Evolving Games and Essential Nets for Affine Polymorphism

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Abstract. This paper presents a game model of Second-order Intuitionistic Multiplicative Affine Logic (IMAL2). We extend Lamarche's essential nets to the second-order affine setting and use them to show that the model is fully and faithfully complete.
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1 Introduction

This paper is about a second-order extension of AJM games [2, 1], which we call *evolving games*. A play begins with O making an opening move, and the two players alternate thereafter. An evolving game has two kinds of tokens: ground and second-order. Ground tokens are standard; they are playable at once (if reachable). *Second-order tokens* are (descriptions of) game evolutions, which cannot be played right away on their own. If a second-order token θ is reachable at a given position, a player may import a game A as an argument for the evolution operator θ, thus causing the current game to grow locally, with the game $\theta(A)$ grafted at where the token θ was. Transported now to the new and expanded game, the same player may play a ground token if one is reachable and so complete the second-order move; or he may continue the evolution process (if a second-order token is reachable) by importing another game, and so on. However after finitely many such evolution steps, the player is required to play a ground token, thus finally completing the second-order move. A version of this approach first appeared in a LICS'97 paper [8] by Hughes. He shows how a fully complete model for System F can be constructed. A more abstract presentation has been considered in [15].

Our goal is to construct a simple evolving game model for Second-order Intuitionistic Multiplicative Affine Logic (IMAL2) (see Figure 1 for the rules of the IMAL2 Sequent Calculus; note that we do not consider unit) in the sytle of [15]. In Section 2 we introduce evolving games and define the playable moves and

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(atom)	$a \vdash a$	**(var)**	$X \vdash X$
(exch)	$\dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$	**(wk)**	$\dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}$
(⊗-l)	$\dfrac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C}$	**(⊗-r)**	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
(⊸-l)	$\dfrac{\Gamma \vdash A \qquad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C}$	**(⊸-r)**	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$
(∀-l)	$\dfrac{\Gamma, A[B/X] \vdash C}{\Gamma, \forall X.A \vdash C}$	**(∀-r)**	$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall X.A}$

where the side condition of (∀-r) is: X does not occur free in Γ.

Fig. 1. The rules defining valid IMAL2 sequents

positions of an evolving game. As our aim is to construct a fully complete model for IMAL2, our treatment here is restricted to a version of free such games, which are in one-one correspondence with closed IMAL2 types. IMAL2 proofs are modelled by strategies which we present in two stages. We consider first the simple scenario in which the games imported by O (as arguments for his second-order tokens) are guaranteed to be singleton games consisting of a ground token. P-strategies for playing such evolving games are called *symbolic*. Strategies that denote proofs are total, (ground) token-reflecting and finitely presentable; in addition, the evolution arguments P imports are determined schematically by those which O has imported thus far in the play. We call symbolic strategies that possess these properties *regular*, and they are introduced in Section 3. Regular strategies cannot be composed; however strategies that are generated from regular strategies by a process of copycat expansion do compose, as we show in Section 4. (It is worth noting that our proof of compositionality is direct and syntax-independent, or rather, independent of the formal system IMAL2.) We call such strategies *good*, and they give rise to a model of IMAL2.

In Section 5 we turn our attention to essential nets [11], which are a kind of oriented proof nets, for IMAL2. We give a *correctness criterion* for such nets, and prove that all correct nets are sequentializable. The main result of the paper is Theorem 2:

> *Evolving games and good strategies are fully and faithfully complete for IMAL2*

which is proved in Section 6. The key step is a correspondence result (Lemma 3 and Proposition 1) which shows that each regular strategy determines a correct essential net for the associated end-sequent. Faithful completeness is proved with respect to a notion of equivalence of such nets.

The only game model for IMAL2 in the literature is given in [1]. Recently Abramsky and Lenisa [3] have constructed a linear combinatory algebra of partial involutions on the natural numbers, arising from Geometry of Interaction constructions; they show that a fully and faithfully complete model for ML polymorphic types of system F can be obtained in this way. To the best of our knowledge, our game model is the first fully (and faithfully) complete model for IMAL2; indeed all results in this paper are new.

2 Evolving games

We assume the notions of games (and singleton games) as defined in [1, 12]; we shall call them ***IMAL games***. Recall that new games can be constructed from old using standard game constructions. We write $s \upharpoonright A$ to mean the subsequence of s consisting only of moves from A, and define $\overline{P} = O$ and $\overline{O} = P$. For a game G, we write $M_G^{\circledast}$ to mean the set of finite alternating sequences of moves from M_G. The first two, *tensor games* $A \otimes B$ and *linear function space games* $A \multimap B$, are standard. For $\circledcirc = \otimes$ and $\multimap$, we have

$$M_{A \circledcirc B} = M_A + M_B$$
$$P_{A \circledcirc B} = \{ s \in M^{\circledast}_{A \circledcirc B} \mid s \upharpoonright A \in P_A, \ s \upharpoonright B \in P_B \}$$

where $\lambda_{A \otimes B}$ is defined to be the canonical map $[\lambda_A, \lambda_B] : M_A + M_B \longrightarrow \{ P, O \}$, and $\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$. Note that it is a consequence of the definition that every $s \in P_{A \otimes B}$ satisfies the *O-Switching Condition*: for each pair of consecutive moves mm' in s, if m and m' are from different components (i.e. one is from A the other from B), then m' is an O-move. Similarly it follows that every $s \in P_{A \multimap B}$ satisfies the *P-Switching Condition* i.e. only P can switch component.

Tokens: ground and second-order Fix an infinite set $\mathcal{T}_g$ of *ground tokens* which are ranged over by a, b, c, etc. The set $\mathcal{G}$ of ***free games*** is generated from the ground tokens by the constructors of IMAL2 as follows:

– every ground token $a \in \mathcal{T}_g$, considered as a singleton game, is in $\mathcal{G}$
– if G_1 and G_2 are in $\mathcal{G}$ then $G_1 \otimes G_2$ and $G_1 \multimap G_2$ are in $\mathcal{G}$
– if $G \in \mathcal{G}$ then for each $e \in \mathcal{T}_g$, the singleton game $[\forall e.G]$ is in $\mathcal{G}$.

(By abuse of notation, we confuse a singleton game with its sole token.) Tokens of the form $[\forall e.G]$ are called ***second-order tokens***. Thus $\mathcal{G}$ contains games generated from both ground and second-order tokens using the constructors $\otimes$ and $\multimap$.

Remark 1. As we shall see shortly, a second-order token $[\forall a.G]$ is a description of the game operation: $A \mapsto G[A/a]$. As usual "α-equivalent" second-order tokens are considered identical. Clearly free games are in one-one correspondence with closed IMAL2 types.

Given any IMAL2 formula F and a map ρ from its free variables to $\mathcal{G}$, there is an obvious denotation $[\![F]\!]_\rho$ in $\mathcal{G}$. E.g. take $F = X \otimes \forall Y.(Y \otimes (X \multimap \forall Z.Z))$ and $\rho : X \mapsto [\forall b.(e \multimap b)] \otimes e$; we have

$$[\![F]\!]_\rho \;=\; ([\forall b.(e \multimap b)] \otimes e) \otimes [\forall Y.(Y \otimes (([\forall b.(e \multimap b)] \otimes e) \multimap [\forall Z.Z]))].$$

Playable moves of an evolving game When considered as an IMAL game, a move of a free game G is either a ground or a second-order token, which can be named by its occurrence in the syntax tree of G. For this purpose the branches of $\otimes$ are labelled l and r, and those of $\multimap$ are labelled L and R. For instance, the tokens of the game $a \otimes (b \multimap [\forall c.c])$ (from left to right) are l, rL and rR.

In the setting of evolving games, starting from G (say), the interim game (which we can think of as the "current game board") grows as the play unfolds, so that over the life time of a play, many more moves than are specified in the initial game G become available. We call these generalized moves *playable moves* of the evolving game G and define it by recursion as follows.

Definition 1. A string $s \in (\{l, r, L, R, \} \cup \mathcal{G})^*$ is a ***playable move*** of an evolving game G if:

– either $s \in \{l, r, L, R\}^*$ names a ground token in G, called the *playable token* of s – then we say that G does not evolve (as a consequence of playing s)
– or there exist $A \in \mathcal{G}$ (called ***evolution argument***), $s_1 \in \{l, r, L, R\}^*$, $s_2 \in (\{l, r, L, R\} \cup \mathcal{G})^*$ such that $s = s_1 A s_2$, s_1 names a second-order token $[\forall e.T]$ in G, and s_2 is a playable move of the game $T[A/e]$. The *playable token* of s is defined to be the same as that of s_2 (although playability is defined with respect to a different game). If s_2 causes $T[A/e]$ to evolve into G', then we say that s causes G to evolve into $G[G'/[\forall e.T]]$.

Note that in the second case above it may take several evolution arguments to make a playable move. We shall write the evolution arguments as superscripts (e.g. $s_1^A \cdots$ above). From the moment such an A is played we say that the ***slot*** s_1 is defined. It may turn out that s_2 begins with a contiguous segment of evolution arguments in which case all of them form part of the definition of slot s_1. If $s_2 = A_1 \cdots A_n s_3$ for $A_i \in \mathcal{G}$ and s_3 begins with one of l, r, L or R, we write $\lceil s_1 \rceil = A A_1 \cdots A_n$. If s is a playable move for G, we call $s \upharpoonright \{l, r, L, R\}$ the ***location*** of s, and we write it as $\lfloor s \rfloor$.

Example 1. (i) $(a \otimes [\forall d.d]) r (b \otimes [\forall c.c]) l$ is a playable move for the singleton game $[\forall b.b]$, which consequently evolves into $a \otimes (b \otimes [\forall c.c])$. The playable token is b and the slots defined by that move are $\lceil \epsilon \rceil = a \otimes [\forall d.d]$ and $\lceil r \rceil = b \otimes [\forall c.c]$.

(ii) The playable move $\underbrace{[\forall e.e] \cdots [\forall e.e]}_{k} a$ causes the singleton game $[\forall b.b]$ to evolve into a. There are k intermediate steps - all equal to $[\forall b.b]$, and a is the playable token. The move defines $\lceil \epsilon \rceil$ to be $\underbrace{[\forall e.e] \cdots [\forall e.e]}_{k} a$.

Observe that in general $\lceil - \rceil : \{l, r, L, R\}^* \rightharpoonup \mathcal{G}^+$, where X^+ means the set of non-empty words over the alphabet-set X. (A playable move m can be seen as defining a set $\lceil s_1^m \rceil, \cdots, \lceil s_{n^m}^m \rceil$ of slot definitions for $s_j^m \in \{l, r, L, R\}^*$.)

Plays of an evolving game We define *positions* of the evolving game G as follows.

Definition 2 (Positions). The empty sequence is a position. If the finite sequence of playable moves, s, is a position and G' is the evolved game at that point, then sm is a position provided

(i) m is a playable move of G', and
(ii) the sequence consisting of the respective locations of the elements in sm is a position of G'' (here in the standard sense of IMAL games [12]), where G'' is the game that has evolved from G' after m is played.

Example 2. We examine two maximal positions for the game

$$[\forall X.((X \multimap X) \otimes [\forall Y.(Y \multimap Y)])] \multimap [\forall X.((X \multimap X) \otimes [\forall Y.(Y \multimap Y)])]$$

(i) We display the position as follows:

$$\begin{array}{lcccccccc} & (R^e lR) & (L^e lR) & (LlL) & (RlL) & (Rr^c R) & (Lr^c R) & (LrL) & (RrL) \\ \text{evolution} & G_1 & G_2 & G_2 & G_2 & G_3 & G_4 & G_4 & G_4 \end{array}$$

where:

$$\begin{array}{ll} G_1 & [\forall X.((X \multimap X) \otimes [\forall Y.(Y \multimap Y)])] \multimap ((e \multimap e) \otimes [\forall Y.(Y \multimap Y)]) \\ G_2 & (e \multimap e) \otimes [\forall Y.(Y \multimap Y)] \multimap (e \multimap e) \otimes [\forall Y.(Y \multimap Y)] \\ G_3 & (e \multimap e) \otimes [\forall Y.(Y \multimap Y)] \multimap (e \multimap e) \otimes (c \multimap c) \\ G_4 & (e \multimap e) \otimes (c \multimap c) \multimap (e \multimap e) \otimes (c \multimap c) \end{array}$$

After the first move we have $\lceil R \rceil = e$, after the second $\lceil L \rceil = e$. The fifth move defines $\lceil Rr \rceil$ to be c and the next makes $\lceil Lr \rceil$ equal c.

(ii) After the first move we have $\lceil R \rceil = e$ and $\lceil Rr \rceil = c$. The second move sets $\lceil L \rceil = e$ and $\lceil Lr \rceil = c$. The position is displayed below:

$$\begin{array}{lcccccccc} & (R^e r^c R) & (L^e r^c R) & (LrL) & (RrL) & (RlR) & (LlR) & (LlL) & (RlL) \\ \text{evolution} & H_1 & H_2 & H_2 & H_2 & H_2 & H_2 & H_2 & H_2 \end{array}$$

where:

$$\begin{array}{ll} H_1 & [\forall X.((X \multimap X) \otimes [\forall Y.(Y \multimap Y)])] \multimap (e \multimap e) \otimes (c \multimap c) \\ H_2 & (e \multimap e) \otimes (c \multimap c) \multimap (e \multimap e) \otimes (c \multimap c) \end{array}$$

Call a string $s \in \{l, r, L, R\}^*$ an ***O-location*** if s contains an even number of L's. Otherwise it is a ***P-location***. The sets containing them are called L_O and L_P respectively. Intuitively an O-location is the occurrence of an O-move in a game. Thus if $\lceil s \rceil$ is defined in a position and s is an O-location (respectively P-location), then it was first defined by O (respectively P). In other words, a player cannot define his opponent's slots.

3 Symbolic strategies

In order to prove a definability result, it is convenient to consider first the simple scenario in which the evolution arguments provided by O are guaranteed to be (sequences of) ground tokens (regarded as singleton games) – we say that O plays *symbolically* in this case. This assumption makes it easy to see that denotable P-strategies have a finite description.

Formally we say that O plays ***symbolically*** in a position if for each $s \in L_O$ defined therein we have $\lceil s \rceil \in \mathcal{T}_g^+$. For the rest of the section we assume that O plays symbolically. We say that a P-strategy σ is ***symbolic*** just in case for every even-length $s \in \sigma$, if sm is a position then $sm \in \sigma$ if and only if the O-move m is symbolic.

The purpose of the next definition is to introduce strategies that are parametric in the sense that the evolution arguments given by P are determined by those given by O in a schematic fashion. First we set $\mathcal{G}(L_O)$ to be the collection of formal objects defined by the following rules:

- every ground token $a \in \mathcal{T}_g$ and every O-location are in $\mathcal{G}(L_O)$
- if G_1 and G_2 are in $\mathcal{G}(L_O)$ then $G_1 \otimes G_2$ and $G_1 \multimap G_2$ are in $\mathcal{G}(L_O)$
- if $G \in \mathcal{G}(L_O)$ then for each $e \in \mathcal{T}_g$, $[\forall e.G]$ is in $\mathcal{G}(L_O)$.

For example we have $c \multimap Rr \otimes [\forall a.(LLr \multimap a \otimes b)] \in \mathcal{G}(L_O)$. (Informally $\mathcal{G}(L_O)$ is the collection of free games in which some ground tokens are replaced by O-locations.)

Definition 3. The pair $f : L_O \rightharpoonup L_P$ and $F : L_P \rightharpoonup (\mathcal{G}(L_O))^+$ define a symbolic strategy σ if for all even-length $sm_O m_P \in \sigma$ (m_O necessarily symbolic):

- $\lfloor m_P \rfloor = f(\lfloor m_O \rfloor)$
- for all $u, v \in (\{l, r, L, R\} \cup \mathcal{G})^*$ and $\overline{A} \in \mathcal{G}^+$ such that $m_P = u^{\overline{A}}v$ where $\lfloor u \rfloor$ is a P-location and v begins with one of l, r, L or R, we have $\overline{A}$ is determined by $F(\lfloor u \rfloor)$ as follows: Suppose the O-locations that appear in $F(\lfloor u \rfloor)$ are $w_1, \cdots, w_k$, then $\overline{A}$ is obtained from $F(\lfloor u \rfloor)$ by replacing each w_i by $\lceil w_i \rceil$, each $\lceil w_i \rceil$ must already be defined in sm_O.

We write $\sigma = \sigma_{f,F}$ if f, F are the least such functions. $\square$

By definition, it is easy to see:

1. f and F are related: each P-location in $dom(F)$ is a prefix of some element in $cod(f)$.

2. By leastness of f and F, elements of $dom(f) \cup cod(f)$ are incomparable.

Note that a symbolic strategy is *location-wise* history-free, in the sense that the location of P's response at any position depends only on the preceding O-move.

To model proofs, f and F should be finite (it is enough to require that of f). The finiteness is needed to eliminate infinite strategies like the one for the game $[\forall X.X] \multimap [\forall Y.Y]$ in which P replies by importing $([\forall X.X] \multimap [\forall Y.Y]) \multimap a$ where a is the token of the last O-move, and then playing the ground token a. This puts O in the same position as at the beginning of the game, and the play could continue indefinitely, if f and F were allowed to be infinite.

Definition 4. A ***regular strategy*** is a total, ground-token-reflecting[1] symbolic strategy $\sigma_{f,F}$ given by some finite f and F, where f is an injection.

It follows that if $sm_O m_P \in \sigma$, $m_P = u^{\overline{A}}v$ for a P-location $\lfloor u \rfloor$, v beginning with one of l, r, L and R, and

$$F(\lfloor u \rfloor) \;=\; G_1(w_1, \cdots, w_k) \cdots G_j(w_1, \cdots, w_k) \;\in\; (\mathcal{G}(L_O))^+$$

(where each $G_i(w_1, \cdots, w_k)$ is in $\mathcal{G}(L_O)$ such that the O-locations that appear in it are among $w_1, \cdots, w_k$) then in the *P-view* [9] of sm_O there must be O-moves which define $\lceil w_i \rceil$ for each i (i.e. the evolution argument which O has supplied at each w_i).

Example 3. The strategy considered in Example 2 given by:

$$f = \{(RlR, LlR), (LlL, RlL), (RrR, LrR), (LrL, RrL)\}$$
$$F = \{(L, R), (Lr, Rr)\}$$

is regular. Note that this is the identity strategy.

Remark 2. In general, it is too weak to require the dependence of evolution arguments on P-views. Consider the following sequent:

$$\vdash \forall X.X \otimes (\forall X.X \multimap b \otimes \forall Y.Y) \multimap b \otimes \forall Y.Y.$$

P's response at the two positions

$$(Rl)(LrRl)(LrL^a) \text{ and } (Rr^a)(LrRr^a)(LrL^a)$$

should be the same, say, $(Ll^{a \otimes a}r)$, but if the way P imports evolution arguments is to depend on P-views, we are free to play $Ll^{a \otimes a}r$ in the first instance and $Ll^{[\forall V.V] \otimes a}r$ in the second. For regular strategies, this is not possible. Since $\lceil Ll \rceil$ must be defined in terms of the evolution arguments O has imported (at certain O-locations) defined in the history of the play, it must depend solely on $\lceil LrL \rceil$. This forces P to import the same evolution argument in both instances.

It is easy to see that regular strategies are finite, in the sense that it is a finite set of positions, and so, every play terminates. In the following we shall see how they can be used to generate possibly infinite plays.

[1] I.e. for every even-length $sm^- m \in \sigma$, if the O-move m^- is a *ground* token a, then the token of m is also a (see [12]).

4 Good strategies for composition

Unfortunately regular strategies cannot be composed. This section is concerned with a notion of strategies that are good for composition. *Good strategies* are generated from regular ones by a process of *copycat expansion*, which is essentially a semantic form of η-expansion [8, 10]. We shall see that good strategies compose by the standard mechanism of "parallel composition with hiding" [2].

Definition 5 (Good strategies). Take a regular strategy $\sigma = \sigma_{f,F}$. The *good strategy generated from* σ, written $\overline{\sigma}$, is defined by the following algorithm:

Suppose the odd-length position $sm \in \overline{\sigma}$ such that $\lfloor m \rfloor = t$. Find $u \leq t$ such that $u \in cod(f) \cup dom(f)$. (Note that a unique such u exists by induction on the length of positions.) This decomposes the O-move m into $m_u m_v$ such that $t = uv$, m_u ends in one of l, r, L and R, and $\lfloor m_u \rfloor = u$. There are two cases:

(i) If $u \in dom(f)$, play $\overline{f(u)} m_v$
(ii) If $u \in cod(f)$, play $f^{-1}(u) m_v$

In case (i) if u is encountered in the play for the first time, $\overline{f(u)}$ is obtained from $f(u)$ by inserting appropriate evolution arguments using F, based on the evolution arguments which O has already given in sm; otherwise $\overline{f(u)} = f(u)$

Remark 3. In a play, the evolution argument for each *slot* (whether O or P) is given explicitly only once. Thus in case (i), $\overline{f(u)} = f(u)$ if the u in question has already been met in the history of play. In case (ii), all slots should have already been defined, since the corresponding instance of case (i) must have been encountered first. This explains the apparent asymmetry in the two cases.

Lemma 1 (Zigzag). *In any position of a good strategy generated by f and F, for any $u \in dom(f)$, f is applied in case (i) first if at all; thereafter each time before (i) is applied again for the same u, an instance of case (ii) must have already occurred for that u.*

Proof. Otherwise, in the subgame rooted at t, we would have two consecutive O-moves. □

It is straightforward to see that good strategies are (location-wise) history-free and total. Good strategies can be infinite, as the following example illustrates.

Example 4. Take the good strategy extending the symbolic identity $\{\epsilon, R^a, R^a L^a\}$ for the game $[\forall X.X] \multimap [\forall Y.Y]$. Suppose O plays $R^{[\forall X.X] \multimap a} R$. P plays copycat and so responds by $L^{[\forall X.X] \multimap a} R$. From this point onwards, both players behave analogously, thus engaging in an infinite exchange.

Now composition (of good strategies) can be defined by the conventional definition. In fact

Theorem 1. *Good strategies compose.* □

We prove the Theorem directly, without recourse to the formal system IMAL2, as a simple consequence of the following lemma. We give its proof in Appendix A.

Lemma 2. *(i) The composite strategy of two good strategies is total i.e. there is no infinite chattering when the two strategies interact.*
(ii) The composite of two good strategies is generated from a regular strategy $\sigma_{f,F}$ for some finite f (and so F is also finite). □

Remark 4. The composition algorithm that works is the standard one [2] but appropriate adjustments should be made to respect the convention that the locations of B-moves in $A \multimap B$ begin with R, whereas those in $B \multimap C$ begin with L. Therefore if σ (for $A \multimap B$) tells P to play (the move whose location is) Ru in B, he should ask τ (for $B \multimap C$) what to do with (the move whose location is) Lu (rather than Ru).

We devote the next sections to the introduction of essential nets for IMAL2 with the aim of proving the main result of the paper:

Theorem 2 (Full Completeness). *Evolving games and good strategies are fully and faithfully complete for IMAL2.* □

5 Essential nets for IMAL2 and IMLL2

We extend Lamarche's essential nets [11, 14] for (quantifier-free) IMLL to the second-order affine fragment. Our correctness criterion is motivated by the evolving game model and corresponds to the rule that the game imported by P is determined "uniformly" by those imported by O earlier. To our knowledge, the results in this section are also new. Proof nets for first-order MLL which extend the DR-criterion [5] for MLL have already been proposed in [6], from which it is an easy step to derive a correctness criterion for MLL2 (i.e. second-order *classical* MLL). Although IMLL2 is a sublogic of MLL2, essential nets are quite a different kind of structure from proof nets; knowing the corresponding DR-criterion is not much of a help in finding a correctness criterion for essential nets.

Essential nets are a graphical representation of derivations of IMAL2 sequents expressed in ***polarized form*** $\Vdash N_1, \ldots, N_k, P$, where the N_is are negative formulas and P is a positive formula; note that the polarized sequents are one-sided. For the sake of completeness we give a definition of IMAL2 sequents in polarized form in Appendix B. In the following we shall assume the correspondence between the two-sided sequents $\vdash$ and one-sided polarized sequents $\Vdash$, and use them interchangeably.

Definition 6. An ***essential net*** for an IMAL2 sequent $\Vdash N_1^-, \cdots, N_k^-, P^+$ in polarized form is a directed graph which is constructed from the following components:

- axiom link, which is an oriented edge from a positive copy to a negative copy of a propositional atom or second-order variable:

$$\begin{tikzcd} a^- & & a^+ \arrow[ll, bend right] & X^- & & X^+ \arrow[ll, bend right] \end{tikzcd}$$

– polarized $\otimes$-nodes and $⅋$-nodes:

$$\begin{tikzcd}[column sep=small] A^- & & B^+ & A^- \arrow[dr] & & B^- \arrow[dl] & A^+ & & B^+ & A^+ & & B^- \arrow[dl] \\ & A^- ⅋^+ B^+ \arrow[ur] & & & A^- ⅋^- B^- & & & A^+ \otimes^+ B^+ \arrow[ul] \arrow[ur] & & & A^+ \otimes^- B^- \arrow[ul] & \end{tikzcd}$$

In each of the four components above, the two nodes situated above are called ***premises*** of the node below (similarly for the $\forall^+$-node and $\forall^-$-node in the following). We call the left premise of a $⅋^+$-node its ***sink***; note that there is no edge linking a $⅋^+$-node to its sink.

– $\forall^+$-nodes:

$$\begin{tikzcd} A^+ \\ \forall^+ X.A^+ \arrow[u] \end{tikzcd}$$

where X is called the ***eigenvariable*** of the node $\forall^+ X.A^+$.

– $\forall^-$-nodes:

$$\begin{tikzcd} A^-[T/X] \arrow[d, "T"] \\ \forall^- X.A^- \end{tikzcd}$$

Note the *labelled* edge. If X occurs in A, the label T can be retrieved from the net. We call T the ***eigentype*** of the $\forall^-$-node.

– and ***weakening nodes*** (note the bar over the formula)

$$\overline{A^-}$$

such that the ***conclusions*** of the net (nodes that are not premises of any node) are exactly $N_1^-, \cdots, N_k^-, P^+$, polarized as indicated. The positive conclusion is called the ***root*** of the net.

Important Convention. In addition we require that:

1. the eigenvariables be distinct
2. every variable that occurs in the net be the eigenvariable of some $\forall^+$-node, otherwise we replace all its occurrences throughout the net by a fresh propositional atom
3. conclusions are closed formulas.

An essential net for an IMLL2 sequent is defined in exactly the same way by construction from the above components but *less* the weakening nodes. Note that an IMLL2 net is also an IMAL2 net. □

It is an easy exercise to check that each IMAL2 derivation (expressed in terms of polarized sequents) determines an essential net for the end-sequent. Of course there are essential nets that do not arise in this way, as the next example shows.

Example 5. We give the "obvious" essential net for the (invalid) sequent $\vdash \forall Z.(a \otimes Z) \multimap (a \otimes \forall X.X)$ in Figure 2. The reader might wish to construct an essential net for the (invalid) sequent $\vdash \forall XV.(X \otimes V) \multimap (\forall Y.Y \otimes \forall Z.Z)$.

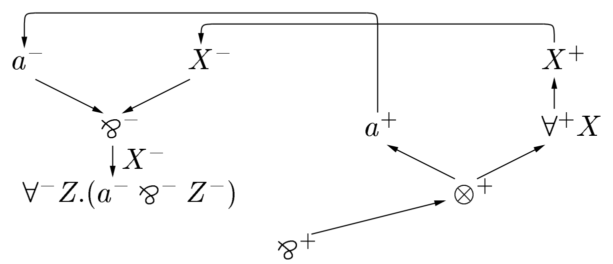


Fig. 2. An essential net for $\vdash \forall^- Z.(a^- \wp^- Z^-) \wp^+ (a^+ \otimes^+ \forall^+ X.X^+)$.

[**A note on figures:** Owing to typographical constraints, we have been economical with the node-labels in our drawing of essential nets, often displaying only the outermost connective rather than the whole formula.]

The main challenge in finding a *correctness criterion* (a characterization of essential nets that arise from derivations) is to capture the side condition of the rule ($\forall$-r). It turns out that the notion of essential nets alone (and in particular the Convention) already rules out several invalid sequents.

Example 6. (i) There is no essential net for $\vdash X^-, \forall^+ X.X^+$ because of the convention that conclusions must be closed formulas – note that X^- may not be replaced by a fresh atom because it is an eigenvariable.
(ii) There is no essential net for

$$\vdash \forall^- Y.(Y^- \wp^- Y^-), \forall^+ X.X^+ \otimes^+ \forall^+ Z.Z^+$$

because X^+ and Z^+ are different and it is impossible to axiom-link each of them with Y^-.

Correctness criterion for IMAL2 and IMLL2 essential nets

It is straightforward to see that if an IMAL2 essential net is acyclic then every node x is reachable from either the root or from a weakening node (or from both). Call the latter a ***source*** of x. In the case of IMLL2 nets, acyclicity implies that every node is reachable from the root.

Definition 7. An IMAL2 essential net is said to be ***correct*** just in case:

(1) the digraph is acyclic
(2) for any $\wp^+$-node p, for any source x of p's sink which is also p's source, every path from x to p's sink passes through p
(3) for any $\forall^-$-node q whose eigentype is T (say), for any free variable X that occurs in T, for any source x of q which is also a source of the $\forall^+$-node r whose eigenvariable is X, every path from x to q passes through r.

(Note that the three statements above are also defined to be the correctness criterion for IMLL2 nets, though for these nets they can be stated more simply since every node has the root as its unique source.) □

Example 7. (i) In the essential net in Figure 2, note that the $\forall^-$-node does not satisfy condition (3): there is a path from the root to the $\forall^-$-node without passing through the $\forall^+$-node.
(ii) The sequent $\vdash \forall Y.Y \otimes \forall Z.Z \multimap \forall X.(X \otimes X)$ has (infinitely) many correct essential nets.

Now for some examples of essential nets that have weakening nodes:

Example 8. (i) Consider the essential net given by the following derivation

$$\cfrac{\cfrac{\cfrac{\vdash Y^- \wp^+ Y^+}{\vdash Y^-,\ Y^- \wp^+ Y^+}}{\vdash \forall^- X.X^-,\ Y^- \wp^+ Y^+}}{\vdash \forall^- X.X^-,\ \forall^+ Y.Y^- \wp^+ Y^+}$$

which has two connected components. The source of the $\forall^-$-node (which has eigentype Y) is the weakening node Y^-. The $\forall^+$-node, whose eigenvariable is Y, has the root as its source. Thus condition (3) of the correctness criterion is vacuously satisfied.
(ii) The sequent $(a \multimap b) \otimes (b \multimap a) \multimap c, d \vdash d$ has no correct essential net in which c is a weakening node.
(iii) The sequent $a \multimap a, b \vdash b$ has no correct essential net which contains no weakening node. (Of course it has a correct essential net in which $a \multimap a$ is a single weakening node.)

Theorem 3 (Sequentialization). *IMAL2 (and hence also IMLL2) essential nets $\mathcal{E}$ satisfying the correctness criterion are sequentializable i.e. there is a derivation of the end-sequent that gives rise to $\mathcal{E}$.* □

We give a proof of the Theorem in Appendix C.

Canonical essential nets

Although essential nets respect the commuting conversions of IMLL, the addition of weakening brings out new ones which are not handled so well. For example the proofs below should be deemed equivalent, but their essential nets differ:

$$\dfrac{z \vdash z \qquad \dfrac{x \vdash x}{y, x \vdash x}}{z, z \multimap y, x \vdash x} \qquad\qquad \dfrac{\dfrac{x \vdash x}{z \multimap y, x \vdash x}}{z, z \multimap y, x \vdash x}$$

Another problem is the commutation of weakening with the $\parr^-$-rule. The two proofs

$$\dfrac{\dfrac{\dfrac{x \vdash x}{y, x \vdash x}}{y, z, x \vdash x}}{y \otimes z, x \vdash x} \qquad \dfrac{x \vdash x}{y \otimes z, x \vdash x}$$

are equivalent, but the corresponding nets are different. This motivates the following notion of equivalence:

Definition 8. Two essential nets are said to be ***equivalent*** if they are identical when restricted to the parts that are reachable from the root (the unique positive conclusion).

Good strategies respect this notion of equivalence. For the purpose of proving a strong definability result, it is convenient to reason in terms of appropriate representatives of equivalence classes of essential nets. We call these representatives ***canonical*** essential nets, and they satisfy an additional condition: If a node is *not* reachable from the root, it is a weakening node. (We know that the converse holds, because weakening nodes have no premises and are of negative polarity, so no edge can lead to them.)

The new condition has a number of consequences:

1. a weakening node cannot be the negative premise of a $\otimes^-$-node, because it would render that $\otimes^-$-node unreachable
2. at most one premise of a $\parr^-$-node may be a weakening node, for otherwise the $\parr^-$-node would be unreachable
3. the premise of a $\forall^-$-node cannot be introduced by weakening
4. all axiom links are reachable.

Canonical essential nets arise "naturally" from sequent calculus derivations in which the use of weakening is delayed for "as long as possible". A derivation may be normalized to a canonical form by first deleting each unreachable connected component and then adding the component back as a *single* weakening node. A nice consequence is that all $\forall^-$-nodes are reachable (from the root), which promises an exact correspondence with strategies.

6 Proof of full and faithful completeness

Our final task is to prove a strong definability theorem:

Theorem 4 (Strong Definability). *Correct canonical IMAL2 essential nets and shortsighted regular strategies are in 1-1 correspondence.* □

A strategy σ is said to be ***shortsighted*** just in case for every even-length $s \in \sigma$, if sm is a position then $sm \in \sigma$ if and only if the O-move m is *enabled* by the last move of s. (See [12] for a definition of the enabling ordering.)

Lemma 3. *Fix an IMAL2 sequent* $\vdash \Gamma$. *(There is no harm in assuming that* Γ *is a formula.) Every shortsighted regular strategy* $\sigma = \sigma_{f,F}$ *for the associated game determines a canonical essential net* $\mathcal{E}_\sigma$ *for* $\vdash \Gamma$. *Further:*

(i) f *and* F *respectively define the axiom links and the eigentypes of* $\mathcal{E}_\sigma$.
(ii) There is a one-one correspondence between positions in σ *and paths (starting from the root) in* $\mathcal{E}_\sigma$ *which end in a leaf.* □

The Lemma can be proved by first establishing a lemma similar to [12, Lemma 19], which we omit. Instead we set out in detail the correspondence in (i) of the Lemma:

- For each $(u, v) \in f$, u is the occurrence (in $\mathcal{E}_\sigma$) of the positive end of a unique axiom link, and v the occurrence of the negative end.
- Each t in the domain of F is the occurrence of a unique $\forall^-$-node. Suppose its eigentype has free variables $Y_1, \cdots, Y_n$ (say), then $F(t)$ is expressed in terms of O-locations which are the occurrences of the $\forall^+$-nodes whose eigenvariables are the Y_is.

and illustrate it by the following example.

Example 9. Consider the IMAL2 sequent $\vdash\ \forall Z.Z \multimap \forall X.(\forall Y.Y \multimap X)$ and a regular strategy $\sigma_{f,F}$ generated by

$$f = \{(RR, LR), (LLR, RLR), (RLL, LLLl)\}$$
$$F = \{(L, ([\forall b.b] \otimes R \multimap R) \multimap R), (RL, R \multimap R), (LLLl, R)\}$$

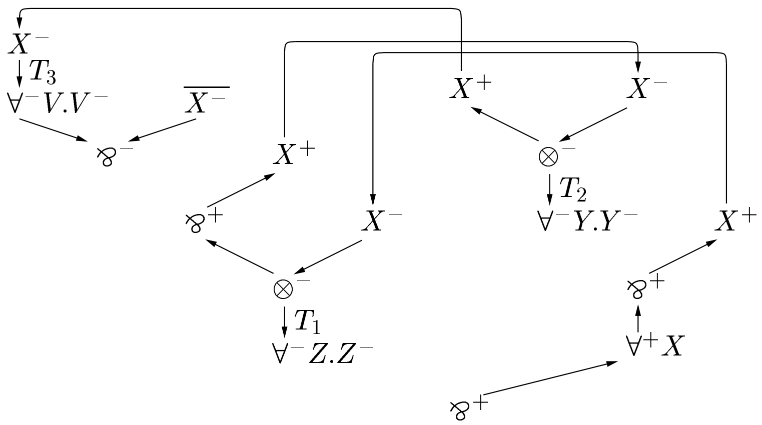
The strategy has the following maximal position

$$(R^a R)\,(L^{([\forall b.b]\otimes a \multimap a)\multimap a} R)\,(LLR)\,(RL^{a \multimap a} R)\,(RLL)\,(LLLl^a)$$

The reader may wish to check that f and F respectively specify the axiom links and the eigentypes of the essential net which is determined by $\sigma_{f,F}$ as shown in Figure 3.

Following on from Lemma 3, we can show that:

Proposition 1. $\mathcal{E}_\sigma$ *is correct.*



where $T_1 = ((\forall^- V.V^- \invamp^- X^-) \invamp^+ X^+) \otimes^- X^-$, $T_2 = X^+ \otimes^- X^-$, $T_3 = X^-$.

Fig. 3. A correct canonical essential net for $\vdash \forall^- Z.Z^- \invamp^+ (\forall^+ X.(\forall^- Y.Y^- \invamp^+ X^+))$.

Proof. For an illustration we prove that $\mathcal{E}_\sigma$ satisfies condition (3) of Definition 7; the other two conditions can be shown to hold by arguments similar to those in the proof of [12, Theorem 22]. Suppose there is a node $\forall^- Z.A$ – call it p – whose eigentype has a free variable Y (say). Since $\mathcal{E}_\sigma$ is canonical, p is reachable from the root. For a contradiction, suppose there is a path from the root to p, whose occurrence (a P-location) in $\mathcal{E}_\sigma$ is t (say), without passing through a $\forall^+ Y.G$-node, whose occurrence (an O-location) is t' (say). By Lemma 3(ii), there is a position (pick the shortest) in σ, $\underline{sm_O m_P}$ say, that ends in a move m_P of the form $u^{\overline{B}} v$ where the location of u is t; further no move from the position has a location that has t' as a prefix. By definition of symbolic strategy and by Lemma 3(i), we have $F(t)$ is expressed in terms of $\lceil t' \rceil$ (among possibly other O-locations), which should already be defined in the position. But this contradicts our assumption. $\square$

This proves Theorem 4 and hence Theorem 2.

Further directions For us, this work is a necessary step towards the construction of a fully complete model for Second-order Intuitionistic Multiplicative *Light* Affine Logic (IMLAL2) ([7, 4]), and so, a game characterization of the PTIME functions. We believe that an appropriate synthesis of this work with our recent discreet strategies model for (quantifier-free) IMLAL (see [13]) should produce the result. It would also be interesting to investigate full completeness for IMLL2, and for IMAL2 with unit.

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A Proof of Lemma 2

We first prove (i). Suppose infinite chattering occurred when composing σ_{f_1,F_1} with σ_{f_2,F_2} and b_0 opens the infinite sequence $b_0, b_1, \cdots$ in the middle component. For each $t_i \in dom\, f_i$ define $\delta_{t_i} = |f(t_i)| - |t_i|$ i.e. if b_{x+1} was made using f_i for t_i we will have $|\lfloor b_{x+1} \rfloor| - |\lfloor b_x \rfloor| = \delta_{t_i}$ in Case (i) and $|\lfloor b_{x+1} \rfloor| - |\lfloor b_x \rfloor| = -\delta_{t_i}$ in Case (ii). In that way, we can define a sequence of integers $\epsilon_1\delta_1, \epsilon_2\delta_2, \cdots, \epsilon_i\delta_i, \epsilon_{i+1}\delta_{i+1}, \cdots$, where $\epsilon_{i+1}\delta_{i+1}$ indicates the difference in length between $\lfloor b_{i+1} \rfloor$ and $\lfloor b_i \rfloor$. Hence, we get:

$$|\lfloor b_x \rfloor| = |\lfloor b_0 \rfloor| + \epsilon_1\delta_1 + \cdots + \epsilon_x\delta_x.$$

By Lemma 1:

$$|\epsilon_1\delta_1 + \cdots + \epsilon_x\delta_x| \leq \sum_{t_1 \in dom\, f_1} |\delta_{t_1}| + \sum_{t_2 \in dom\, f_2} |\delta_{t_2}|,$$

because for each t_i and $j < k$, if $\epsilon_j\delta_j \equiv \epsilon\delta_{t_i}$ and $\epsilon_l\delta_l \equiv \epsilon\delta_{t_i}$, there exists k such that $j < k < l$ and $\epsilon_k\delta_k \equiv (-\epsilon)\delta_{t_i}$. Now observe that $\sum_{t_1 \in dom\, f_1} |\delta_{t_1}| + \sum_{t_2 \in dom\, f_2} |\delta_{t_2}|$ is a constant (depends only on σ_{f_1,F_1} and σ_{f_2,F_2} and the fact that they are finite), so $|\lfloor b_x \rfloor| \leq C$ and $b_0, b_1, b_2, \cdots$ cannot be infinite.

Now for (ii). Suppose we have two regular strategies $\sigma_1 : A \multimap B$ and $\sigma_2 : B \multimap C$ given by $\langle f_1, F_1 \rangle$ and $\langle f_2, F_2 \rangle$ respectively. The Lemma is equivalent to saying: when O plays symbolically in A and C, all interaction sequences are finite.

For a contradiction, suppose there is such an infinite interaction sequence S.

Observe that for any element e of $dom(F_i)$ and in any position of σ_i, the information contained in $F_i(e)$ is actually used only once (afterwards once an appropriate evolution argument has been exported, there will be no need to access $F_i(e)$ again). This is also reflected by our notation – once a slot has been defined, its content never changes.

Because F_i are finite and S is infinite, from some point (say the n-th move of S) moves are made without consulting F_i for $i = 1, 2$.

Because there can be no infinite chattering (Lemma 2(i)), there must be moves in A or C in S after the n-th move. Consider the first such in A or C (say the m-th move, $m \geq n$). Let G_1 and G_2 be the evolved games by then (from $A \multimap B$ and $B \multimap C$ respectively).

Important Claim: Players will not import any games other than singleton games after the m-th move.

Consequently G_1 and G_2 cannot evolve into games with depth greater than the one they already have. (By *depth* we mean the depth of the syntactic tree of the game in which "[ ]" are ignored; e.g. the depth of $a \multimap b$ is 2, and the depth of $a \multimap [\forall X.(X \multimap b)]$ is 3.)

We justify the Claim as follows. As neither F_1 nor F_2 is used after the m-th move, all that the players can do in the middle component and elsewhere is to copy the games (as in m_v in Definition 5). But since O plays symbolically in A and C, from the m-th move onwards, all games being imported are ground tokens.

Hence, the evolution preserves the depth of G_1 and G_2.

The above shows that the evolution in S must stop at some point, because the depth of the evolved game is bounded and so it is finite. Thus the interaction sequence S cannot be infinite, because this would require the game to be infinite. $\square$

B Polarized IMLL2 sequents

An IMAL2 formula A can be transformed to a *positive polarized form* $\ulcorner A \urcorner$ and a *negative polarized form* $\llcorner A \lrcorner$. A polarized form is constructed from *polarized* atoms and variables (namely $a^+, a^-, b^+, b^-, \cdots, X^+, X^-, \cdots$) and polarized connectives (namely $\otimes^+, \otimes^-, \invamp^+, \invamp^-, \forall^+$ and $\forall^-$). They are defined by mutual recursion by the following rules:

$$
\begin{array}{rcl}
\ulcorner a \urcorner = a^+ & & \llcorner a \lrcorner = a^- \\
\ulcorner X \urcorner = X^+ & & \llcorner X \lrcorner = X^- \\
\ulcorner A \multimap B \urcorner = \llcorner A \lrcorner \invamp^+ \ulcorner B \urcorner & & \llcorner A \multimap B \lrcorner = \ulcorner A \urcorner \otimes^- \llcorner B \lrcorner \\
\ulcorner A \otimes B \urcorner = \ulcorner A \urcorner \otimes^+ \ulcorner B \urcorner & & \llcorner A \otimes B \lrcorner = \llcorner A \lrcorner \invamp^- \llcorner B \lrcorner \\
\ulcorner \forall X.A \urcorner = \forall^+ X.\ulcorner A \urcorner & & \llcorner \forall X.A \lrcorner = \forall^- X.\llcorner A \lrcorner
\end{array}
$$

Given an IMAL2 sequent $C_1, \cdots, C_n \vdash A$, we define the corresponding ***polarized*** sequent to be the following one-sided sequent:

$$\Vdash \llcorner C_1 \lrcorner, \cdots, \llcorner C_n \lrcorner, \ulcorner A \urcorner.$$

C Proof of Theorem 3

We show the order in which nodes can be eliminated by using the correctness criterion for IMAL essential nets:

> IMAL essential nets satisfying the conditions in Definition 7 (only (i) and (ii) therein are applicable) are exactly those that are sequentializable.

which can be proved using the techniques of in [11]. (However Lamarche's treatment of weakening therein is quite different: In his system a sequent may only be weakened by a ?-formula; also in his nets each ?-node has a link to the root.)

Take an IMAL2 essential net $\mathcal{E}$. Immediate eliminations can be made if there is a conclusion that has one of the following shapes:

1. $A \invamp^+ B$

2. $\forall^+ X.A$ (after the elimination all occurrences of X throughout the net must be replaced by a fresh atom)
3. $A \otimes^+ B$ is splitting
4. $A \parr^- B$
5. $\forall X^-.A$ and its eigentype contains no free variable
6. $\overline{A^-}$

Let us assume that there is no conclusion in $\mathcal{E}$ that has any of the above shapes and the net is not an axiom link. The positive conclusion is either a non-splitting $A \otimes^+ B$ or an atom a^+. The negative conclusions have shapes a^-, $A \otimes^- B$ or $\forall Z^-.A$ whose eigentype T contains a free variable Y. We aim to show that there is at least one conclusion of the form $A \otimes^- B$ which is splitting.

Consider the IMAL essential net $\mathsf{erase}(\mathcal{E})$ which is obtained by erasing the second-order components (i.e. all the $\forall^+$-nodes, $\forall^-$-nodes and the associated eigentypes). By the sequentialization theorem for IMAL nets, $\mathsf{erase}(\mathcal{E})$ has a splitting $A \otimes^- B$ node, but we have to make sure that it remains splitting in the original net $\mathcal{E}$. It suffices to eliminate the possibility of such a node being in the scope of a quantifier.

For each conclusion of the shape $\forall^- Z.A$, and for each free variable Y of its eigentype, we append a positive copy of a fresh atom τ using $\otimes^-$ as in $\tau^+ \otimes^- \forall Z^-.A$, and append a negative copy τ^- using $\parr^+$ to the $\forall^+$-node whose eigenvariable is Y. We write the transformation as $\mathcal{F} \mapsto \mathcal{F}^+$. For instance, if the eigentype of the $\forall^-$-node is $X^- \parr^- Y^-$ in the following (call it $\mathcal{F}$)

$$\begin{aligned} \vdash \quad & a^-, a^+ \otimes^- (b^- \parr^- c^-), \forall^- Z.(Z^- \parr^+ Z^+) \otimes^- Z^-, \\ & \forall^+ X.((((\forall^+ Y.Y^+) \otimes^- X^-) \parr^+ X^+) \otimes^+ b) \otimes^+ c \end{aligned}$$

then $\mathcal{F}^+$ is

$$\begin{aligned} \vdash \quad & a^-, a^+ \otimes^- (b^- \parr^- c^-), \tau^+ \otimes^- \forall^- Z.(Z^- \parr^+ Z^+) \otimes^- Z^-, \\ & \forall^+ X.((((\tau^- \parr^+ (\forall^+ Y.Y^+)) \otimes^- X^-) \parr^+ X^+) \otimes^+ b) \otimes^+ c \end{aligned}$$

We check that the transformation preserves the correctness criterion. It is easy to see that acyclicity is preserved, but what about condition (2) of Definition 7 for the new $\parr^+$-node (call it p)? Suppose x is a source of p's sink which is also a source of p itself. By construction, the only way to reach p's sink is via $\forall^- Z.A$ and then via the fresh $\otimes^-$-node and finally along the fresh axiom link from τ^+ to τ^-. Thus x is a source of $\forall^- Z.A$ which is also a source of $\forall^+ Y.G$. By applying the correctness criterion (3) to $\mathcal{F}$, we see that condition (2) is satisfied by the new $\parr^+$-node p of $\mathcal{F}^+$.

Now note that the fresh $\otimes^-$-nodes cannot be splitting, and because the positive conclusion is not a $\forall^+$-node, none of the cases listed above applies to the transformed sequent.

By applying the sequentialization theorem to the IMAL net $\mathsf{erase}(\mathcal{E}^+)$, obtained by erasing the second-order components from the transformed IMAL2 net, we obtain a splitting $\otimes^-$-node. This node cannot be one of the fresh $\otimes^-$-nodes, nor can it be inside a quantifier in the original net, so it must also be splitting for the original IMAL2 net $\mathcal{E}$. $\square$