Coverability in 2-VASS with One Unary Counter

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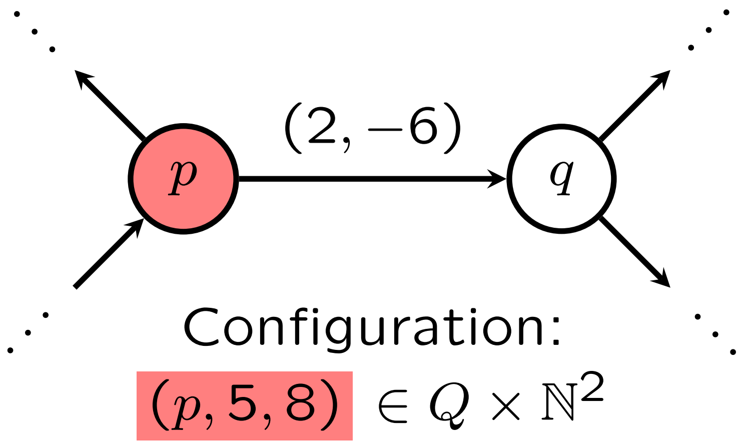
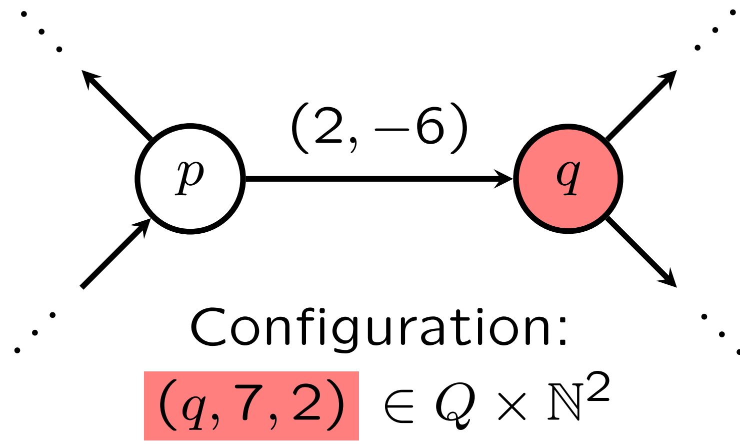
INTRODUCTION

Vector Addition Systems with States (2-VASS)





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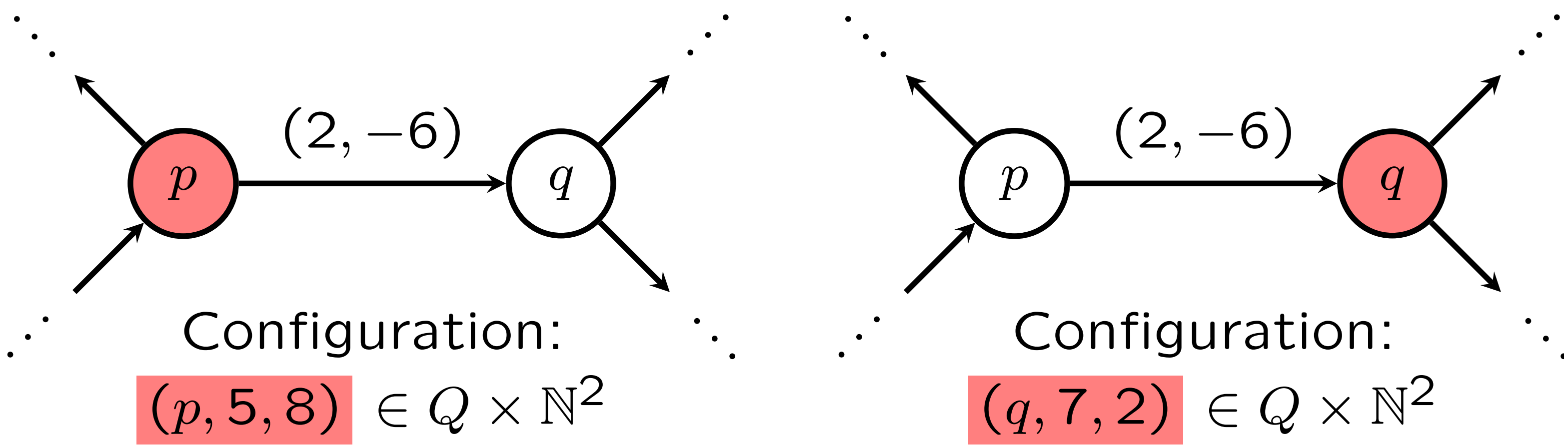
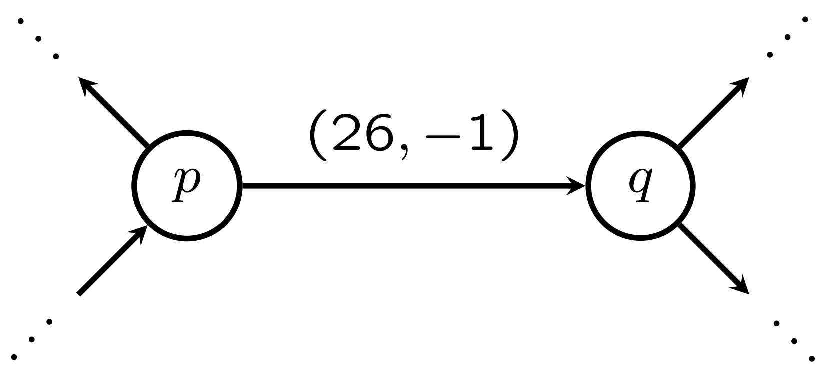


Reachability does there exist a *run* in V from $(p, \mathbf{u})$ to $(q, \mathbf{v})$?

Coverability does there exist a *run* in V from $(p, \mathbf{u})$ to $(q, \mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$?

CONTRIBUTION

We consider **2-VASS with one unary counter**, the restricted variant where one counter receives unary updates $\{-1, 0, +1\}$.



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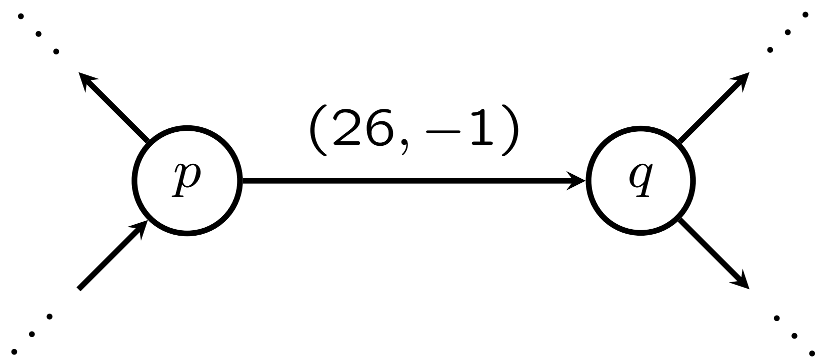

Coverability in 2-VASS with one unary counter is in NP.

MOTIVATION

Counter Programs: sequences of commands on counters.

$$x += 1 \qquad x -= 1 \qquad \texttt{zero}(x) \qquad \texttt{goto}\ \ell\ \texttt{or}\ \ell' \qquad \texttt{halt}$$

Even for 2 counters, halt is undecidable. [Minsky '67]

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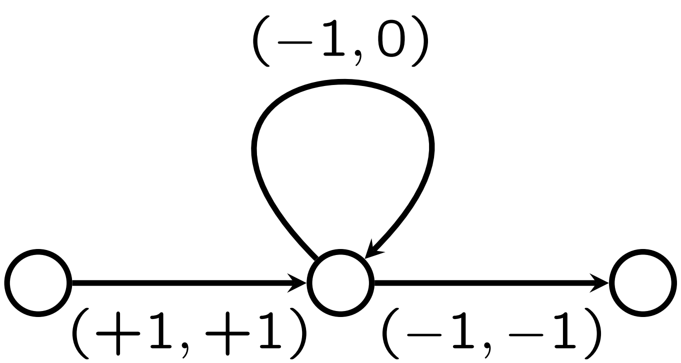

1: $x += 1$, $y += 1$,

2: goto 3 or 5,

3: $x -= 1$,

4: goto 2 or 2,

5: $x -= 1$, $y -= 1$,

6: **halt**.

MOTIVATION

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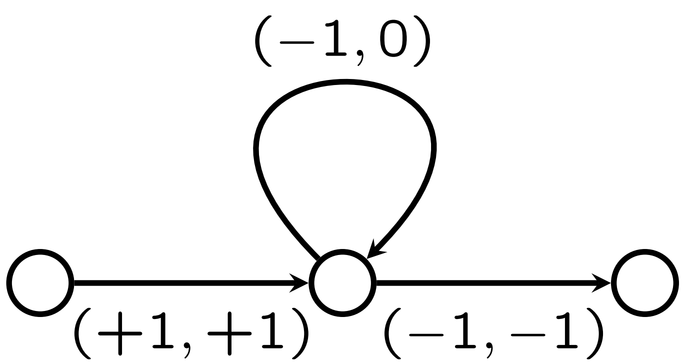


```
1:  x += 1, y += 1,
2:  goto 3 or 5,
3:  x -= 1,
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```

For **VASS**, reachability is decidable. [Mayr '81]

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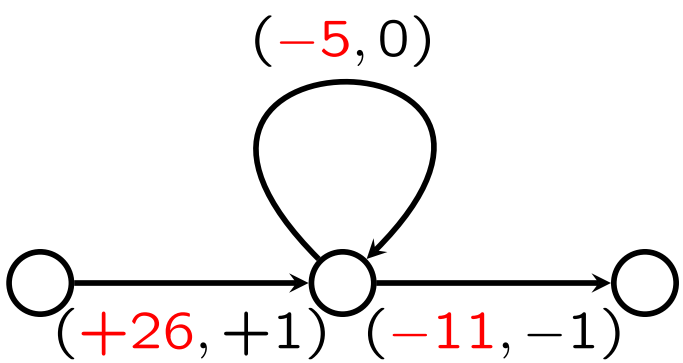

1: $x += 26$, $y += 1$,
2: goto 3 or 5,
3: $x -= 5$,
4: goto 2 or 2,
5: $x -= 11$, $y -= 1$
6: halt.

For **VASS**, reachability is decidable. [Mayr '81]

RELATED WORK

One counter can be zero-tested:

- **Reachability** is decidable. [Reinhardt '08]
- Simpler proof for **VASS** model. [Bonnet '11]
- For two counters only, **reachability** is PSPACE-complete. [Leroux and Sutre '20]

RELATED WORK

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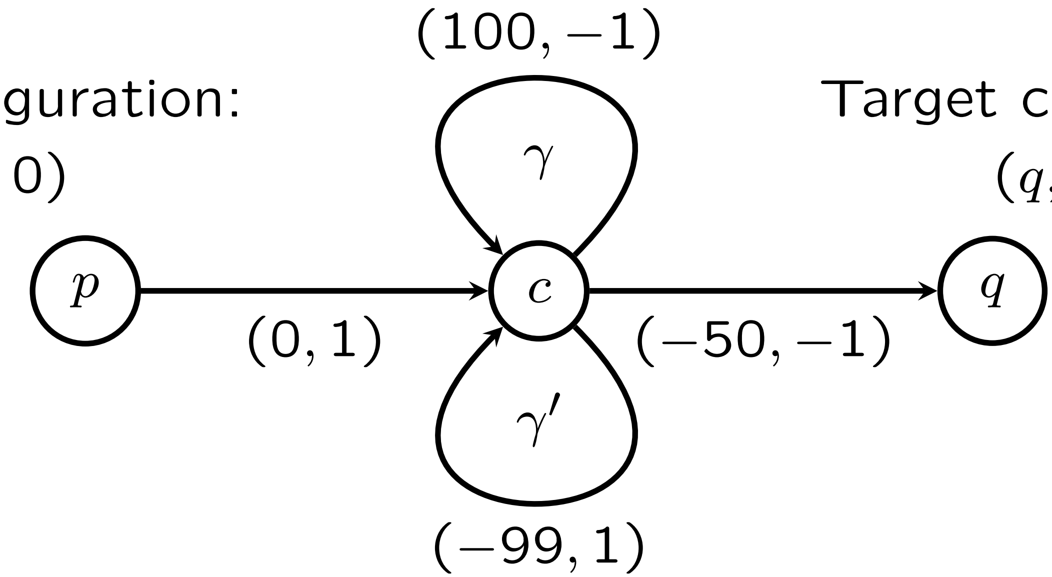
One counter and one stack (1-PVASS):

- **Coverability** is decidable. [Leroux, Sutre, and Totzke '15]
- **Reachability** and **coverability** are PSPACE-hard. [Englert, Hofman, Lasota, Lazić, Leroux, and Straszyński '21]

EXAMPLE

Initial configuration:

$(p, 0, 0)$



Target configuration:

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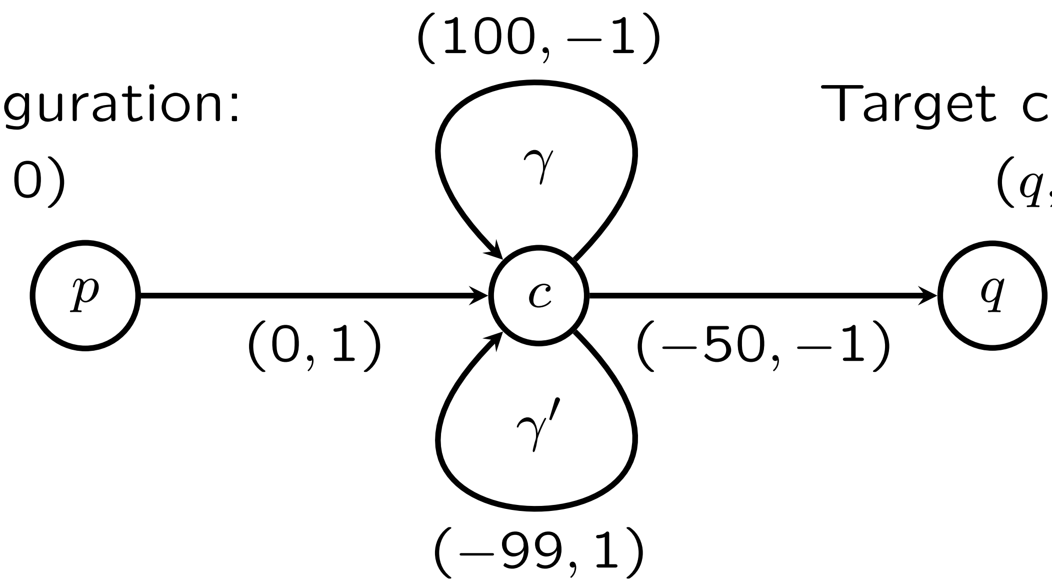

Target configuration:

$(q,10,0)$

$$
\begin{aligned}
(p,0,0) &\longrightarrow (c,0,1) \xrightarrow{\gamma} (c,100,0) \xrightarrow{\gamma'} (c,\ \ 1,1) \longrightarrow \cdots \\
&\cdots \xrightarrow{\gamma} (c,101,0) \xrightarrow{\gamma'} (c,\ \ 2,1) \longrightarrow \cdots \\
&\qquad\vdots \\
&\cdots \xrightarrow{\gamma} (c,159,0) \xrightarrow{\gamma'} (c,60,1) \longrightarrow \cdots \quad (\nearrow (q,10,0)) \\
&\cdots \xrightarrow{\gamma} (c,160,0) \xrightarrow{\gamma'} (c,61,1) \longrightarrow \cdots \\
&\qquad\vdots
\end{aligned}
$$

BACKGROUND

Reachability and **coverability** are EXPSPACE-hard. [Lipton '76]

Coverability is in EXPSPACE. [Rackoff '78]

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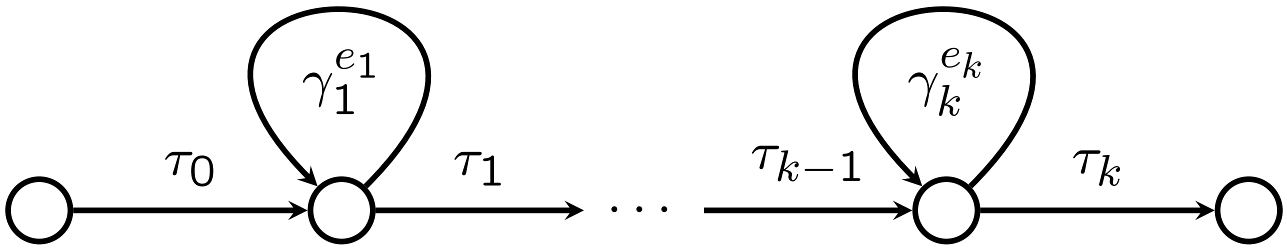
Complexities of fixed dimension **reachability** and **coverability**:

	Unary encoding	Binary encoding
1-VASS	NL-complete	NP-complete NL-hard and in $NC^2 \subseteq P$
2-VASS	NL-complete	PSPACE-complete

DEFINITIONS

Linear form paths



The paths τ_i connect disjoint cycles γ_i iterated e_i many times.

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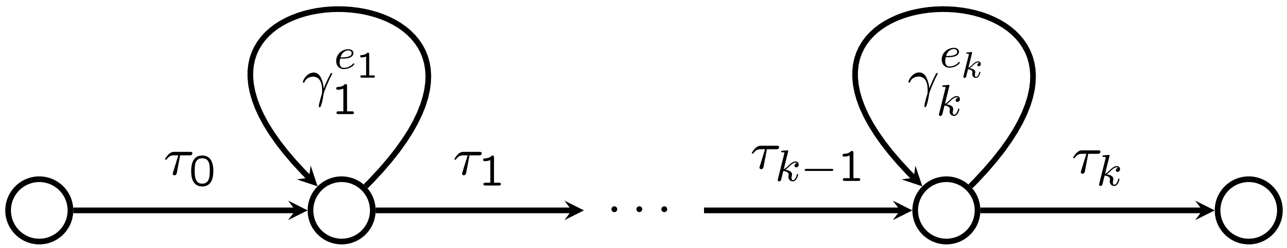

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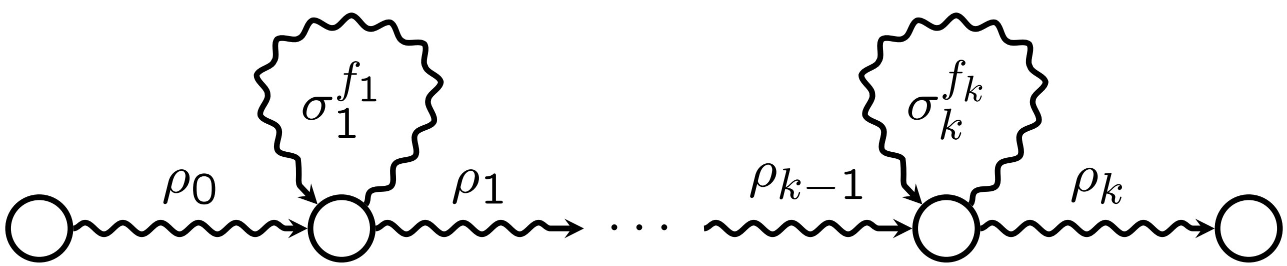
<u>Compressed</u> linear form paths



For succinctness, the paths ρ_i and cycles σ_i are in linear form.

RESULTS

Theorem: Given a **2-VASS with one unary counter** V and suppose there exists a run in V from $(p, \mathbf{u})$ to $(q, \mathbf{v})$. Then there exists compressed linear form path of *polynomial size* inducing a run from $(p, \mathbf{u})$ to $(q, \mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$.

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$\Rightarrow$ **Coverability in 2-VASS with one unary counter is in NP.**

... just guess and check compressed linear form paths.

PROOF IDEA OF THEOREM

Given a run in V from $(p, \mathbf{u})$ to $(q, \mathbf{v})$, consider the underlying path π of connected transitions in V...

Case 1, π has a polynomial number of transitions:

$\Longrightarrow$ $\pi = \tau_0$ is a *poly-size* linear form path.

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Case 2, success with newly developed techniques:

Cycles are carefully moved and bundled together γ^e.

$$\implies \pi' = \tau_0 \gamma_1^{e_1} \tau_1 \cdots \tau_{k-1} \gamma_k^{e_k} \tau_k \text{ is a } \textit{poly-size} \text{ linear form path.}$$

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Case 3, failure with newly developed techniques:

Implies existance of ‘pumpable’ linear form cycle σ.

$$\Longrightarrow \pi'' = \rho \sigma^x \tau \text{ is a } \textit{poly-size} \text{ \underline{compressed} linear form path.}$$

CONCLUSION

Coverability in 2-VASS with one unary counter is in NP.

Unfortunately, we lack a matching NP-hard lower bound.

Conjecture: **coverability** in P.

- We have only been verify this for linear path schemes (using a dynamic programming algorithm).

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Future Work: is **reachability** also in NP?

- If this is true, then it is NP-complete since **reachability** in binary 1-VASS is already NP-hard.

Questions?

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Presented by Henry Sinclair-Banks