DA-STATS

Topic 04: Basic Probability and Bayes ' Theorem

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Outline

- What is Probability?
- Basic Concepts
- Sets and Probability
- Axioms of Probability
- Conditional Probability
- Bayes’ Theorem

Books

Probability, Random Variables, Statistics, and Random Processes

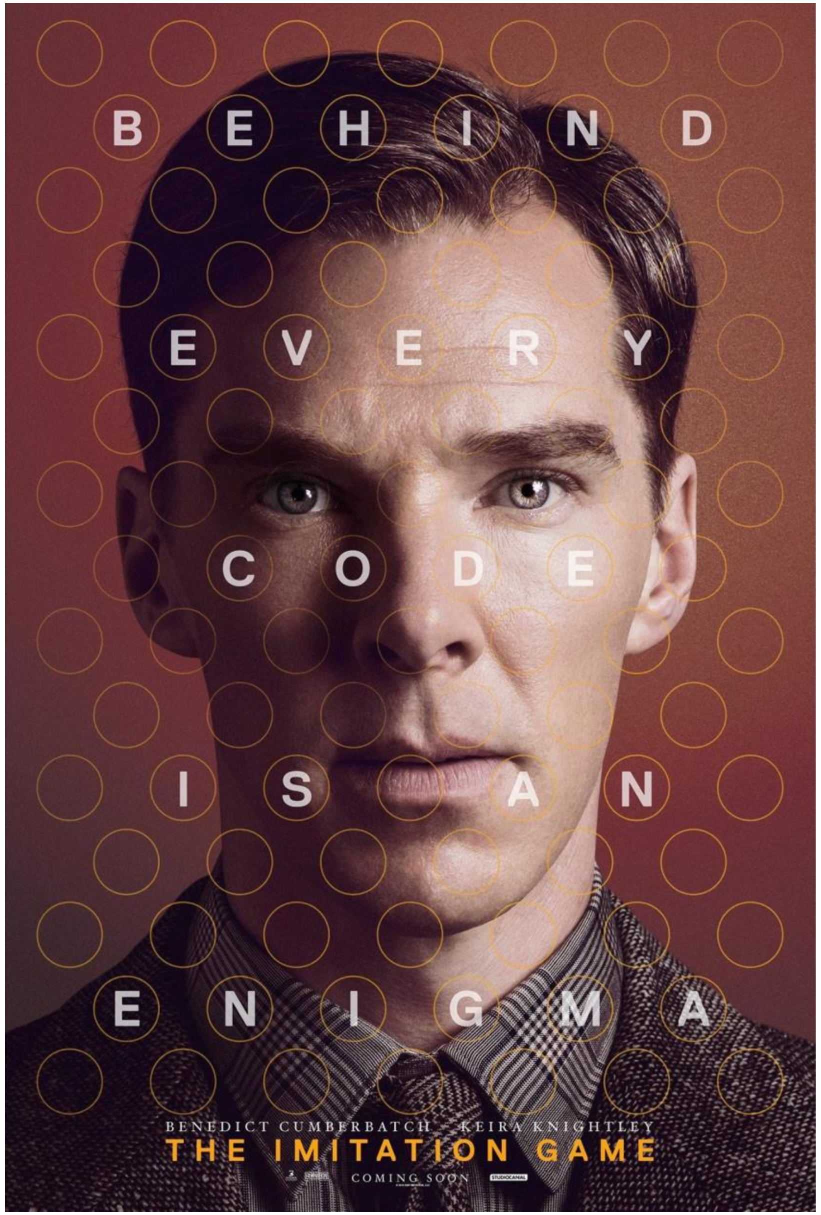
Fundamentals & Applications

Ali Grami

Probability

- Probability is the measure of
  - ➞ chance
  - ➞ randomness
  - ➞ likelihood of an event occurring
- Applications in
  - ➞ Science
  - ➞ Engineering
  - ➞ Business
  - ➞ Medicine

“Every day we performed our bloodsoaked calculus. Every day we decided who lived and who died. And every day we guided the Allied armies to victory without anyone knowing.”



Games of Chance

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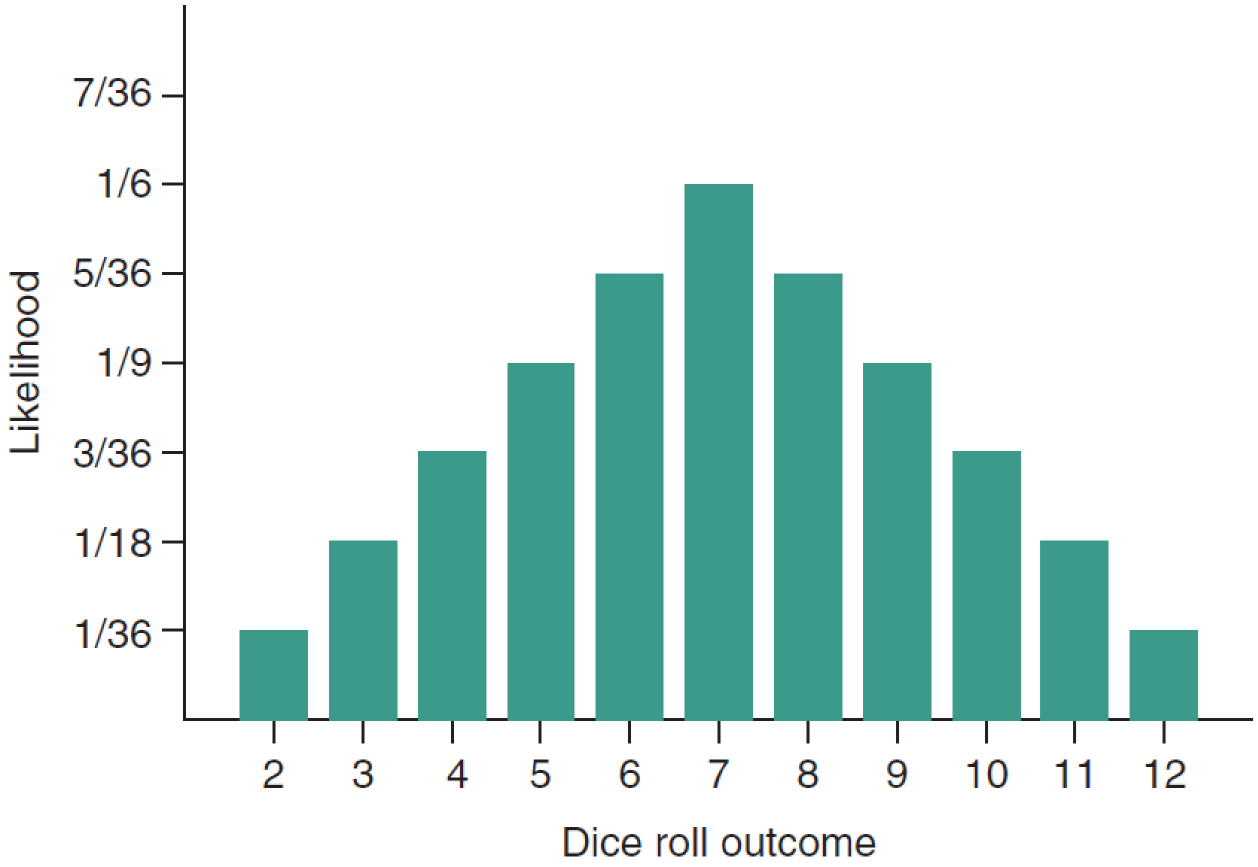
- Roll of dice or location of a card in a deck
  - → regulated and law-like if preconditions and processes were available
  - → precise information could not be fully known
- Key
  - → willfully ignorant of the myriad of small effects
  - → focus on more general truths e.g., what tends to happen
  - → construct a representation of set of outcomes
  - → long-term regularities

Probability Distribution



A probability distribution representing a single roll of a pair of six-sided dice

Basic Concepts

- An **experiment** is a measurement procedure or observation process.
- The **outcome** is the end result of an experiment.
- An **event** is a single outcome or collection of outcomes.

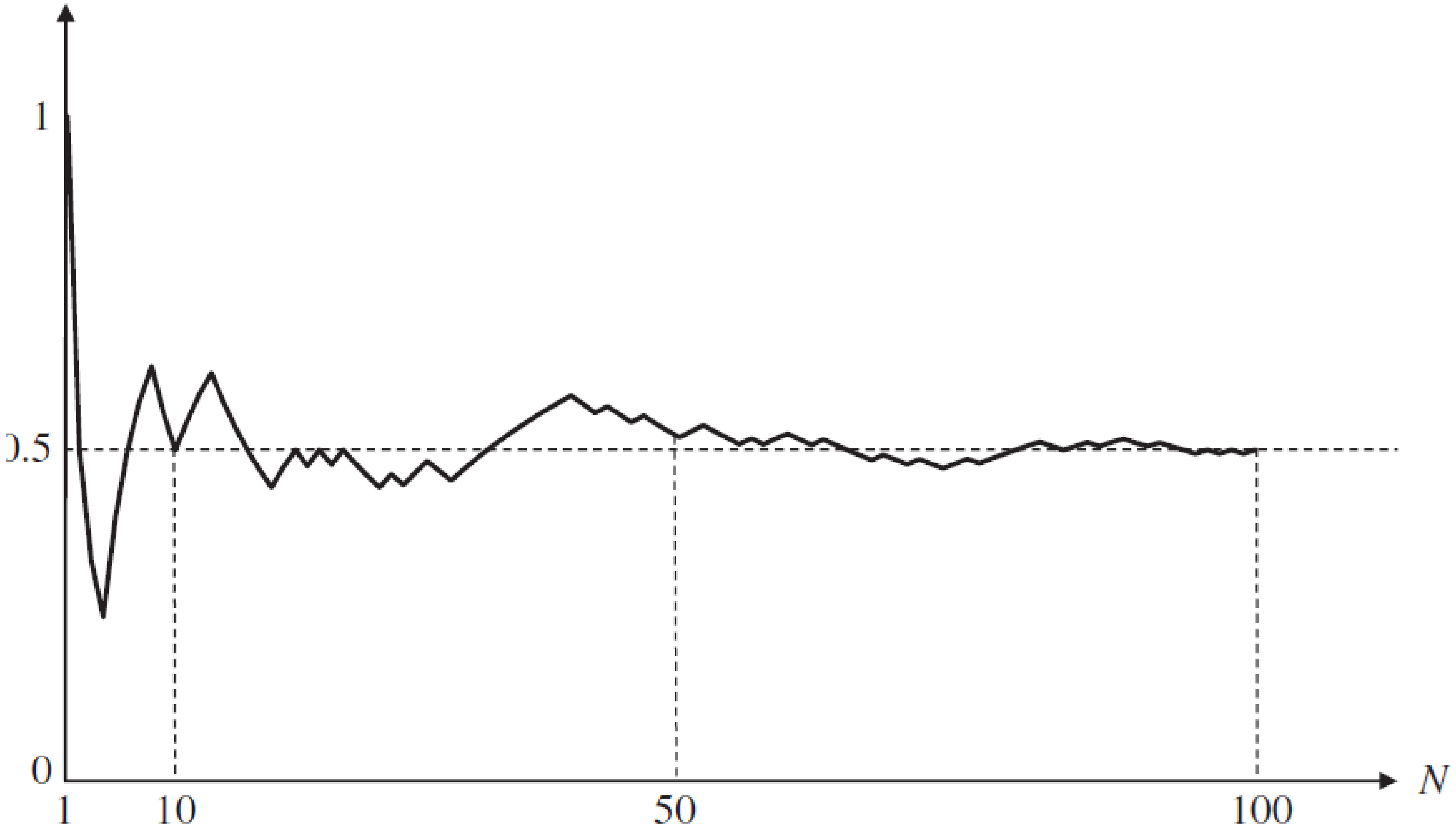
Basic Concepts

- **Deterministic experiment** – if the outcome of an experiment is certain.
- **Random experiment –** the outcome may unpredictably vary.
- **Trial** – each repetition of the random experiment.
- **Independent Trial** – the outcome of one trial has no bearing on the other

Probability Model

- Simplified approximation to an actual random experiment.
- Averages obtained in long sequences of independent trials of random experiments almost always give rise to the same value.

Relative frequency in a tossing of a fair coin



Probability of Favourable Event

$$P = \frac{number\ of\ favourable\ events}{total\ number\ of\ events}$$

The Basic Principle of Counting

- Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together how many possible outcomes are expected of the two experiments?

$$\begin{array}{cccc} (1,1), & (1,2), & \ldots, & (1,n) \\ (2,1), & (2,2), & \ldots, & (2,n) \\ \vdots & & & \\ (m,1), & (m,2), & \ldots, & (m,n) \end{array}$$

$$mn$$

The Basic Principle of Counting

- A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

$$3 \times 4 \times 5 \times 2 = 120$$

Permutations

- How many different ordered arrangements of 3 objects are possible?

$$3 \times 2 \times 1 = 6$$

- How many different ordered arrangements of n objects are possible?

$$n(n-1)(n-2)\ldots(3)(2)(1) = n!$$

$n!$ different permutations of the n objects.

Permutations

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- How many different batting orders are possible for a baseball team consisting of 9 players?

$$9! = 362{,}880$$

Permutations

- How many different letter arrangements can be formed from the letters PEPPER, when P’s and E’s are distinguished from one another?

$$6! = 720$$

What if we do not distinguish P’s and E’s from one another?

$$\text{Possible arrangements of P's and E's} = 3!\,2!$$

$$\text{Possible letter arrangements of PEPPER} = \frac{6!}{3!2!} = 60$$

Combinations

- How many different groups of 3 could be selected from the 5 items A, B, C, D, and E?
  - ➞5 ways to select first item
  - ➞4 ways to select next item
  - ➞3 ways to select final item
  - ➞Therefore, $5 \times 4 \times 3$, if the order of selection is relevant
  - ➞However, group of A, B, C will be counted 6 times
  - ➞The total number of groups that can be formed is

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

Combinations

- Determine the number of different groups of r objects that could be formed from a total of n objects.

$$\frac{n(n-1)\ldots.(n-r+1)}{r!}=\frac{n!}{(n-r)!\,r!}$$

Combinations

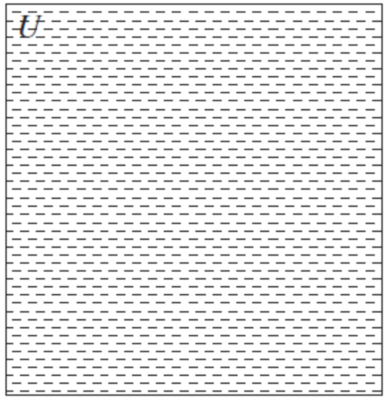
- A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

$$\binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$$

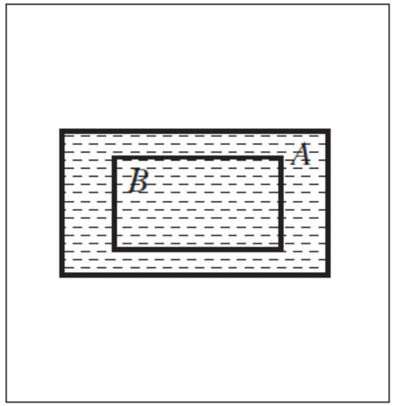
Set Theory and Its Applications to Probability

- A **set** is a collection of objects or things, which are called **elements** or **members**.
- Generally represented by symbol **{}**
- Notation:
  - ➞ $x \in A$ (x is a member of set A)
  - ➞ $x \notin A$(x is not a member of set A)
- **Cardinality** of a set is the number of distinct elements in a set A represented as $|A|$
- The **empty set** or **null set**, denoted by $\emptyset$,

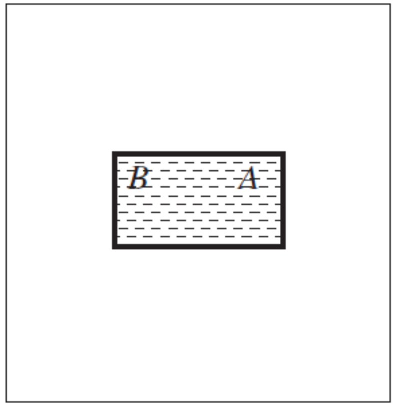
Sets and their relationships



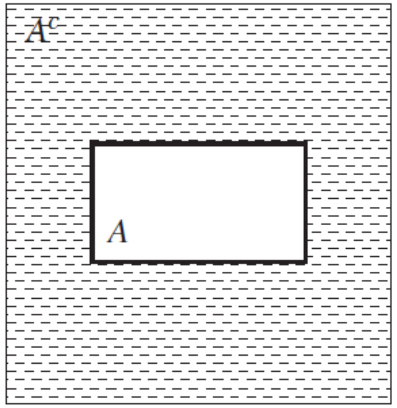
Universal set



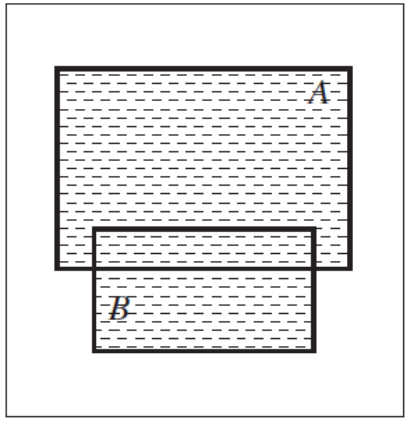
Subset ⊂



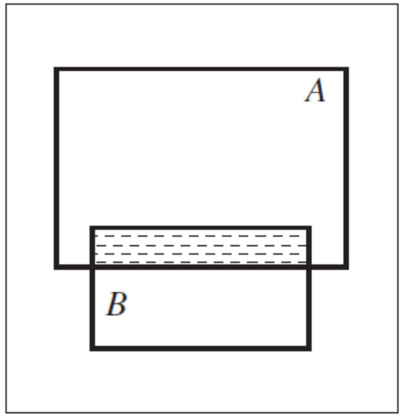
Equal sets



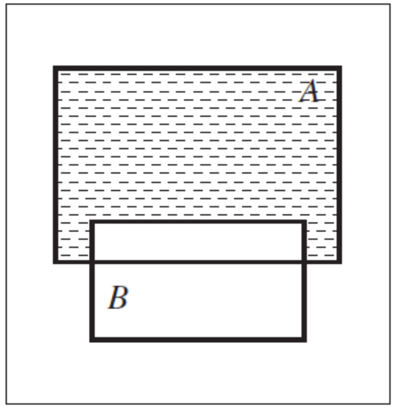
Complement of a set



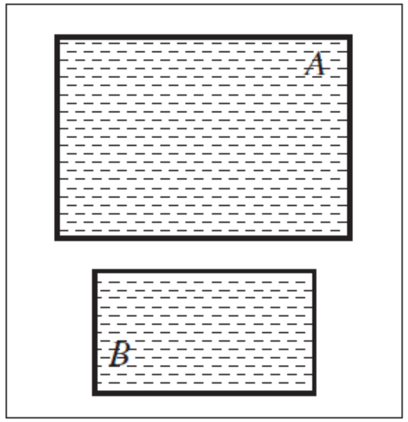
Union of Sets ∪



Intersection of Sets ∩



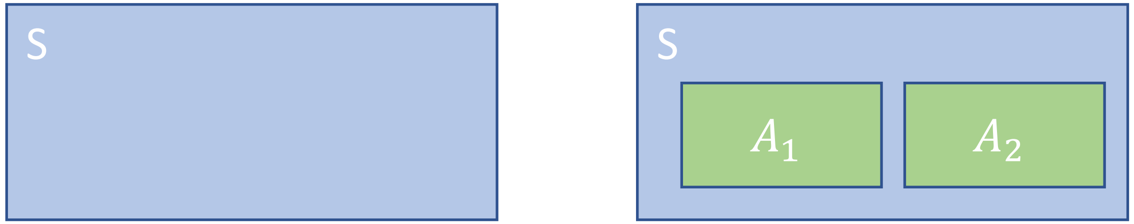
Difference of Sets



Mutually Exclusive

Axioms of Probability

- Axiom 1: For every event A, $P(A) \geq 0$
- Axiom 2: $P(S) = 1$, S is the outcome space, set of all possible outcomes.
- Axiom 3:If $A_1, A_2, \ldots$ is a countable sequence of events such that $\mathrm{A}_i \cap A_j = \emptyset$ for all $i \neq j$ where $\emptyset$ is the null event, that is they are pairwise disjoint (mutually exclusive) events, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$



Sets and Axioms of Probability

- Let $S = \{1, 2, 3, \ldots, n\}$
- Let $A \subset S, B \subset S, A \cap B = \emptyset$
- $P(A) = |A|/n$
- Axiom 1: $P(A) \geq 0$, for $A \subset S$, $|A| \geq 0 \Rightarrow P(A) \geq 0$
- Axiom 2: $P(S) = 1$, $|S| = n$, $P(S) = \frac{n}{n} = 1$
- Axiom 3: $P(A \cup B) = P(A) + P(B) = \frac{|A|}{n} + \frac{|B|}{n}$

The Addition Rule

$$P(A \cup B) = P(A) + P(B)$$

Assume that we roll one six-sided dice. What is the probability of coming up with a 2 (event A) or a 5 (event B)?

$$P(A) = \frac{number\ of\ favourable\ events}{total\ number\ of\ events} = \frac{1}{6} = 0.17$$

$$P(B) = \frac{number\ of\ favourable\ events}{total\ number\ of\ events} = \frac{1}{6} = 0.17$$

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 0.33$$

The Addition Rule

- Suppose we are about to win a game if we roll either an even number (event A) or a number greater than 4 (event B). What are our chances of winning?

$$P(A) = \frac{3}{6} = 0.5,\ P(B) = \frac{2}{6} = 0.333$$

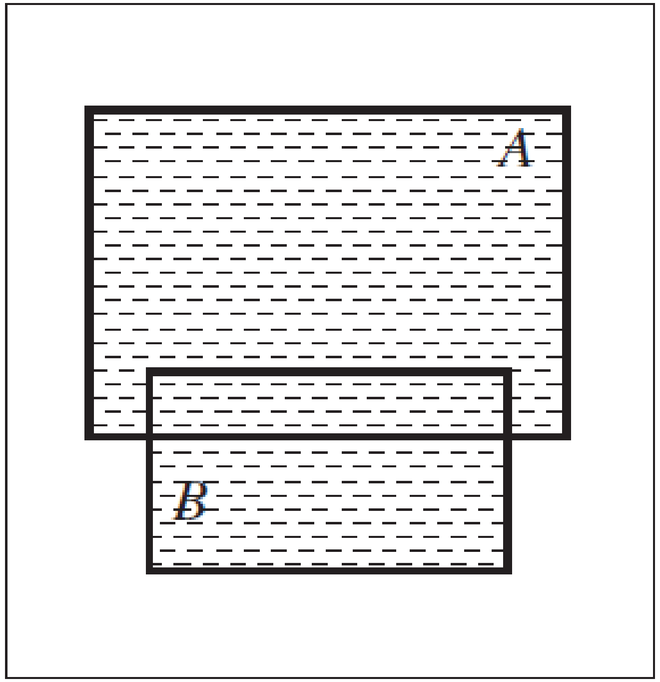
$$P(A \cup B) = ?$$

$$A = \{2,4,6\}, B = \{5,6\},\ A \cup B = \{2,4,5,6\},\ A \cap B = \{6\}$$

$$P(A \cup B) = \frac{4}{6} = 0.667$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.5 + 0.333 - 0.1667 = 0.667$$



Not Mutually Exclusive

The Multiplication Rule

- Co-occurrence of two events
- Probabilistic independence
  - ➙occurrence of one event has no effect on determining the probability of occurrence of a second event

$$P(A\ and\ B) = P(A)P(B)$$

- Probabilistic dependence
  - ➙occurrence of one event changes the determination of the probability of occurrence for the second event

$$P(A\ and\ B) = P(A|B)P(B)$$

$$P(A) = \frac{3}{6}, P(B) = \frac{2}{6}$$

$$P(A)P(B) = \frac{6}{36} = 0.1667$$

The Multiplication Rule

What is the probability of randomly selecting a 4 (event A) and an 8 (event B) on two successive draws from a deck of cards?

Since sampling with replacement is used, one card is randomly drawn from the deck and then put back into the deck, and then a second card is randomly selected.

$$P(A) = \frac{4}{52} = 0.0769, P(B) = \frac{4}{52} = 0.0769$$

$$P(A\ and\ B) = P(A)P(B) = (0.0769)(0.0769) = 0.0059$$

The Multiplication Rule

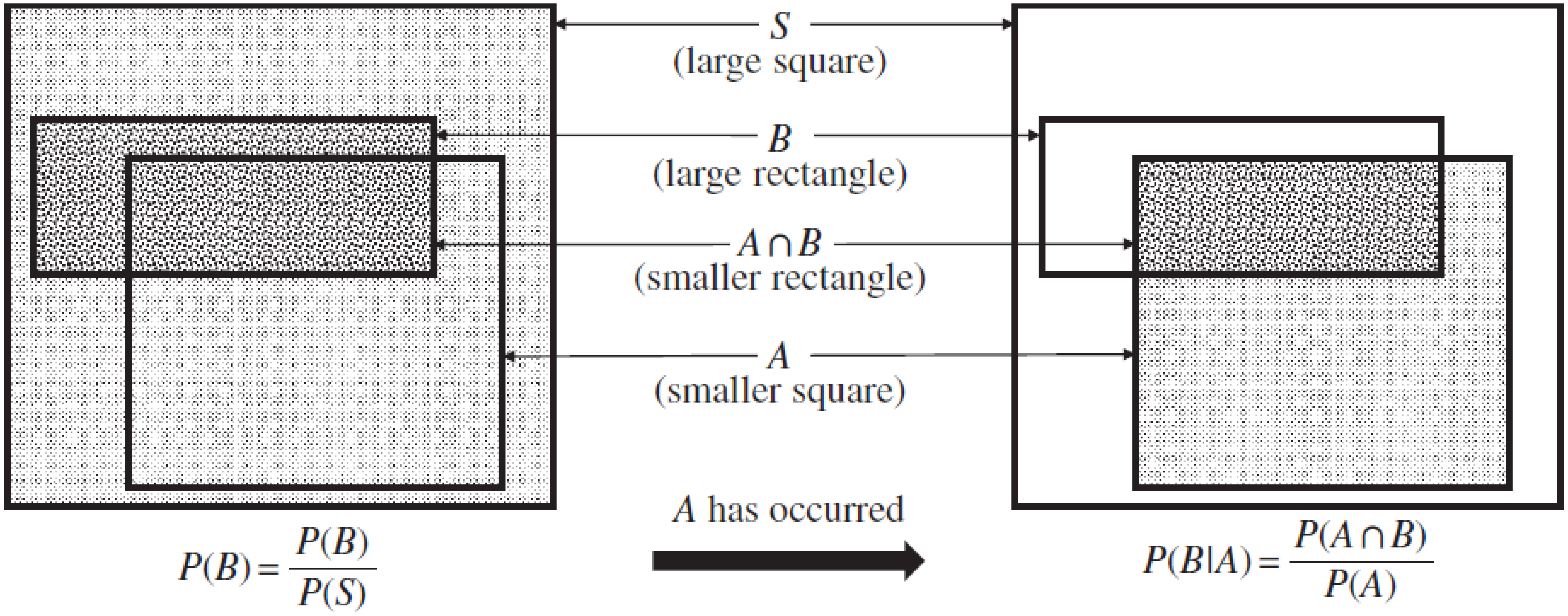
What is the probability of randomly selecting a person from the campus student population that is both a biological female (event A) and a psychology major (event B)?

Given: Suppose it is known that 10% of the student population are psychology majors, 80% of the psychology majors are biological females, and 60% of the entire student population are biological females.

- If the events are independent, then we can use
$$P(A\ and\ B) = P(A)P(B) = (0.6)(0.1) = 0.06$$
- The events are not independent
$$P(A\ and\ B) = P(A|B)P(B) = (0.8)(0.1) = 0.08$$

Conditional Probability

- $P(A|B)$ Probability of event A given B has occurred.
  - ➞Probability of A updated on basis of B has occurred.



Conditional Probability

- $P(A|B) if\ B \subset A \Rightarrow\ A \cap B = B$?

  ➞$P(A|B) = 100\%\ or\ 1$

- $P(A|B) if\ A \cap B = \emptyset$

  ➞$P(A|B) = P(A)$

$\Rightarrow if\ P(A|B) \neq P(A)$, then events A and B are dependent.

Conditional Probability

- Determine the conditional probability that a family with two children has two girls, given they have at least one girl. Assume the probability of having a girl is the same as the probability of having a boy.
- Four possibilities: {*GG*, *GB*, *BG*, and *BB*}
  - ➞each reflecting the order of birth, are equally likely

Conditional Probability

- Event $A = \{GG\}, \rightarrow P(A) = \frac{1}{4}$
- Event $C = \{GG, GB, BG\} \rightarrow P(C) = \frac{3}{4}$, we know one of the children is a girl.
- $A \cap C = \{GG\} \rightarrow P(A \cap \mathrm{C}) = \frac{1}{4}$
- $P(A|C) = \frac{P(A \cap \mathrm{C})}{P(C)} = \frac{(1/4)}{(3/4)} = 1/3$

Mutually Independent Events

■ **Question** *Given the following probabilistic situation involving balls in an urn, are events Green and X independent? Are events Yellow and Y independent?*

15	Green – X
15	Green – Y
10	Red – X
20	Red – Y
25	Yellow – X
15	Yellow – Y

$$P = \frac{number\ of\ favourable\ events}{total\ number\ of\ events}$$

$$P(A\ and\ B) = P(A)P(B)$$

$$P(A\ and\ B) = P(A|B)P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Mutually Independent Events

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Step 1. To determine if events "Green" and "X" are independent, we need to first determine the values corresponding to Formulas 6.5 and 6.6. Let us assign "Green" to be event A and "X" to be event B. These are $P(Green\ and\ X)$, $P(Green)$, $P(X)$, $P(Green|X)$.

$P(Green\ and\ X)$	$=.15$
$P(Green)$	$=.3$
$P(X)$	$=.5$
$P(Green|X)$	$=.15/.5 = .3$

15	Green – X
15	Green – Y
10	Red – X
20	Red – Y
25	Yellow – X
15	Yellow – Y

Step 2. Then we need to "run" both formulas. (Recall that we assigned "Green" to be event A and "X" to be event B.)

$$P(A\ and\ B) = P(A)P(B)$$

$$P(Green\ and\ X) = P(Green)P(X) = .3(.5) = .15$$

$$P(A\ and\ B) = P(A|B)P(B)$$

$$P(Green\ and\ X) = P(Green|X)P(X) = .3(.5) = .15$$

This means "Green" and "X" are independent.

Mutually Independent Events

WARWICK

■ **Question** *Given the following probabilistic situation involving balls in an urn, are events Green and X independent? Are events Yellow and Y independent?*

15	Green – *X*
15	Green – *Y*
10	Red – *X*
20	Red – *Y*
25	Yellow – *X*
15	Yellow – *Y*

$$P = \frac{number\ of\ favourable\ events}{total\ number\ of\ events}$$

$$P(A\ and\ B) = P(A)P(B)$$

$$P(A\ and\ B) = P(A|B)P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Mutually Independent Events

WARWICK

Step 1. To determine if events "Yellow" and "*Y*" are independent, we need to first determine the values corresponding to Formulas 6.5 and 6.6. Let us assign "Yellow" to be event *A* and "*Y*" to be event *B*. These are $P(Yellow\ and\ Y)$, $P(Yellow)$, $P(Y)$, $P(Yellow|Y)$.

$P(Yellow\ and\ Y)$	$=.15$
$P(Yellow)$	$=.4$
$P(Y)$	$=.5$
$P(Yellow\|Y)$	$=.15/.5 = .3$

15	Green – *X*
15	Green – *Y*
10	Red – *X*
20	Red – *Y*
25	Yellow – *X*
15	Yellow – *Y*

Step 2. Then we need to "run" both formulas. (Recall that we assigned "Yellow" to be event *A* and "*Y*" to be event *B*.)

$$P(A\ and\ B) = P(A)P(B)$$

$$P(Yellow\ and\ Y) = P(Yellow)P(Y) = .4(.5) = .2$$

$$P(A\ and\ B) = P(A|B)P(B)$$

$$P(Yellow\ and\ Y) = P(Yellow|\,Y)P(X) = .3(.5) = .15$$

This means "Yellow" and "Y" are dependent.

Independence and Mutual Exclusivity Are Different

- If two events are mutually exclusive, they cannot both occur in the same trial:
  - ➞The probability of their intersection is zero.
  - ➞The probability of their union is the sum of their probabilities.

Independence and Mutual Exclusivity Are Different

- If two events are independent, they can both occur in the same trial (except possibly if at least one of them has probability zero).
  - ➞ The probability of their intersection is the product of their probabilities.
  - ➞ The probability of their union is less than the sum of their probabilities, unless at least one of the events has probability zero.

Bayes' Theorem

- Is $P(A|B) = P(B|A)$?
- Conditional Probability Fallacy or Confusion of inverse

- If we know $P(A|B)$, we can calculate $P(B|A)$ given some further information.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|not\ B)P(notB)}$$

Bayes' Theorem

- If we know $P(A|B)$ and want to determine the $P(B|A)$, we will additionally need the $P(B)$, $P(not\ B)$ and $P(A|not\ B)$
- If we know $P(B)$, we can determine $P(not\ B)$
  - ➞ $P(B) + P(not\ B) = 1$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|not\ B)P(notB)}$$

Bayes’ Theorem

■ **Question** *Imagine it is true that 1% of 40-year-old women who participate in a routine screening have breast cancer. Further imagine that 80% of women with breast cancer will receive a positive reading from the mammogram screen procedure. However, 9.6% of women without breast cancer will also receive a positive reading from the mammogram screening procedure (this is sometimes referred to as a “false positive” result). Now suppose a 40-year-old woman is told that her mammogram screening is positive for breast cancer. What is the likelihood that she actually has breast cancer?*

Bayes' Theorem

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$P(A\|B)$	$=P(\textit{positive reading}\|\textit{breast cancer})$
$P(B)$	$=P(\textit{breast cancer})$
$P(A\|notB)$	$=P(\textit{positive reading}\|\textit{not breast cancer})$

It follows then that

$P(A\|B)$	$=.8$
$P(B)$	$=.01$
$P(A\|notB)$	$=$ .096

And we can deduce that

$P(notB)$	$=.99$

Bayes' Theorem

■ **Question** *Imagine it is true that 1% of 40-year-old women who participate in a routine screening have breast cancer. Further imagine that 80% of women with breast cancer will receive a positive reading from the mammogram screen procedure. However, 9.6% of women without breast cancer will also receive a positive reading from the mammogram screening procedure (this is sometimes referred to as a "false positive" result). Now suppose a 40-year-old woman is told that her mammogram screening is positive for breast cancer. What is the likelihood that she actually has breast cancer?*

Use Bayes' theorem to solve the equation

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|notB)P(notB)}$$

$$P(B|A) = \frac{.8(.01)}{.8(.01) + .096(.99)} = \mathbf{.078} \textit{ or about } \mathbf{7.8\%}$$

"Anything at all is possible. Some things are unlikely. Some things will never happen. But they always could, at any time."

— Ashly Lorenzana