The Complexity of Coverability in Fixed Dimension VASS with Various Encodings

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OFCOURSE
17th November 2022
MPI–SWS, Kaiserslautern, Germany
CHAPTER ZERO
INTRODUCTION

Warm-up Example
Problem Statement
Background and Motivation
Fun-Road-Trip Checklist

✓ always at least one friend, and
✓ never negative money!

Friends: 4
Money: €100
Fun-Road-Trip Checklist
✓ always at least one friend, and
✓ never negative money!

Friends: 4
Money: €100

GOAL
Friends: ≥ 5
Money: ≥ €10
**Fun-Road-Trip Checklist**

- ✔ always at least one friend, and
- ✔ never negative money!

**GOAL**
- Friends: ≥ 5
- Money: ≥ €10

**Starting Point**
- Friends: 4
- Money: €100

**Pathway**
- €60
- +€20
- -1 Friend
- -€25
- -€5
- +1 Friend

**End Point**
- -1 Friend
- -1 Friend
- -1 Friend

**Route Highlights**
- Gas station: -€60
- +1 Friend
- -1 Friend
- Coffee shop: +1 Friend, -€5
- -1 Friend

**Total END**
- Friends: 0
- Money: -€25

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Fun-Road-Trip Checklist
✓ always at least one friend, and
✓ never negative money!
Fun-Road-Trip Checklist

✓ always at least one friend, and
✓ never negative money!

GOAL
Friends: ≥ 5
Money: ≥ €10

Friends: 5
Money: €40

-€60

+1 Friend

-1 Friend

+€20

-1 Friend

-€25

+1 Friend

-€5

-1 Friend

-1 Friend

-1 Friend

-1 Friend
Fun-Road-Trip Checklist
✓ always at least one friend, and
✓ never negative money!

GOAL
Friends: ≥ 5
Money: ≥ €10

Friends: 4
Money: €40
Fun-Road-Trip Checklist
✓ always at least one friend, and
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GOAL
Friends: ≥ 5
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Fun-Road-Trip Checklist
✓ always at least one friend, and
✓ never negative money!

GOAL
Friends: $\geq 5$
Money: $\geq \€10$

-€60

+€20

-1 Friend

-1 Friend

-€25

+1 Friend

-€5

-1 Friend

Friends: 2
Money: €60

-1 Friend

-1 Friend

-1 Friend
Fun-Road-Trip Checklist

- always at least one friend, and
- never negative money!

GOAL
Friends: ≥ 5
Money: ≥ €10

Friends: 1
Money: €60

Friends: -1
Money: -€5

Friends: +1
Money: +€20

Friends: -1
Money: -€60

Friends: -1
Money: -€25
Fun-Road-Trip Checklist

☑ always at least one friend, and
☑ never negative money!

GOAL
Friends: ≥ 5
Money: ≥ €10

Friends: 5
Money: €40
Fun-Road-Trip Checklist

✓ always at least one friend, and
✓ never negative money!
PROBLEM STATEMENT

Vector Addition Systems with States (k-VASS)

Configuration: $p(5,\ldots,8) \in Q \times \mathbb{N}^k$

Configuration: $q(7,\ldots,2) \in Q \times \mathbb{N}^k$
PROBLEM STATEMENT

Vector Addition Systems with States (k-VASS)

Configuration:

\[ p(5, \ldots, 8) \in Q \times \mathbb{N}^k \]

\[ q(7, \ldots, 2) \in Q \times \mathbb{N}^k \]

Reachability
does there exist a run from \( p(\bar{u}) \) to \( q(\bar{v}) \)?

Coverability
does there exist a run from \( p(\bar{u}) \) to \( q(\bar{w}) \)
for some \( \bar{w} \geq \bar{v} \)?
BACKGROUND

Coverability in non-fixed dimension VASS is EXPSPACE-complete, regardless of the encoding. [Lipton '76] [Rackoff '78]
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Coverability in binary $k$-VASS is in PSPACE. [Rackoff '78]

Coverability in unary $k$-VASS is in NL. [Rackoff '78]
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Coverability in binary 2-VASS is PSPACE-hard. [Blondin, Finkel, Göller, Haase, and McKenzie ’15]
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Coverability in binary $k$-VASS is in PSPACE.  

Coverability in unary $k$-VASS is in NL.  

Coverability in binary 2-VASS is PSPACE-hard.  

Coverability in binary 1-VASS is in NC$^2$.  

[Lipton ’76] [Rackoff ’78]  

[Rackoff ’78]  

[Rackoff ’78]  

[Blondin, Finkel, Göller, Haase, and McKenzie ’15]  

[Almagor, Cohen, Pérez, Shirmohammadi, and Worrell ’20]
# TABLE OF RESULTS

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<tr>
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MOTIVATION

Petri nets are an equivalent model of computation.

Coverability has applications in verification of safety conditions.

Reachability tools are often applied to coverability benchmarks.
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Coverability has applications in verification of safety conditions.

Reachability tools are often applied to coverability benchmarks.

Related problems (with asymmetric treatment of counters):

Coverability in 1-VASS with a pushdown stack is PSPACE-hard and is decidable.  
[Lesfic, Sutre, and Totzke '15]
[Englert, Hofman, Lasota, Lazic, Leroux, and Straszynski '20]

Reachability in 2-VASS where one counter can be zero-tested is PSPACE-complete.  
[Leroux and Sutre '20]
CHAPTER ONE
UPPER BOUNDS

Our Contribution
The Overall Approach

Technique: “Polynomially Many Short Cycles”
OUR CONTRIBUTION

Theorem: Coverability in 2-VASS with one binary counter and one unary counter is in NP. [our result]
OUR CONTRIBUTION

**Theorem:** Coverability in 2-VASS with two binary counters is PSPACE-complete. [Blondin, Finkel, Göller, Haase, and McKenzie ’15]

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Theorem: Coverability in 2-VASS with two unary counters is NL-complete. [Rackoff ’78]
OUR CONTRIBUTION

**Theorem:** Coverability in 2-VASS with one binary counter and one unary counter is in NP. [our result]
FINDING SMALL WITNESSES

Start with a path $\pi = \tau_0 \gamma_1^e \tau_1 \cdots \tau_{k-1} \gamma_k^e \tau_k$ such that all paths $\tau_i$ and cycles $\gamma_i$ are short, and $p(\vec{u}) \xrightarrow{\pi} q(\vec{v})$. 
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Replacement Lemma
(based on “polynomially many short cycles” technique)
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Start with a path \( \pi = \tau_0 \gamma_1^1 \tau_1 \cdots \tau_{k-1} \gamma_k^k \tau_k \) such that all paths \( \tau_i \) and cycles \( \gamma_i \) are short, and \( p(\overline{u}) \xrightarrow{\pi} q(\overline{v}) \).

Replacement Lemma
(based on “polynomially many short cycles” technique)

Obtain a path \( \rho = \tau_0 \sigma_1^1 \tau_1 \cdots \tau_{k-1} \sigma_k^k \tau_k \) such that there are polynomially many distinct short cycles \( \sigma_i \), and \( p(\overline{u}) \xrightarrow{\rho} q(\overline{w}) \) where \( \overline{w} \geq \overline{v} \).
FINDING SMALL WITNESSES

Start with a path \( \pi = \tau_0 \gamma_1^{e_1} \tau_1 \cdots \tau_{k-1} \gamma_k^{e_k} \tau_k \) such that all paths \( \tau_i \) and cycles \( \gamma_i \) are short, and \( p(\vec{u}) \xrightarrow{\pi} q(\vec{v}) \).

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Reshuffling Lemma

Obtain a polynomial size compressed linear form run \( \varsigma \) such that \( p(\vec{u}) \xrightarrow{\varsigma} q(\vec{x}) \) where \( \vec{x} \geq \vec{w} \geq \vec{v} \).
Start with a path $\pi = \tau_0 \gamma_1^e \tau_1 \cdots \tau_{k-1} \gamma_k^e \tau_k$ such that all paths $\tau_i$ and cycles $\gamma_i$ are short, and $p(\tilde{u}) \xrightarrow{\pi} q(\tilde{v})$.

**Replacement Lemma**
(based on “polynomially many short cycles” technique)

Obtain a path $\rho = \tau_0 \sigma_1^e \tau_1 \cdots \tau_{k-1} \sigma_k^e \tau_k$ such that there are polynomially many distinct short cycles $\sigma_i$, and $p(\tilde{u}) \xrightarrow{\rho} q(\tilde{w})$ where $\tilde{w} \geq \tilde{v}$.

**Reshuffling Lemma**

Obtain a *polynomial size compressed linear form run* $\varsigma$ such that $p(\tilde{u}) \xrightarrow{\varsigma} q(\tilde{x})$ where $\tilde{x} \geq \tilde{w} \geq \tilde{v}$. 
CHARACTERISATION OF A SHORT CYCLE

$\alpha$ is the $p-q$ prefix
$\beta$ is the $q-p$ suffix
CHARACTERISATION OF A SHORT CYCLE

\( \alpha \) is the \( p-q \) prefix
\( \beta \) is the \( q-p \) suffix

(a) Start-end state \( p \),

\( (-100, -1) \) \( \rightarrow \) \( (20, -1) \) \( \rightarrow \) \( (-50, 1) \)

\( (15, 1) \) \( \rightarrow \) \( (-10, -1) \) \( \rightarrow \) \( (-20, 1) \) \( \rightarrow \) \( (75, 1) \)
CHARACTERISATION OF A SHORT CYCLE

(a) Start-end state $p$,

(b) State where minimum binary effect observed is $q$,
CHARACTERISATION OF A SHORT CYCLE

\[ (-100, -1) \rightarrow (20, -1) \rightarrow (-50, 1) \rightarrow (75, 1) \rightarrow (15, 1) \rightarrow (-10, -1) \rightarrow (-20, 1) \rightarrow (-100, -1) \]

\(\alpha\) is the \(p-q\) prefix
\(\beta\) is the \(q-p\) suffix

(a) Start-end state \(p\),
(b) State where minimum binary effect observed is \(q\),
(c) Length of \(\alpha\) is 3,
CHARACTERISATION OF A SHORT CYCLE

\[
\begin{align*}
(15, 1) & \rightarrow (20, -1) \rightarrow (-50, 1) \\
(-100, -1) & \rightarrow (-10, -1) \rightarrow (-20, 1) \\
\end{align*}
\]

$\alpha$ is the $p-q$ prefix

$\beta$ is the $q-p$ suffix

(a) Start-end state $p$,
(b) State where minimum binary effect observed is $q$,
(c) Length of $\alpha$ is 3,
(d) Length of $\beta$ is 4,
CHARACTERISATION OF A SHORT CYCLE

(a) Start-end state $p$,
(b) State where minimum binary effect observed is $q$,
(c) Length of $\alpha$ is 3,
(d) Length of $\beta$ is 4,
(e) Unary effect of $\alpha$ is $-1$, 

$\alpha$ is the $p$-$q$ prefix
$\beta$ is the $q$-$p$ suffix

$\begin{align*}
(15, 1) & \quad \rightarrow (20, -1) \\
(-100, -1) & \quad \rightarrow (-50, 1) \\
(-10, -1) & \quad \rightarrow (75, 1) \\
(15, 1) & \quad \rightarrow (-20, 1)
\end{align*}$
CHARACTERISATION OF A SHORT CYCLE

\[ \alpha \] is the \( p-q \) prefix
\[ \beta \] is the \( q-p \) suffix

(a) Start-end state \( p \),
(b) State where minimum binary effect observed is \( q \),
(c) Length of \( \alpha \) is 3,
(d) Length of \( \beta \) is 4,
(e) Unary effect of \( \alpha \) is \(-1\),
(f) Unary effect of \( \beta \) is 2,
CHARACTERISATION OF A SHORT CYCLE

\( (\alpha, \beta) \) is \( p - q \) prefix

\( \beta \) is the \( q - p \) suffix

(a) Start-end state \( p \),
(b) State where minimum binary effect observed is \( q \),
(c) Length of \( \alpha \) is 3,
(d) Length of \( \beta \) is 4,
(e) Unary effect of \( \alpha \) is \(-1\),
(f) Unary effect of \( \beta \) is 2,
(g) Minimum unary effect over \( \alpha \) is \(-2\),
CHARACTERISATION OF A SHORT CYCLE

(a) Start-end state $p$,
(b) State where minimum binary effect observed is $q$,
(c) Length of $\alpha$ is 3,
(d) Length of $\beta$ is 4,
(e) Unary effect of $\alpha$ is $-1$,
(f) Unary effect of $\beta$ is 2,
(g) Minimum unary effect over $\alpha$ is $-2$, and
(h) Minimum unary effect over $\beta$ is 0.
SHORT CYCLE REPLACEMENT

Suppose $\gamma_1$ and $\gamma_2$ have the same characterisation and consider $
\sigma = \alpha_i \beta_j$ where $i, j \in \{1, 2\}$ selected for greatest binary effect.
Suppose \( \gamma_1 \) and \( \gamma_2 \) have the same characterisation and consider \( \sigma = \alpha_i \beta_j \) where \( i, j \in \{1, 2\} \) selected for greatest binary effect.

**Idea:** replace all iterations of \( \gamma_1 \) and \( \gamma_2 \) in a run with iterations of \( \sigma \), the run remains executable and has at least the effect.

\[
\pi = \tau_1 \gamma_1 \tau_2 \gamma_2 \tau_3 \quad \leadsto \quad \rho = \tau_1 \sigma \tau_2 \sigma \tau_3
\]

If \( p(\vec{u}) \xrightarrow{\pi} q(\vec{v}) \), then \( p(\vec{u}) \xrightarrow{\rho} q(\vec{w}) \) and \( \vec{w} \geq \vec{v} \).
NUMBER OF CHARACTERISATIONS

(a) Start-end state,

(b) State where minimum binary effect observed,

(c) Length of the prefix \( \alpha \),

(d) Length of the suffix \( \beta \),

(e) Unary effect of the prefix \( \alpha \),

(f) Unary effect of the suffix \( \beta \),

(g) Minimum unary effect over the prefix \( \alpha \), and

(h) Minimum unary effect over the suffix \( \beta \).

How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state,
(b) State where minimum binary effect observed,
(c) Length of the prefix $\alpha$,
(d) Length of the suffix $\beta$,
(e) Unary effect of the prefix $\alpha$,
(f) Unary effect of the suffix $\beta$,
(g) Minimum unary effect over the prefix $\alpha$, and
(h) Minimum unary effect over the suffix $\beta$.

How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state, $|Q|$

(b) State where minimum binary effect observed, $|Q|$

(c) Length of the prefix $\alpha$, $|Q| + 1$

(d) Length of the suffix $\beta$, $|Q| + 1$

(e) Unary effect of the prefix $\alpha$, $|Q|$

(f) Unary effect of the suffix $\beta$, $|Q|$

(g) Minimum unary effect over the prefix $\alpha$, and $|Q|$

(h) Minimum unary effect over the suffix $\beta$. $|Q|$

How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state, \(|Q|\)

(b) State where minimum binary effect observed, \(|Q|\)

(c) Length of the prefix \(\alpha\), \(|Q| + 1\)

(d) Length of the suffix \(\beta\),

(e) Unary effect of the prefix \(\alpha\),

(f) Unary effect of the suffix \(\beta\),

(g) Minimum unary effect over the prefix \(\alpha\), and

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How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state, \(|Q|\)

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(d) Length of the suffix \(\beta\), \(|Q| + 1\)

(e) Unary effect of the prefix \(\alpha\),

(f) Unary effect of the suffix \(\beta\),

(g) Minimum unary effect over the prefix \(\alpha\), and

(h) Minimum unary effect over the suffix \(\beta\).

How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state, \( |Q| \)
(b) State where minimum binary effect observed, \( |Q| \)
(c) Length of the prefix \( \alpha \), \( |Q| + 1 \)
(d) Length of the suffix \( \beta \), \( |Q| + 1 \)
(e) Unary effect of the prefix \( \alpha \), \( 2|Q| + 1 \)
(f) Unary effect of the suffix \( \beta \),
(g) Minimum unary effect over the prefix \( \alpha \), and
(h) Minimum unary effect over the suffix \( \beta \).

How many different characterisations?
NUMBER OF CHARACTERISATIONS

(a) Start-end state, $|Q|$
(b) State where minimum binary effect observed, $|Q|$
(c) Length of the prefix $\alpha$, $|Q| + 1$
(d) Length of the suffix $\beta$, $|Q| + 1$
(e) Unary effect of the prefix $\alpha$, $2|Q| + 1$
(f) Unary effect of the suffix $\beta$, $2|Q| + 1$
(g) Minimum unary effect over the prefix $\alpha$, and
(h) Minimum unary effect over the suffix $\beta$.

How many different characterisations?
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(a) Start-end state, \(|Q|\)
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(g) Minimum unary effect over the prefix \(\alpha\), and \(|Q| + 1\)
(h) Minimum unary effect over the suffix \(\beta\).

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How many different characterisations?
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(f) Unary effect of the suffix \(\beta\), \(2|Q| + 1\)
(g) Minimum unary effect over the prefix \(\alpha\), and \(|Q| + 1\)
(h) Minimum unary effect over the suffix \(\beta\). \(|Q| + 1\)

How many different characterisations?

\[ \leq |Q|^2(|Q| + 1)^4(2|Q| + 1)^2 \]
TECHNIQUE

“Polynomially Many Short Cycles”

The cycle replacement idea gives runs witnessing coverability that only contain one short cycle (that may be iterated many times) for each characterisation.

There are a polynomial number of different characterisations.

**Conclusion:** no more than a polynomial number of distinct short cycles need exist in any executable run witnessing coverability.
CHAPTER TWO
LOWER BOUNDS

Combinations of Encodings
Open Problems and Our Contributions
Technique: “Dual Counters”
## COMPLEXITY OF COVERABILITY

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<th>Binary encoded counter updates</th>
<th>Unary encoded counter updates</th>
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| Binary encoded initial and target vectors | $k \geq 2$: PSPACE-complete  
$k = 1$: only gap between $\text{NL}$ and in $\text{NC}^2$ | | |
| Unary encoded initial and target vectors | | $k \geq 1$: NL-complete  
No complexity gaps. | |
## Complexity of Coverability

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| **Binary encoded initial and target vectors** | $k \geq 2$: PSPACE-complete  
$k = 1$: only gap between NL and in $\text{NC}^2$ |  
| **Unary encoded initial and target vectors** | Reduces from above:  
*New initial and final states, add initial vector at start, and subtract target vector at end. Ask coverability to and from $\emptyset$.* | $k \geq 1$: NL-complete  
No complexity gaps. |
# Complexity of Coverability

## Various Encodings

<table>
<thead>
<tr>
<th>Encoding</th>
<th>Binary encoded counter updates</th>
<th>Unary encoded counter updates</th>
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<tbody>
<tr>
<td>Binary encoded</td>
<td>$k \geq 2$: PSPACE-complete</td>
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<tr>
<td>initial and target vectors</td>
<td>$k = 1$: only gap between NL and in NC$^2$</td>
<td>$k \geq 4$: NP-hard</td>
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<tr>
<td></td>
<td></td>
<td>$k \geq 8$: PSPACE-hard</td>
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OPEN PROBLEMS

Problem: Coverability in $k$-VASS with $k$ unary counters and binary encoded initial and target vectors.
OPEN PROBLEMS

**Problem:** Coverability in $k$-VASS with $k$ unary counters and binary encoded initial and target vectors.

**Problem:** Binary coverability in unary $k$-VASS.
OPEN PROBLEMS

**Problem:** Coverability in $k$-VASS with $k$ unary counters and binary encoded initial and target vectors.

**Problem:** Binary coverability in unary $k$-VASS.

**Complexity Gaps:**

- $k = 1$: NL-hard and in $\text{NC}^2$.
- $k = 2$: NL-hard and in $\text{NP}$.
- $k = 3$: NL-hard and in $\text{PSPACE}$.
- $4 \leq k \leq 7$: NP-hard and in $\text{PSPACE}$.
- $k \geq 8$: PSPACE-complete.
HARDNESS OF REACHABILITY

**Theorem**: Unary reachability in unary 3-VASS is NP-hard.

[Czerwiński and Orlikowski '22+]

*Proof approach*: reduce from SAT.
HARDNESS OF REACHABILITY

Theorem*: Unary reachability in unary 3-VASS is NP-hard.
[Czerwiński and Orlikowski '22+]

Proof approach: reduce from SAT.

Theorem: Unary reachability in unary 5-VASS is PSPACE-hard.
[Czerwiński and Orlikowski '22]

Proof approach: reduce from reachability in exponentially bounded two-counter automata.
OUR CONTRIBUTIONS

**Theorem**: Binary coverability in unary 4-VASS is NP-hard.

[our result]

*Proof approach*: reduce from unary reachability in unary 3-VASS using “dual counters” technique.
OUR CONTRIBUTIONS

**Theorem**: Binary coverability in unary 4-VASS is NP-hard.  
[our result]

*Proof approach*: reduce from unary reachability in unary 3-VASS using “dual counters” technique.

**Theorem**: Binary coverability in unary 8-VASS is PSPACE-hard.  
[our result]

*Proof approach*: reduce from unary reachability in unary 5-VASS using “dual counters” technique.
Consider a unary counter $c$, define its dual counter $d$ such that

- Whenever $c$ increments, $d$ decrements:

- Whenever $c$ decrements, $d$ increments:

- Whenever $c$ holds its value, so does $d$: 

$$ (1, -1) \rightarrow (0, 0) $$

If $c$ is initialised with $u$, then $d$ is initialised with $M^u$, where $M$ is at least the maximum possible value that $c$ can observe.
TECHNIQUE
“Dual Counters”

Consider a unary counter $c$, define its dual counter $d$ such that

- Whenever $c$ increments, $d$ decrements:
  \[
  (1, -1) \xrightarrow{\bullet} \bullet
  \]

- Whenever $c$ decrements, $d$ increments:
  \[
  (-1, 1) \xrightarrow{\bullet} \bullet
  \]

- Whenever $c$ holds its value, so does $d$:
  \[
  (0, 0) \xrightarrow{\bullet} \bullet
  \]

If $c$ is initialised with $u$, then $d$ is initialised with $M - u$, where $M$ is at least the maximum possible value that $c$ can observe.

Coverability targets $c \geq v$ and $d \geq M - v$ implies $c = v$ must hold.
REDUCTION CHALLENGES

Unary reachability in unary 3-VASS is NP-hard, so by taking all dual counters, binary coverability in unary 6-VASS is NP-hard.
REDUCTION CHALLENGES

Unary reachability in unary 3-VASS is NP-hard, so by taking all dual counters, binary coverability in unary 6-VASS is NP-hard.

Similarly, unary reachability in unary 5-VASS is PSPACE-hard, so by taking all dual counters, binary coverability in unary 10-VASS is PSPACE-hard.
REDUCTION CHALLENGES

Unary reachability in unary 3-VASS is NP-hard, so by taking all dual counters, binary coverability in unary 6-VASS is NP-hard.

Similarly, unary reachability in unary 5-VASS is PSPACE-hard, so by taking all dual counters, binary coverability in unary 10-VASS is PSPACE-hard.

Which dual counters are really necessary?
CONCLUSION

**Theorem:** Coverability in 2-VASS with one binary counter and one unary counter is in NP. [our result]

**Open Problem:** Is reachability also in NP?
CONCLUSION

**Theorem:** Coverability in 2-VASS with one binary counter and one unary counter is in NP.

**Open Problem:** Is reachability also in NP?

**Open Problem:** is there a $k < 4$ such that binary coverability in unary $k$-VASS is NP-hard?

**Open Problem:** is there a $k < 8$ such that binary coverability in unary $k$-VASS is PSPACE-hard?
CONCLUSION

**Theorem:** Coverability in 2-VASS with one binary counter and one unary counter is in NP. [our result]

**Open Problem:** Is reachability also in NP?

**Open Problem:** is there a $k < 4$ such that binary coverability in unary $k$-VASS is NP-hard?

**Open Problem:** is there a $k < 8$ such that binary coverability in unary $k$-VASS is PSPACE-hard?

THANK YOU!

*Presented by Henry Sinclair-Banks, University of Warwick* 🇬🇧

*For OFCOURSE, MPI–SWS, Kaiserslautern* 🇩🇪