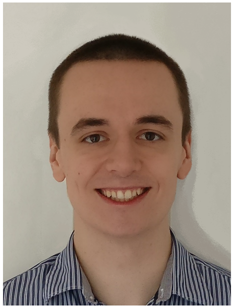
Coverability in 2-VASS with One Unary Counter is in NP



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Germany

FoSSaCS: Counters

25th April 2023

Sorbonne Université, Paris, France

Fun-Road-Trip Checklist

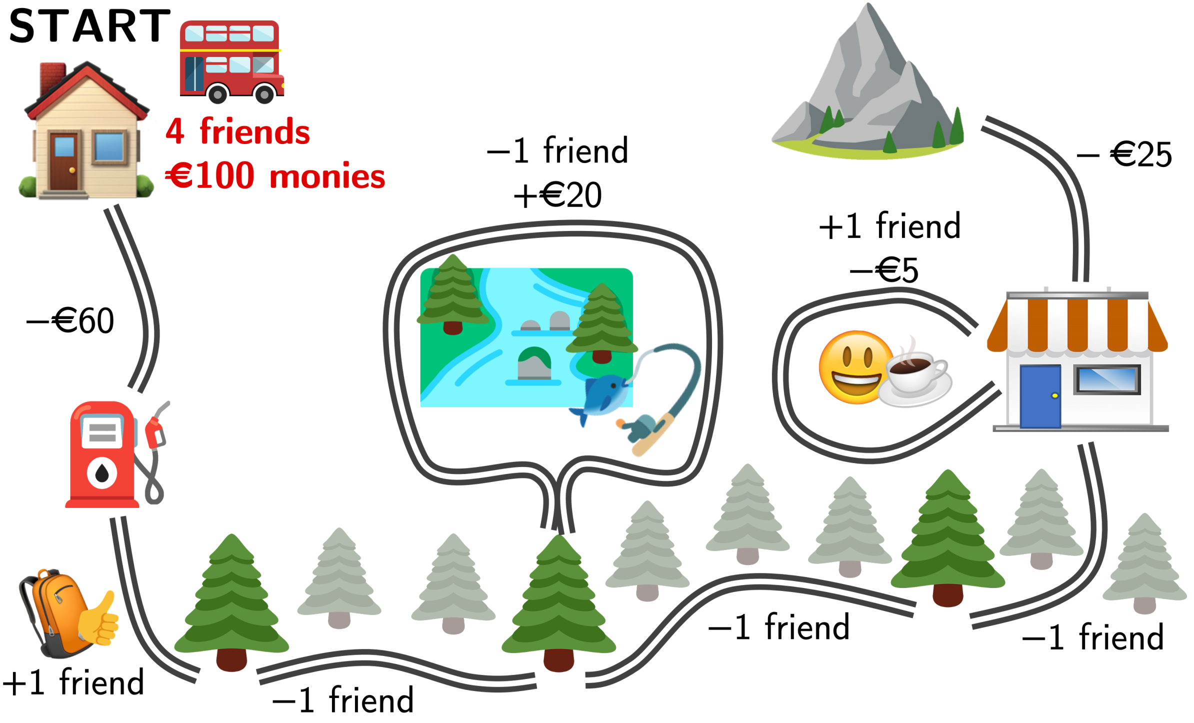
✓ always at least one friend, and

✓ never negative money!

TARGET

≥ 5 friends

≥ €10 monies



START

4 friends

€100 monies

−€60

+1 friend

−1 friend

−1 friend

+€20

−1 friend

+1 friend

−€5

−1 friend

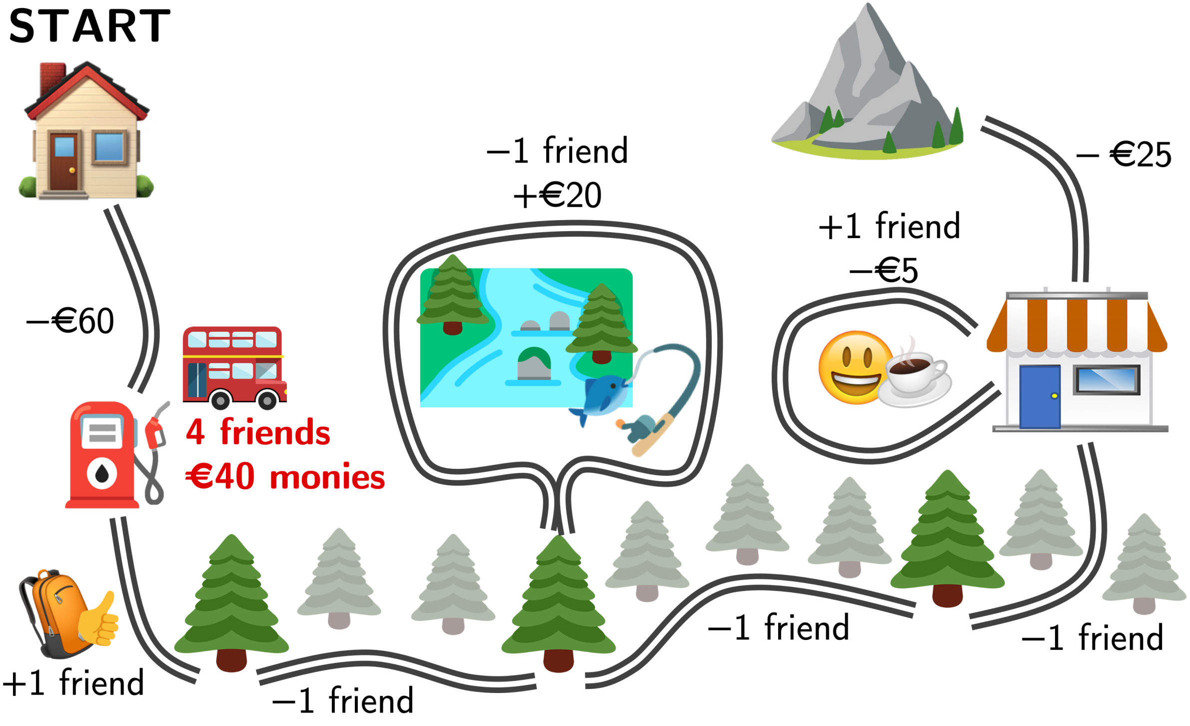
− €25

−1 friend

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



TARGET

≥ 5 friends

≥ €10 monies

START

−€60

4 friends

€40 monies

+1 friend

−1 friend

−1 friend

+€20

−1 friend

+1 friend

−€5

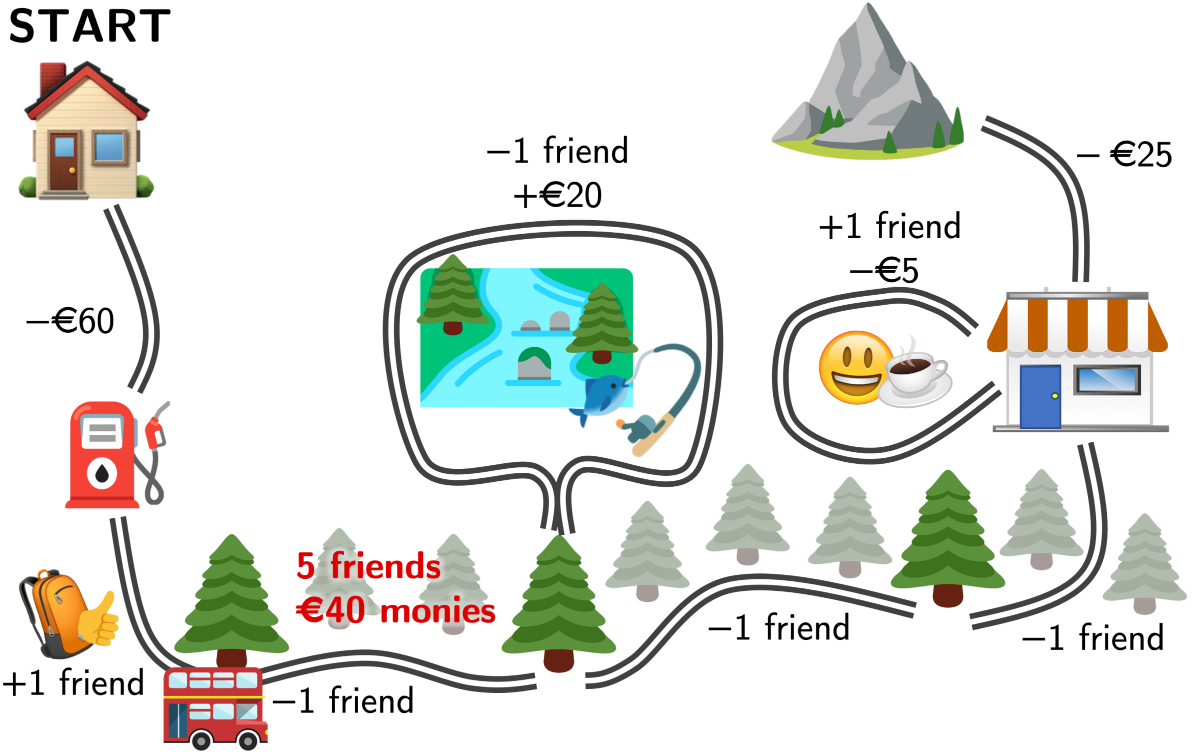
− €25

−1 friend

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



TARGET

$\geq$ 5 friends

$\geq$ €10 monies

START

−€60

+1 friend

−1 friend

5 friends

€40 monies

−1 friend

+€20

−1 friend

+1 friend

−€5

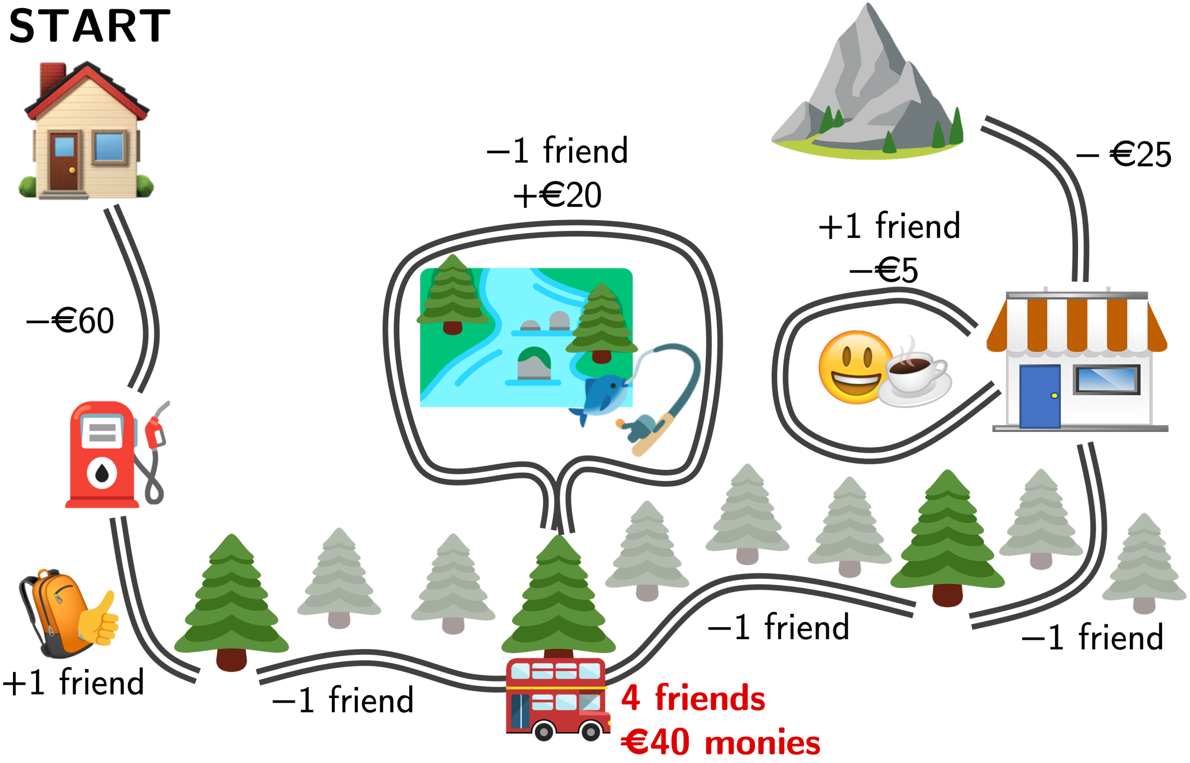
− €25

−1 friend

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



START

TARGET

≥ 5 friends

≥ €10 monies

−€60

+1 friend

−1 friend

−1 friend

+€20

−1 friend

+1 friend

−€5

− €25

−1 friend

−1 friend

4 friends

€40 monies

Fun-Road-Trip Checklist

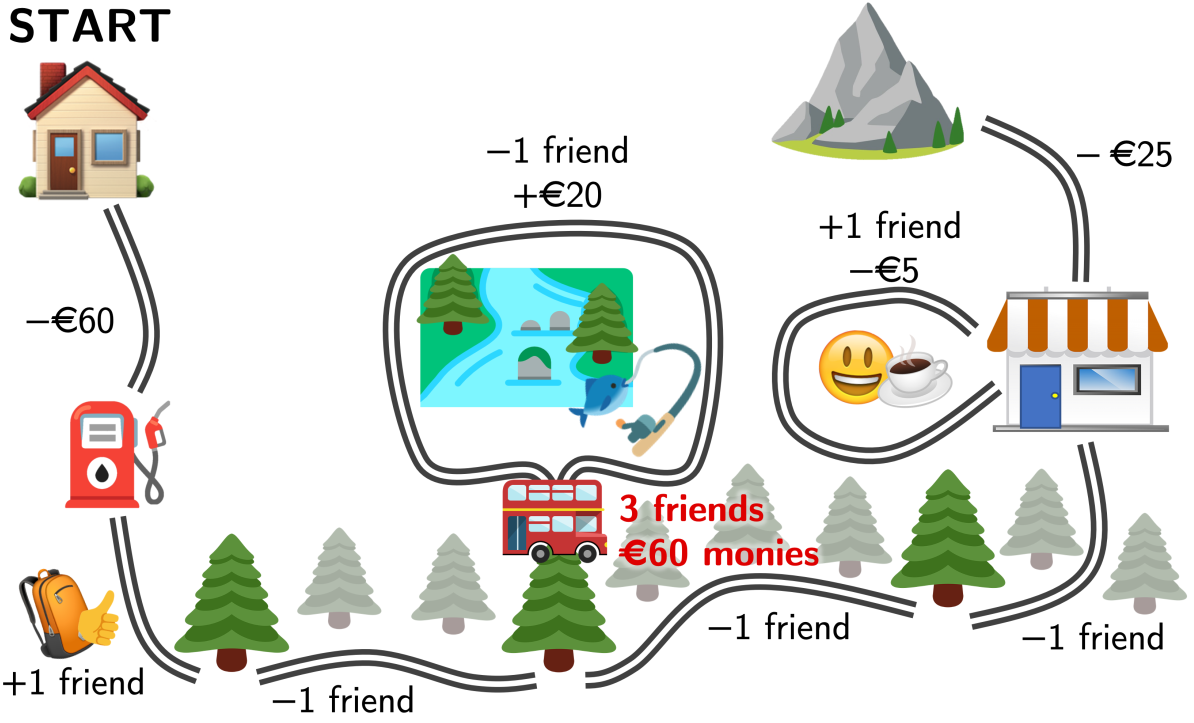
✓ always at least one friend, and

✓ never negative money!

TARGET

≥ 5 friends

≥ €10 monies



START

−€60

+1 friend

−1 friend

−1 friend

+€20

3 friends

€60 monies

−1 friend

−1 friend

+1 friend

−€5

− €25

Fun-Road-Trip Checklist

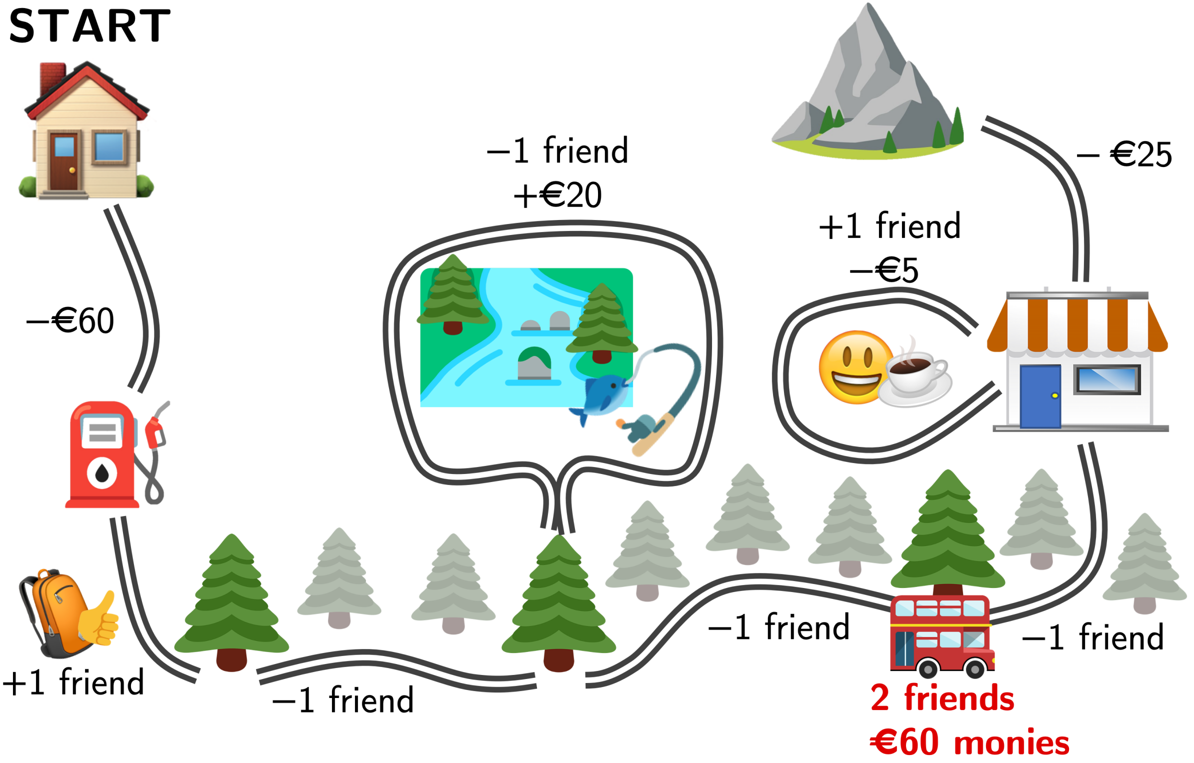
✓ always at least one friend, and

✓ never negative money!

TARGET

≥ 5 friends
≥ €10 monies

START



−€60

+1 friend

−1 friend

−1 friend
+€20

−1 friend

+1 friend
−€5

− €25

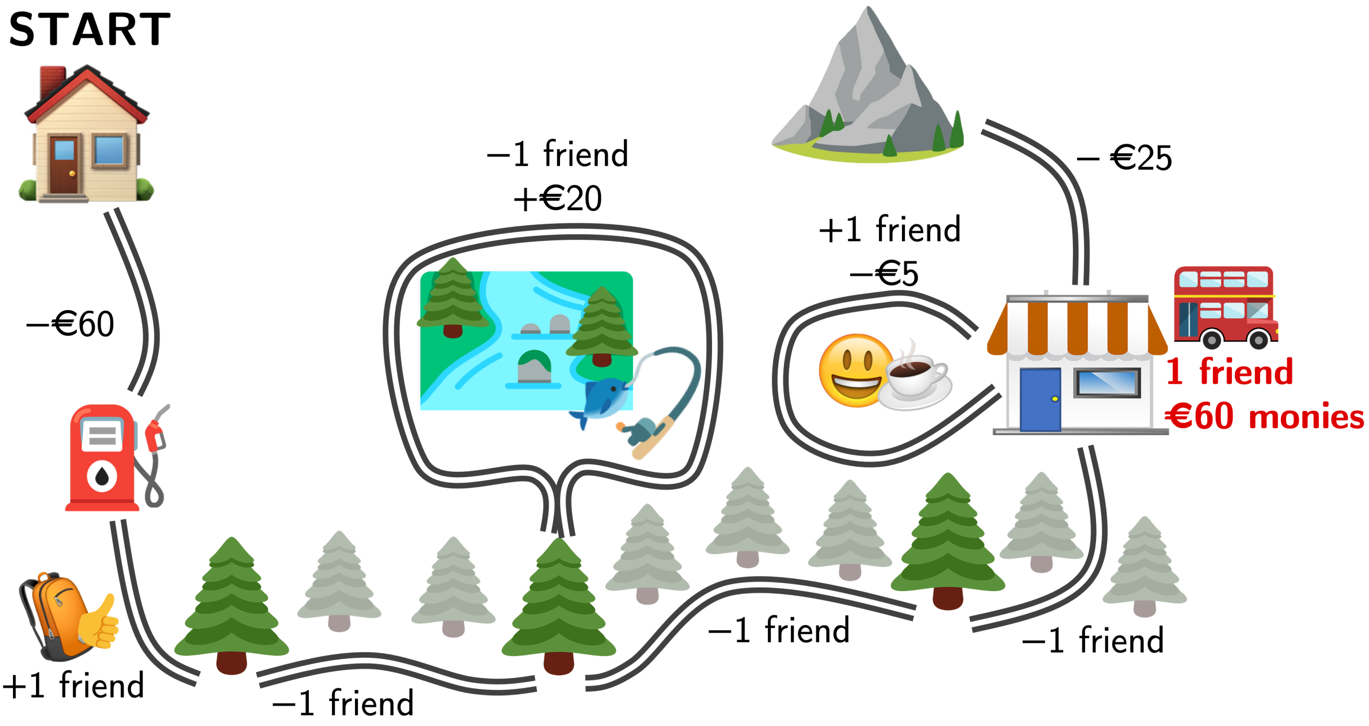
−1 friend

2 friends
€60 monies

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



TARGET

≥ 5 friends

≥ €10 monies

START

−€60

+1 friend

−1 friend

−1 friend

+€20

−1 friend

−1 friend

+1 friend

−€5

– €25

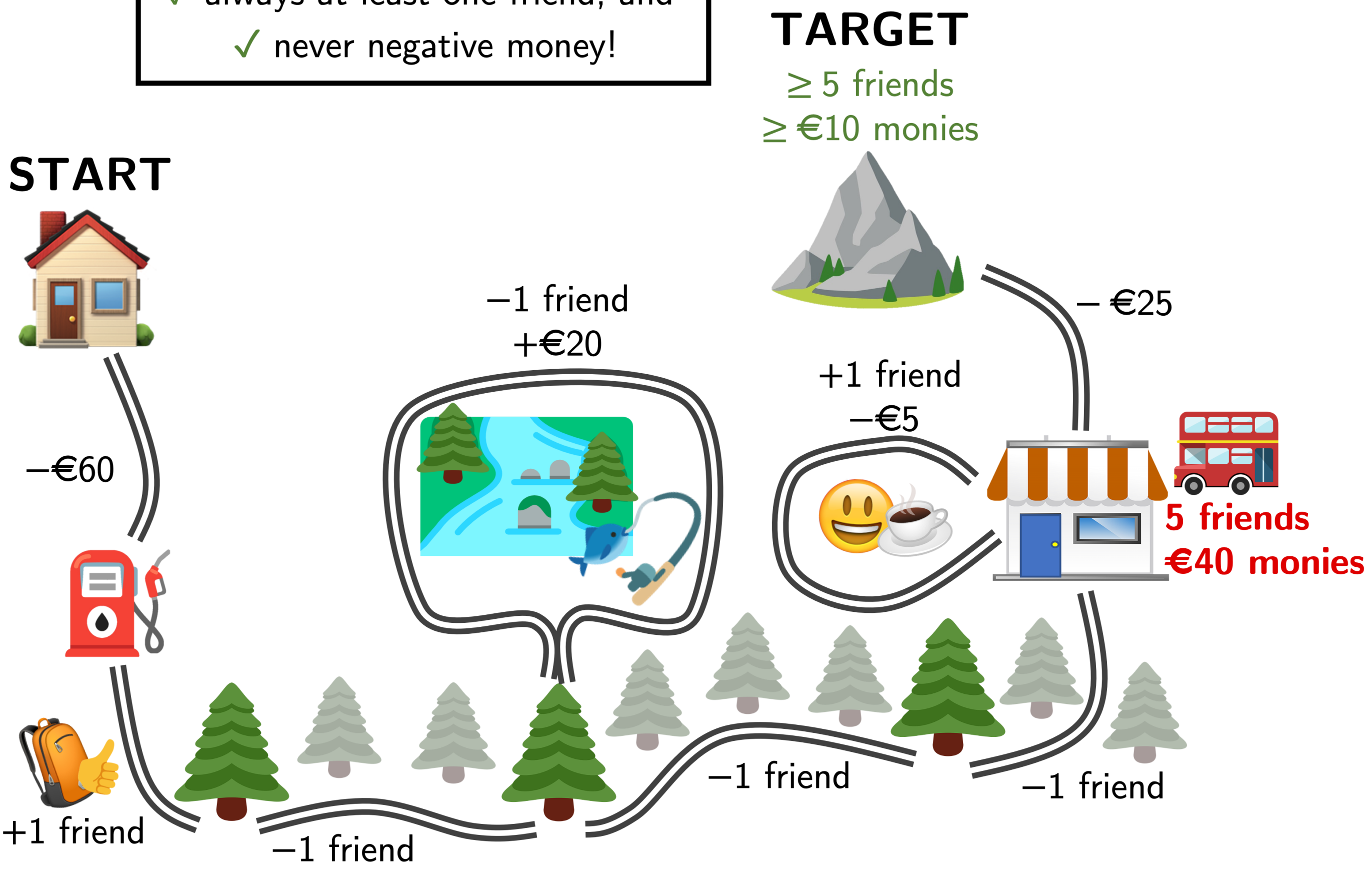
1 friend

€60 monies

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



TARGET

≥ 5 friends

≥ €10 monies

START

−€60

−1 friend

+€20

+1 friend

−€5

− €25

5 friends

€40 monies

+1 friend

−1 friend

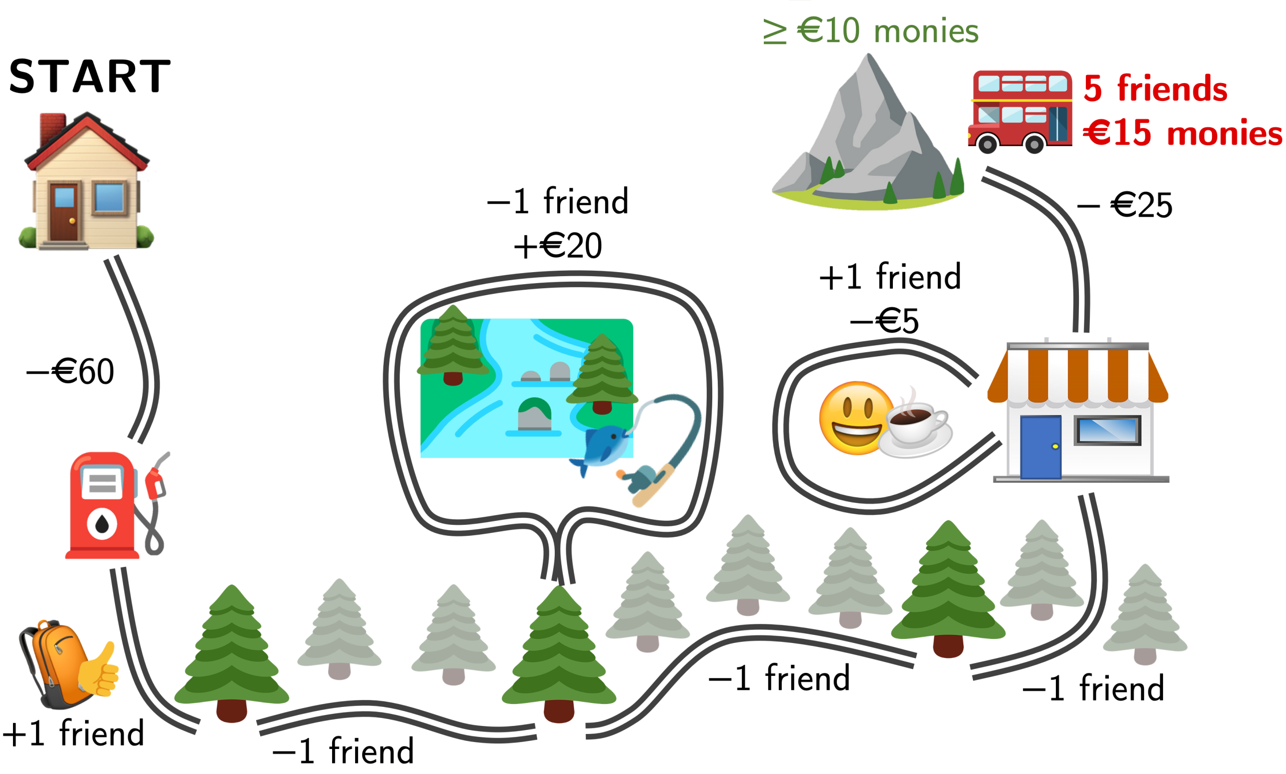
−1 friend

−1 friend

Fun-Road-Trip Checklist

✓ always at least one friend, and

✓ never negative money!



TARGET

≥ 5 friends

≥ €10 monies

5 friends

€15 monies

START

−€60

+1 friend

−1 friend

−1 friend

+€20

−1 friend

+1 friend

−€5

− €25

−1 friend

Coverability in 2-VASS with One Unary Counter

Coverability problem

Input: VASS $\boldsymbol{\mathcal{V}}$, initial configuration $\boldsymbol{p}(\vec{\boldsymbol{u}})$, and target configuration $\boldsymbol{q}(\vec{\boldsymbol{v}})$.

Question: does there exist a run from $\boldsymbol{p}(\vec{\boldsymbol{u}})$ to $\boldsymbol{q}(\vec{\boldsymbol{w}})$ where $\vec{\boldsymbol{w}} \geq \vec{\boldsymbol{v}}$?

Specific features

One component is encoded in unary and the other is encoded in binary.

Initial counter values $\vec{\boldsymbol{u}}$ and target counter values $\vec{\boldsymbol{v}}$ are encoded in binary.

Complexity of Coverability

Theorem: Coverability in general VASS is EXPSPACE-complete.

[Lipton '76] [Rackoff '78]

		Number of unary counters		
Number of binary counters		**0**	**1**	$\geq \mathbf{2}$
	0	NL-complete [folklore]	NL-complete [Valiant and Paterson '75]	NL-complete [Rackoff '78]
	1	in NC^2 [Almagor, Cohen, Pérez, Shirmohammadi, and Worrell '20]	**in NP** [this paper]	. . .
	$\geq \mathbf{2}$	PSPACE-complete [Blondin, Finkel, Göller, Haase, and McKenzie '15]	PSPACE-complete	PSPACE-complete [Rackoff '78]

Motivation and Related Problems

VASS can be used to modestly model concurrent systems.

Coverability can be used to query safety conditions.

Study low dimension VASS to find new techniques in general.

Coverability in binary encoded 2-VASS is already PSPACE-hard.

1-PVASS: one binary counter and one pushdown stack.

2-TVASS: one of two binary counters can be zero-tested.

Our Contribution

Theorem: Coverability in 2-VASS with one unary counter is in NP. [this paper]

Our Contribution

Theorem: Coverability in binary encoded 2-VASS is PSPACE-complete.

[Blondin, Finkel, Göller, Haase, and McKenzie '15] [Rackoff '78]

Theorem: Coverability in 2-VASS with one unary counter is in NP. [this paper]

Theorem: Coverability in unary encoded 2-VASS is NL-complete. [Rackoff '78]

Common Approach: Small Witnesses

each cycle γ_i is iterated e_i many times

Linear Form Path $$\underline{\boldsymbol{\tau}_0}\,\boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\,\underline{\boldsymbol{\tau}_1}\,\cdots\,\underline{\boldsymbol{\tau}_{\boldsymbol{k}-1}}\,\boldsymbol{\gamma}_{\boldsymbol{k}}^{\boldsymbol{e}_{\boldsymbol{k}}}\,\underline{\boldsymbol{\tau}_{\boldsymbol{k}}}$$

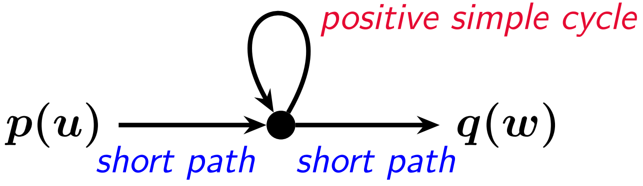
each path τ_i connects cycles γ_i and γ_{i+1}

Size $= |\boldsymbol{\tau}_0| + \ldots + |\boldsymbol{\tau}_{\boldsymbol{k}}| + |\boldsymbol{\gamma}_1| + \ldots + |\boldsymbol{\gamma}_{\boldsymbol{k}}| + \log(\boldsymbol{e}_1) + \ldots + \log(\boldsymbol{e}_{\boldsymbol{k}})$

Claim: Coverability in 1-VASS is witnessed by poly-size linear form paths.

$$\boldsymbol{p}(\boldsymbol{u}) \xrightarrow[\textit{short path}]{} \boldsymbol{q}(\boldsymbol{w})$$

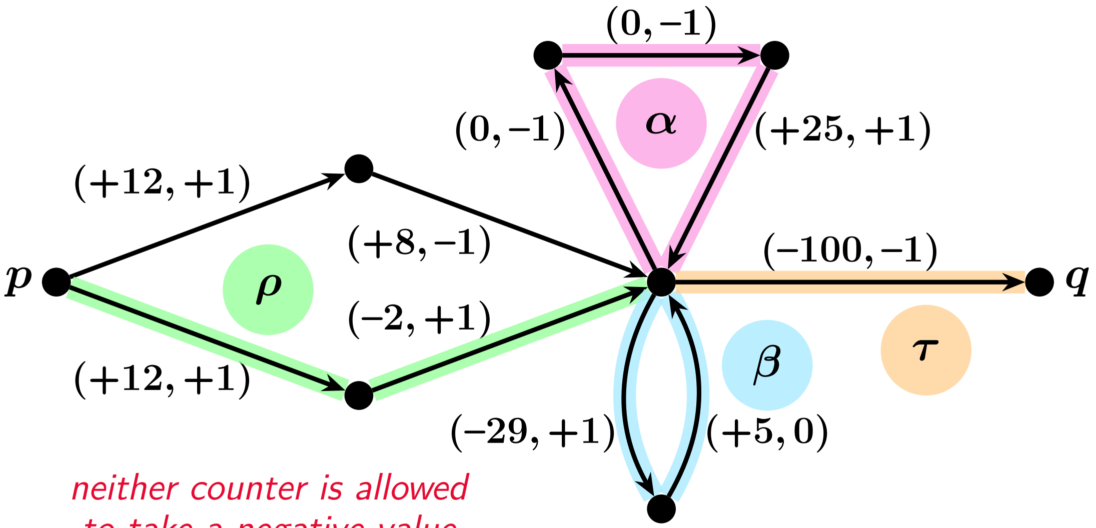
OR



$\boldsymbol{\tau}_0$

$\boldsymbol{\tau}_0\ \boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\ \boldsymbol{\tau}_1$

Motivating Example



neither counter is allowed
to take a negative value

Does there exist a run from $p(0,0)$ to $q(x,y)$ where $x \geq 20$ and $y \geq 10$?

$$p(0,0) \xrightarrow{\rho} c(10,2) \xrightarrow{\alpha} c(35,1) \xrightarrow{\beta} c(11,2)$$

$$c(10,2) \xrightarrow{(\alpha\beta)^{350}} c(360,2) \xrightarrow{\beta^{10}} c(120,12) \xrightarrow{\tau} q(20,11)$$

Our Approach: New Small Witnesses

*each **linear form** cycle σ_i is iterated f_i many times*

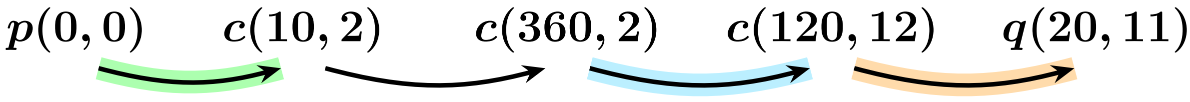
Compressed Linear Form Path $\underline{\boldsymbol{\rho}_0}\,\boxed{\boldsymbol{\sigma}_1^{\boldsymbol{f}_1}}\,\underline{\boldsymbol{\rho}_1}\cdots\underline{\boldsymbol{\rho}_{k-1}}\,\boxed{\boldsymbol{\sigma}_k^{\boldsymbol{f}_k}}\,\underline{\boldsymbol{\rho}_k}$

*each **linear form** path ρ_i connects σ_i and σ_{i+1}*

Size $= |\boldsymbol{\rho}_0| + \ldots + |\boldsymbol{\rho}_k| + |\boldsymbol{\sigma}_1| + \ldots + |\boldsymbol{\sigma}_k| + \log(\boldsymbol{f}_1) + \ldots + \log(\boldsymbol{f}_k)$

Succinctness gains: $(\boldsymbol{\tau}_0\,\boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\,\boldsymbol{\tau}_1)\,(\boldsymbol{\tau}_0\,\boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\,\boldsymbol{\tau}_1)\cdots(\boldsymbol{\tau}_0\,\boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\,\boldsymbol{\tau}_1) = (\boldsymbol{\tau}_0\,\boldsymbol{\gamma}_1^{\boldsymbol{e}_1}\,\boldsymbol{\tau}_1)^{\boldsymbol{f}}$

Example



$\boldsymbol{p}(\mathbf{0},\mathbf{0})$ $\boldsymbol{c}(\mathbf{10},\mathbf{2})$ $\boldsymbol{c}(\mathbf{360},\mathbf{2})$ $\boldsymbol{c}(\mathbf{120},\mathbf{12})$ $\boldsymbol{q}(\mathbf{20},\mathbf{11})$

$\underline{\boldsymbol{\rho}}$ $\underline{(\boldsymbol{\alpha}^{\mathbf{1}}\boldsymbol{\beta}^{\mathbf{1}})^{\mathbf{350}}}$ $\underline{\boldsymbol{\beta}^{\mathbf{10}}\qquad\boldsymbol{\tau}}$

small linear form path *small linear form cycle iterated 350 times* *small linear form path*

Technical Contribution

Theorem: Suppose $\boldsymbol{p}(\vec{\boldsymbol{u}}) \xrightarrow{*} \boldsymbol{q}(\vec{\boldsymbol{v}})$ in a given 2-VASS with one unary counter then there exists a poly-size compressed linear form path $\boldsymbol{\pi}$ that induces a run $\boldsymbol{p}(\vec{\boldsymbol{u}}) \xrightarrow{\pi} \boldsymbol{q}(\vec{\boldsymbol{w}})$ where $\vec{\boldsymbol{w}} \geq \vec{\boldsymbol{v}}$. [this paper]

Technique 1

Only polynomially many short cycles need to occur in a run witnessing coverability.

Requirements:

- Coverability objective.
- At most one binary component.

Technique 2

Reshuffle short paths and cycles to obtain a poly-size linear form path or find a pumpable cycle.

Requirements:

- At most two dimensions.
- At least one unary component.

Conclusion

Theorem: Coverability in 2-VASS with one unary counter is in NP. [this paper]

Proof idea: Guess and check a poly-size compressed linear form path.

Open Problem: is *coverability* in 2-VASS with one unary counter NP-hard?

Open Problem: is *reachability* in 2-VASS with one unary counter in NP?

Thank You!

Presented by Henry Sinclair-Banks, University of Warwick, Coventry, UK 🇬🇧

FoSSaCS'23 in Sorbonne Université, Paris, France 🇫🇷