d) comprises an O-thread and finitely many P-threads (all at depth d).

(p2) ***Network and Thread Creation***: Only O can open a new network, and he does so whenever he starts a new O-thread; whenever P opens a new thread in response to an O-move from a network, a new P-thread is added to that network.

(p3) ***!-Network***: A network whose O-thread is of **!**-type (call the network a **!**-*network*) has at most one P-thread which must also be of **!**-type.

(p4) No thread can belong to more than one network.

(p5) ***Switching Condition***: Only O can switch network i.e. revisit one opened earlier or enter the *threadless universe*[a]. Only P can switch from one thread to another existing thread within the same network.

[a] At depth d, the threadless universe of G consists of moves of depth $< d$.

In the following we motivate and illustrate the protocol rules by considering the game denotations of several simple IMLAL sequents.

Example 5. (i) Take the game $a \multimap \,!\,a$. Any opening move starts an O-thread and hence a network. The only P-move of the game is from the threadless universe, and P responding with that would violate protocol (p5). There is no total strategy for the game $!\,!\,a \multimap \,!\,a$ because (p2) would be violated at depth 2; similarly for $!\,a \multimap a$ and $!\,a \multimap \,!\,!\,a$. The preceding analysis applies in each case where $!$ is replaced by $\S$. Thus neither of the box constructors is a comonad.

(ii) Consider the token-reflecting strategies for the respective games denoting the sequents:

$$\vdash \S\, a \otimes \S\, b \multimap \S\,(a \otimes b)$$
$$\nvdash \S\,(a \otimes b) \multimap \S\, a \otimes \S\, b$$

For the first game, it is an easy exercise to check that the positions of the token-reflecting strategy obey the protocol. For the second, there are two O-threads (named respectively by $\star lR$ and $\star rR$) and the one P-thread belongs to both the respective networks specified by the O-threads, thus violating (p4).

(iii) Consider the respective game denotations of the sequents:

$$\nvdash \mathbf{!}\, a \otimes \mathbf{!}\, b \multimap \mathbf{!}\,(a \otimes b)$$
$$\vdash \mathbf{!}\,(a \otimes b) \multimap \mathbf{!}\, a \otimes \mathbf{!}\, b$$

The positions $(a, l1R)\,(a, 1lL)$ and $(b, r1R)\,(b, 1rL)$ belong to the token-reflecting total strategy for the first game. The respective first moves $(a, l1R)$ and $(b, r1R)$ belong to the same O-thread of the game, and hence, the same network, but the respective second moves $(a, 1lL)$ and $(b, 1rL)$ belong to different P-threads, thus violating (p3). The second sequent is provable in IMLAL but not in LLL.

(iv) Consider the game $\mathbf{!\,!}\,(a \multimap a) \multimap \mathbf{!\,!}\,(a \multimap a)$ of Example 4. The play therein violates (p3): the $\mathbf{!}$-network at depth 2 has two P-threads. However it is easy to see that the formula is provable. Indeed the game has a position that complies with the protocol, for instance

$$(a, R53R)\ (a, R97L)\ (a, L97L)\ (a, L53R).$$

For a related example, take the game $\mathbf{!\,!}\,(a \multimap a) \multimap \mathbf{!}\,\S\,(a \multimap a)$ (whose corresponding formula is provable). The play

$$(a, R{\star}1R)\ (a, R32L)\ (a, L32L)\ (a, R92L)\ (a, L92L)\ (a, L{\star}1R)$$

corresponds to the play in Example 4 but it complies with the protocol.

(v) Consider the respective token-reflecting strategies for the games denoting the following sequents:

(1)	$\nvdash\ (\mathbf{!}\, a \multimap \mathbf{!}\, b) \multimap \mathbf{!}\,(a \multimap b)$	(3)	$\nvdash\ (\S\, a \multimap \S\, b) \multimap \S\,(a \multimap b)$
(2)	$\nvdash\ \mathbf{!}\,(a \multimap b) \multimap (\mathbf{!}\, a \multimap \mathbf{!}\, b)$	(4)	$\vdash\ \S\,(a \multimap b) \multimap (\S\, a \multimap \S\, b)$

In the first and the third games, the fourth move violates protocol (p5) as it is a case of P attempting to switch back to the original network. In the second game, protocol (p3) is violated by the fourth move as P tries to open a second $\mathbf{!}$-thread of a $\mathbf{!}$-network. The strategy for the fourth game complies with the protocol.

Here are some useful consequences of the network protocol.

Lemma 1. *1. Each P-move has the same depth as the preceding O-move.*
2. Each network at depth $d+1$ is embedded into a single network at d.
3. P's switching condition is a consequence of O's. □

We leave the largely straightforward proofs as an exercise.

Networked positions and discreet strategies. We are now in a position to give a formal definition of strategies that comply with the network protocol. We denote the set of threads of a game G by T_G, which partitions into T_G^O (the set of O-threads) and T_G^P (the set of P-threads), and we write $T_{G,i}$ for the set of i-threads. We formalize the network protocol with the help of two (kinds of) functions.

A *thread function* (at depth i) is a partial function $t_{G,i} : M_G \rightharpoonup T_{G,i}$ that maps a move to the i-thread in which it occurs. Whenever the game G and depth i are clear from the context, we will abbreviate $t_{G,i}(m)$ to t_m.

Network function (at depth i) $\eta_{G,i}$. We say that a partial function

$$\eta_{G,i} : T^P_{G,i} \rightharpoonup T^O_{G,i}$$

networks a position $s \in P_G$ of even length at depth i (with respect to $t_{G,i}$) just in case for each odd j, $t_{G,i}(s_j)$ is defined if and only if $t_{G,i}(s_{j+1})$ is, and if both are defined, one of the following holds (we drop the subscripts from $\eta_{G,i}$ and $t_{G,i}$ whenever we safely can):

$$\begin{array}{llll} (i) & \eta(t(s_j)) = t(s_{j+1}) & (ii) & t(s_j) = \eta(t(s_{j+1})) \\ (iii) & \eta(t(s_j)) = \eta(t(s_{j+1})) & (iv) & t(s_j) = t(s_{j+1}) \end{array}$$

(This is just to say that at depth i the P-move s_{j+1} is in the same network as the O-move s_j: there are four cases depending on whether s_j and s_{j+1} are in a P-thread or an O-thread.) In addition, for any O-thread t^O, if t^O is of $!$-type then the inverse image of t^O, written $\eta^{-1}(t^O)$, contains at most one thread which must also be of $!$-type. □

The astute reader will notice that we have not yet captured all the protocol rules. What is missing is a ***compactness condition*** which is part of (p1): $\eta^{-1}(t^O)$ is finite for every O-thread t^O.

Example 6. To see why the compactness condition is desirable, consider the (first-order) Church numerals game (see [4, 3] for further details)

$$!\,(a \multimap a) \;\multimap\; \S\,(a \multimap a).$$

Let η_n be the (least) function that networks any fixed n-th Church numeral strategy (any two such n-th Church numeral strategies are $\approx$-equivalent); and let t^O be the unique O-thread of the game. It is easy to see that $\eta_n^{-1}(t^O)$ has exactly n P-threads. The "infinity numeral" strategy is ruled out by the compactness condition.

We say that a strategy σ on G is *networked at depth* i if there exists a function $\eta_{G,i}$ that networks every $s \in \sigma$ at depth i. Further we say that it is *compactly networked at depth* i if the compactness condition is satisfied. A strategy σ is said to be ***discreet*** if it is compactly networked at every depth up to the depth of G.

5 A model of IMLAL

We are interested in token-reflecting, injective history-free, $\approx$-reflexive, total discreet strategies. To save writing, we call such strategies ***winning***. Let $\mathbb{G}$ be the category whose objects are free games, and whose maps $A \longrightarrow B$ are $\approx_{A \multimap B}$ partial equivalence classes of winning strategies of $A \multimap B$. Our first task is to check that $\mathbb{G}$, as specified by the above data, is a well-defined category – the main technical problem is to prove that total discreet strategies compose. Secondly we show that $\mathbb{G}$ is a model of IMLAL.

Total discreet strategies compose. Consider the games A, B and C and suppose u is an interaction sequence of two positions $p_1 \in P_{A\multimap B}$ and $p_2 \in P_{B\multimap C}$ which are networked by $\eta_{A\multimap B,i}$ and $\eta_{B\multimap C,i}$ respectively at a given depth i. First we would like to show that $u \upharpoonright (A,C)$ can be networked, and that all positions of the composite strategy are networked by the same network function $\eta_{A\multimap C,i}$.

Notation. Fix a network function η_G. We shall write $t' \prec_G t$ to mean $\eta_G(t) = t'$, so for each P-thread t there can be at most one O-thread t' such that t and t' are in the same network. Our argument in the following is with respect to a fixed depth i. For ease of writing, we shall drop all references to the depth and omit the subscript i from $\eta_{A\multimap C,i}$, $t_{A\multimap C,i}$ etc.

First we define the network function $\eta_{A\multimap C}$: for any $t^O \in T^O_{A\multimap C}$ and $t^P \in T^P_{A\multimap C}$, we decree that $\eta_{A\multimap C}(t^P) = t^O$ (i.e. $t^O \prec_{A\multimap C} t^P$) just in case there exist $b_1, \cdots, b_k \in T_B$ such that

$$t^O \prec_1 b_1 \prec_2 b_2 \prec_3 \cdots \prec_{k-1} b_{k-1} \prec_k b_k \prec_{k+1} t^P \tag{1}$$

where each $\prec_i \in \{\prec_{A\multimap B}, \prec_{B\multimap C}\}$, and for each $i \in \{1,\cdots,k\}$, $\prec_i \neq \prec_{i+1}$. (Each b_i in (1) above is actually the embedding image of the B-thread in $T_{A\multimap B}$ or $T_{B\multimap C}$ as appropriate – see Remark 1, but we stick to b_i by abuse of notation.) Note that if t^O and t^P are both A-threads (respectively C-threads) and $t^O \prec_{A\multimap B} t^P$ (respectively $t^O \prec_{B\multimap C} t^P$), then $t^O \prec_{A\multimap C} t^P$ which is the case of $k = 0$.

Because $\eta_{A\multimap B}$ and $\eta_{B\multimap C}$ are partial functions, we first observe that $\eta_{A\multimap C}$ is a partial function from $T^P_{A\multimap C}$ to $T^O_{A\multimap C}$.

Proposition 1. *Take any contiguous segment $m_O m_1 \cdots m_k m_P$ of u such that $m_O \in M^O_{A\multimap C}$, $m_P \in M^P_{A\multimap C}$, and for $i = 1,\cdots,k$, $m_i \in M_B$. If m_O is part of a thread then m_O and m_P are in the same network with respect to $\eta_{A\multimap C}$ i.e. writing t^O as the O-thread of m_O's network, we have either $t^O = t_{m_P}$ or $t^O \prec_{A\multimap C} t_{m_P}$.* □

Note that $\eta_{A\multimap C}$ has been defined only in terms of $\eta_{A\multimap B}$ and $\eta_{B\multimap C}$. Moreover, if σ and τ satisfy (p3), so does $\sigma\,;\,\tau$, because if

$$t^O \prec_1 b_1 \prec_2 b_2 \prec_3 \cdots \prec_{k-1} b_{k-1} \prec_k b_k \prec_{k+1} t^P$$

and t^P is a $!$-thread, then by (p3) applied alternately to σ and τ, we deduce that $b_1,\cdots,b_k$ and t^O are unique and are all $!$-threads. Thus if strategies σ and τ are networked at depth i by $\eta_{A\multimap B}$ and $\eta_{B\multimap C}$ respectively, $\sigma\,;\,\tau$ is networked by $\eta_{A\multimap C}$ at depth i. Since the preceding argument is independent of i, we can conclude:

Theorem 3. *Strategies that are networked at all depths compose.* □

By first proving the following

Proposition 2. *Suppose $\sigma : A \multimap B$ and $\tau : B \multimap C$ are networked strategies. There exists no interaction sequence u over (A,B,C) such that for some $d \geqslant 1$:*

1. *the number of (distinct) d-indices of threads occurring in* $u \upharpoonright B$ *is infinite*
2. *the number of d-indices of O-threads occurring in* $u \upharpoonright A$ *and* $u \upharpoonright C$ *is finite*
3. $u \upharpoonright (A, B) \in \sigma$ *and* $u \upharpoonright (B, C) \in \tau$. □

we can deduce that total *compactly* networked strategies compose. To summarize, we have

Theorem 4 (Compositionality). *Discrete strategies compose.* □

$\mathbb{G}$ is a model of IMLAL. For any winning strategy of $A \multimap B$ defined by the function f, and for any $\mathbb{N}$-*involution* α (i.e. α is an idempotent bijection from $\mathbb{N}$ to $\mathbb{N}$) we define a strategy of $!\,A \multimap\, !\,B$ given by the function $!\,(f, \alpha)$. Suppose $f(m, \delta) = (m', \delta')$ where $\delta, \delta' \in \{ L, R \}$, we define

$$!\,(f,\alpha) \; : \; ((m,i),\delta) \quad \mapsto \quad \begin{cases} ((m',i),\delta') & \text{if } \delta = \delta' \\ ((m',\alpha(i)),\delta') & \text{otherwise} \end{cases}$$

It is straightforward to show that $!\,(f, \alpha)$ defines a winning strategy, and that $\sigma_{!\,(f,\alpha)} \approx \sigma_{!\,(f,\beta)}$ for any $\mathbb{N}$-involution β. Indeed $!$ extends to a functor $\mathbb{G} \longrightarrow \mathbb{G}$. We leave the definition of the functor $\S : \mathbb{G} \longrightarrow \mathbb{G}$ as an easy exercise.

Proposition 3. *1.* $\S\,, ! : \mathbb{G} \longrightarrow \mathbb{G}$ *are functorial.*
2. There are canonical $\mathbb{G}$*-maps, natural in* A *and* B *as follows:*

$$\begin{array}{rcl} \mu_{A,B} : \S\, A \otimes \S\, B & \longrightarrow & \S\,(A \otimes B) \\ \nu_A : \quad !\, A & \longrightarrow & \S\, A \\ \Delta_A : \quad !\, A & \longrightarrow & !\, A \otimes !\, A \end{array}$$

By now the reader should have no difficulty in constructing the canonical maps. By abuse of notation, we use μ also for the canonical map

$$\mu \; : \; \S\, A_1 \otimes \cdots \otimes \S\, A_n \longrightarrow \S\,(A_1 \otimes \cdots \otimes A_n).$$

As $\mathbb{G}$ is symmetric monoidal closed, and in view of the Proposition, we can say that $\mathbb{G}$ is a model of IMLAL.

6 Full Completeness

The section concerns a full completeness result:

Theorem 5 (Full Completeness). *For any winning strategy* σ *for the free game* G *given by an IMLAL sequent* $\Gamma \vdash C$, *there is a derivation* Ξ *of the sequent such that* σ *is the denotation of* Ξ. □

In the following we sketch a proof of the Theorem. First we give two useful lemmas.

Lemma 2 (Deboxing). *In the category* $\mathbb{G}$*:*

1. *For any map $F : \mathbf{!}\,A \longrightarrow \mathbf{!}\,B$ there is a unique map $f : A \longrightarrow B$ such that $F = \mathbf{!}\,f$.*
2. *If the map $F : \mathbf{!}\,A_1 \otimes \cdots \otimes \mathbf{!}\,A_m \otimes \S\,B_1 \otimes \cdots \otimes \S\,B_n \longrightarrow \S\,C$ uses each $\mathbf{!}\,A_i$ once (in the sense that the depth-1 network of any winning strategy that represents F comprises one 1-thread of $\mathbf{!}\,A_i$), then there is a unique map*

$$f \;:\; A_1 \otimes \cdots \otimes A_m \otimes B_1 \otimes \cdots \otimes B_n \longrightarrow C$$

such that $F = (\underbrace{\nu \otimes \cdots \otimes \nu}_{m} \otimes \underbrace{\mathsf{id} \otimes \cdots \otimes \mathsf{id}}_{n})\ ;\ \mu\ ;\ \S\,f$, where μ and ν are the canonical maps defined in Proposition 3. □

Let σ be a IMAL-winning strategy for a free game given by an IMAL-sequent $\Delta \vdash P \otimes Q$ where $\Delta = D_1, \cdots, D_n$.We say that σ is ***splitting*** just in case there is a partition of Δ into (Δ_1, Δ_2) and IMAL-winning strategies σ_1 and σ_2 for $\Delta_1 \vdash P$ and $\Delta_2 \vdash Q$ respectively such that $\sigma = \sigma_1 \otimes \sigma_2$.

Lemma 3 (Pivot). *If σ is not splitting, then there is some $D_i = A \multimap B$ in Δ (which we shall call a **pivot**) and IMAL-winning strategies τ and υ for $\Theta \vdash A$ and $B, \Xi \vdash P \otimes Q$ respectively, where (Θ, Ξ) is a partition of $\Delta \setminus \{ D_i \}$, such that σ can be defined in terms of τ and υ.* □

We prove the Theorem by induction on the size of the sequent $\Gamma \vdash C$ that defines the free game G. W.l.o.g. we assume that no formula in $\Gamma = C_1, \cdots, C_n$ is a tensor, and every C_i is *σ-reachable* in the sense that there is a position in σ that contains some C_i-move.

Step 1. Decontract at depth 1, if necessary, to get a corresponding winning strategy σ_1 for $\Gamma_1 \vdash C$ so that every formula in Γ_1 is used once by σ_1.

Step 2. It suffices to consider the following cases of C:

I. $C = P \otimes Q$
II. $C = \Box P$, a *box formula* i.e. $\Box = \mathbf{!}$ or $\S$
III. $C = a$, an atom.

For Case I, if $P \otimes Q$ is splitting, then split it to get two smaller instances (of a winning strategy for a free game). Otherwise, we transform the sequent by adding a fresh atom as "$x \multimap -$" inside each box-formula from Γ_1 and adding a copy of "$x \multimap -$" of negative polarity inside the O-box of the corresponding network. E.g. the sequent $\S\,c, \mathbf{!}\,(c \multimap d), \S\,(d \multimap e), \S\,e \multimap \mathbf{!}\,a \otimes \mathbf{!}\,b, \mathbf{!}\,c \vdash \S\,(c \otimes a) \otimes \S\,b$ is transformed to

$$\begin{array}{l} \S\,(x \multimap c), \mathbf{!}\,(y \multimap (c \multimap d)), \S\,(z \multimap (d \multimap e)), \\ \S\,(x \multimap (y \multimap (z \multimap e))) \multimap \mathbf{!}\,a \otimes \mathbf{!}\,b, \mathbf{!}\,(v \multimap c) \vdash \S\,(v \multimap c \otimes a) \otimes \S\,b \end{array}$$

Call the new sequent $\Gamma' \vdash C'$ and the corresponding strategy σ'. Note (i) σ' is a winning strategy; (ii) the new $\multimap$-formulas added cannot be pivots as they communicate with the O-threads of their respective networks. Now consider a second transformation to (say) $\Gamma'' \vdash C''$ which is defined by "forgetting all the boxes" and turning it into a free IMAL-game, and call the corresponding strategy σ''. Observe that if σ'' is splitting in IMAL then σ' is also splitting in IMLAL, because all formulas in Γ'' are $\multimap$-formulas or atoms. Suppose σ'' is not splitting in IMAL. Then by Lemma 3 there must be a pivot in Γ''. By construction of Γ', that pivotal $\multimap$-formula cannot occur inside a box in Γ_1, so the pivot is also a pivot for $\Gamma_1 \vdash P \otimes Q$. With the pivot, we obtain two smaller instances, to which we can apply the induction hypothesis.

For Case II, if Γ_1 does not contain a $\multimap$-formula, then every formula in it must be a box-formula; we use Lemma 2 to strip off the outermost boxes, and so obtain a smaller instance, and then apply the induction hypothesis. Otherwise, $\Box P$ is part of some network $\Box G_1, \cdots, \Box G_l$. For each $G = P, G_1, \cdots G_l$, we replace $\Box G$ by $\Box G \otimes \Box G$. E.g. the sequent $c, c \multimap \,!\, b, \S\,(b \multimap a) \vdash \S\, a$ is transformed to

$$c, c \multimap (!\, b \otimes !\, b), \S\,(b \multimap a) \otimes \S\,(b \multimap a) \vdash \S\, a \otimes \S\, a$$

Let σ' be the corresponding strategy for the transformed game $\Gamma' \vdash \Box P \otimes \Box P$; σ' is not splitting (because the $\multimap$-formula in Γ' is reachable from both $\Box P$ on the right). This is an instance of the preceding case. Finally a similar (but simpler) transformation can be applied to reduce Case III to Case I.

Further work An obvious direction is to extend the construction to IMLAL2, the second-order system. Another is to check that the model is not just fully but also *faithfully* complete. We expect to prove this with respect to an appropriate notion of equality between IMLAL proof nets. Also worth developing is a convenient syntax for IMLAL2 as a PTIME intermediate (or meta-) programming language.

Acknowledgments We are grateful to Hanno Nickau for discussions on LLL.

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