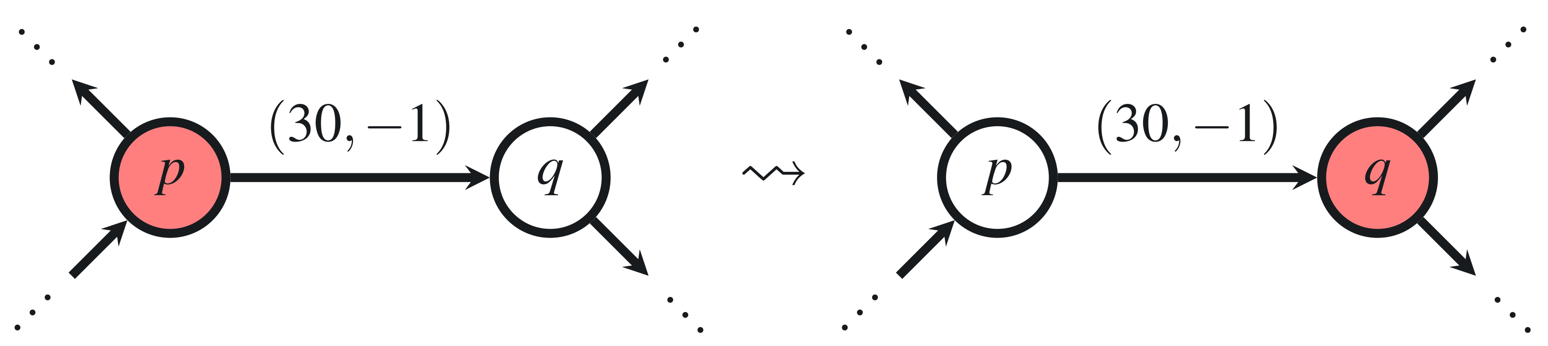
Coverability in 2-VASS with One Unary Counter

Filip Mazowiecki — University of Warsaw, Poland
Henry Sinclair-Banks — University of Warwick, United Kingdom
Karol Węgrzycki — Saarland University & Max Planck Institute for Informatics, Saarbrücken, Germany

PROBLEM STATEMENT

Vector Addition Systems with States (2-VASS)



Configuration: $p(21,8) \in Q \times \mathbb{N}^2$ Configuration: $q(51,7) \in Q \times \mathbb{N}^2$

Given initial configuration $p(\mathbf{u})$ and target configuration $q(\mathbf{v})$ …

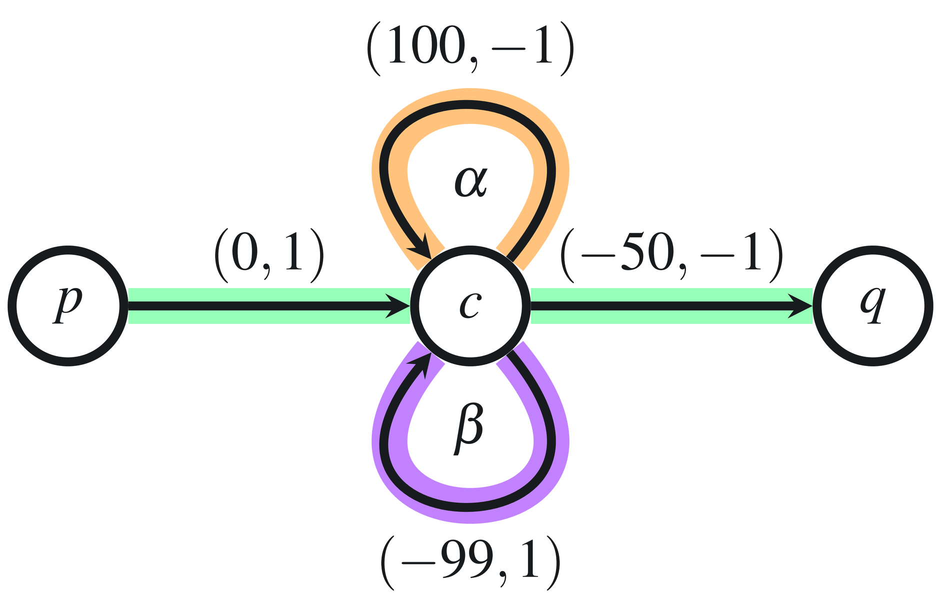
- **Reachability** does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v})$?
- **Coverability** does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$?

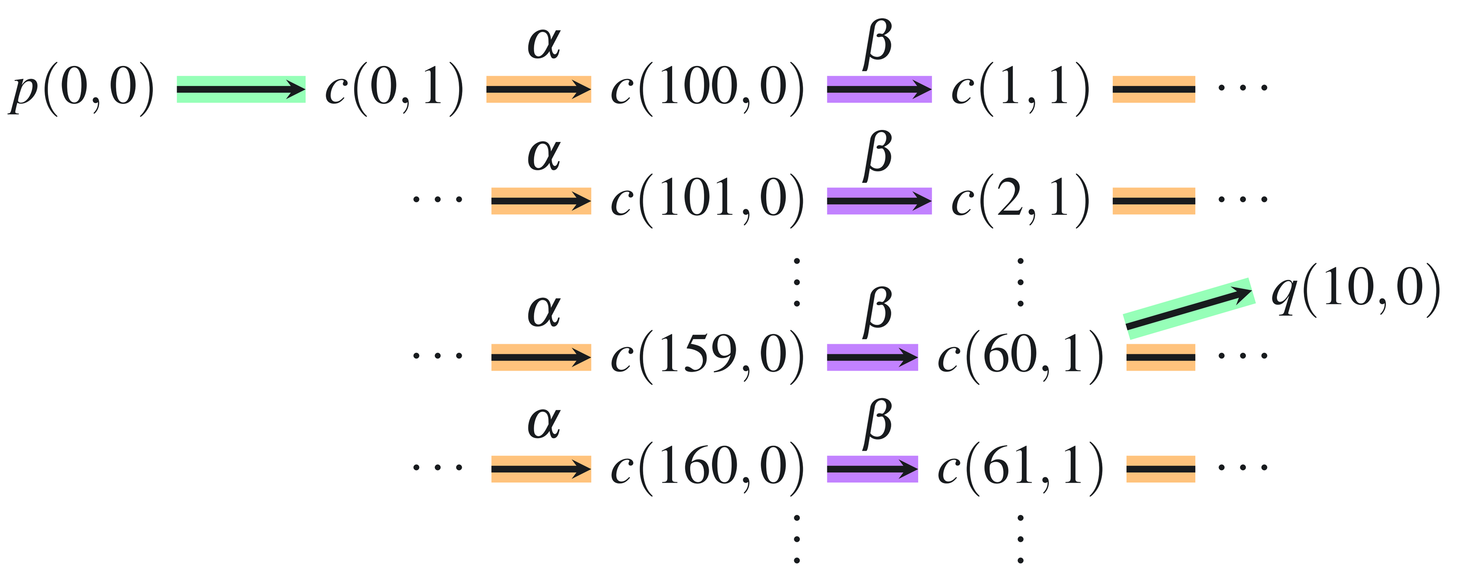
We consider the restricted variant **2-VASS with one unary counter**, where one counter only receives unary updates $\{-1, 0, +1\}$.

COVERABILITY EXAMPLE

Does there exist a run from $p(0,0)$ to $q(b,u)$ such that $b \geq 10$ and $u \geq 0$?

This is a positive instance.





OUR CONTRIBUTION

Theorem: Suppose there exists a run from from $p(\mathbf{u})$ to $q(\mathbf{v})$ in a given **2-VASS with one unary counter**, then there exists a *compressed linear form path of polynomial size* that induces a run from $p(\mathbf{u})$ to $q(\mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$.

$\Longrightarrow$ **Coverability in 2-VASS with one unary counter is in NP.**

MOTIVATION

Similar Problems

The complexities of **coverability** in fixed, low dimension VASS:

	Unary counters	Binary counters
1-VASS	NL-complete [Valiant and Patterson '75] [Rosier and Yen '85]	NL-hard and in $NC^2 \subseteq P$ [Almagor, Boker, Hofman, and Totzke '20]
2-VASS	NL-complete [Englert, Lazić, and Totzke '16]	PSPACE-complete [Blondin, Finkel, Göller, Haase, and McKenzie '15]

Complexities of reachability coincide, except for 1-VASS with a binary counter that is NP-complete.

Related Work

1-VASS with a binary counter and a pushdown stack (1-PVASS):

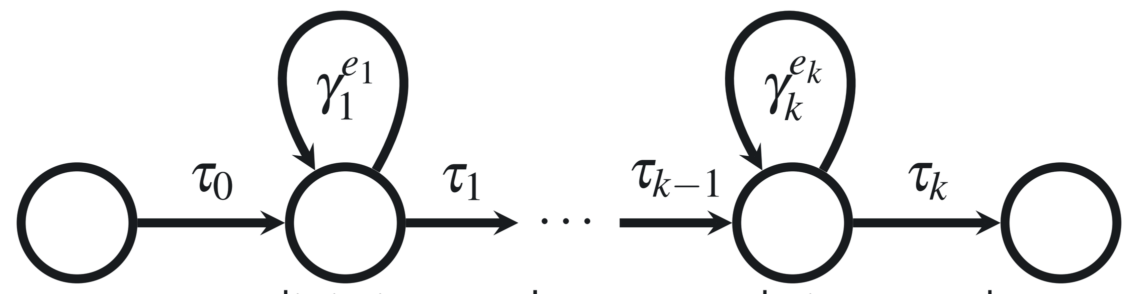
- **Coverability** is decidable. [Leroux, Sutre, and Totzke '15]
- **Reachability** and **coverability** are PSPACE-hard. [Englert, Hofman, Lasota, Lazić, Leroux, and Straszyński '21]

VASS where one counter can be zero tested (TVASS):

- **Reachability** is decidable. [Reinhardt '08], [Bonnet '11]
- **Reachability** in 2-TVASS is PSPACE-complete. [Leroux and Sutre '20]

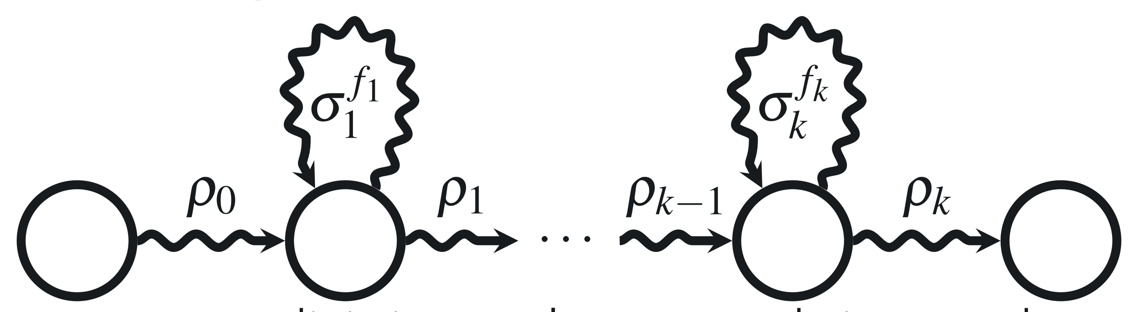
DEFINITIONS

Linear Form Paths (Standard)



The paths τ_i connect disjoint cycles γ_i, each iterated e_i many times.

Compressed Linear Form Paths

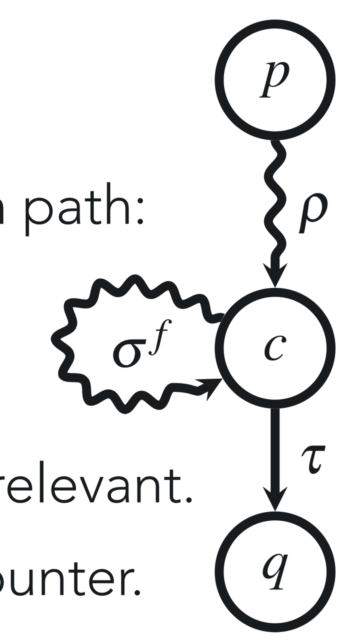


The paths ρ_i connect disjoint cycles σ_i, each iterated f_i many times. Each path and cycle is in linear form (hence compressed).

PROOF APPROACH

There exists a polynomial size compressed linear form path:

- ρ, σ, and τ are linear form paths and cycles.
- σ has positive and zero effect on the two counters.
- f is large enough such that one counter becomes irrelevant.
- τ is a path for an instance of coverability with one counter.

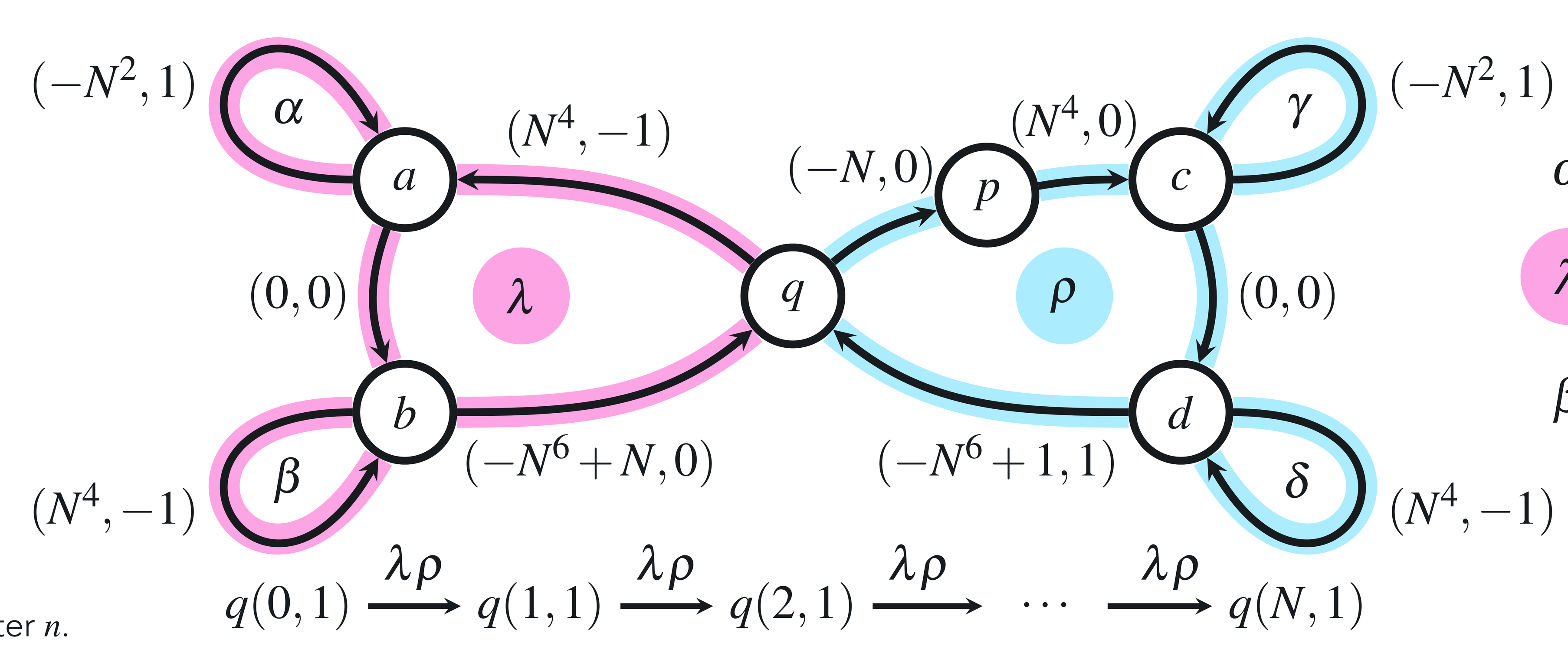


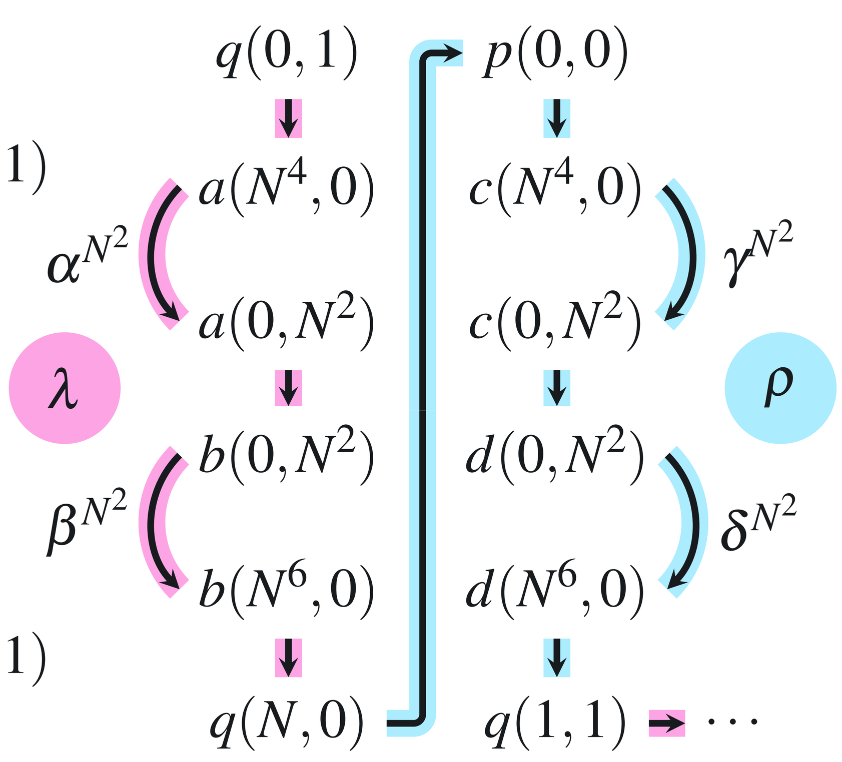
COMPRESSIBLE EXAMPLE

Does there exist a run from $q(0,1)$ to $q(b,u)$ such that $b \geq N$ and $u \geq 1$?

This is a positive instance.

Polynomial size witness: no linear form path, but there is a compressed linear form path.





$N = 2^n$, for some binary encoded input parameter n.