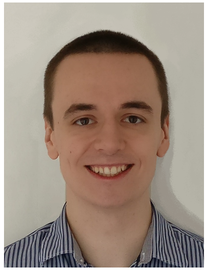
The Complexity of Coverability in Vector Addition Systems with States

Henry Sinclair-Banks

About joint work with Marvin Künnemann, Filip Mazowiecki, Lia Schütze, and Karol Węgrzycki.

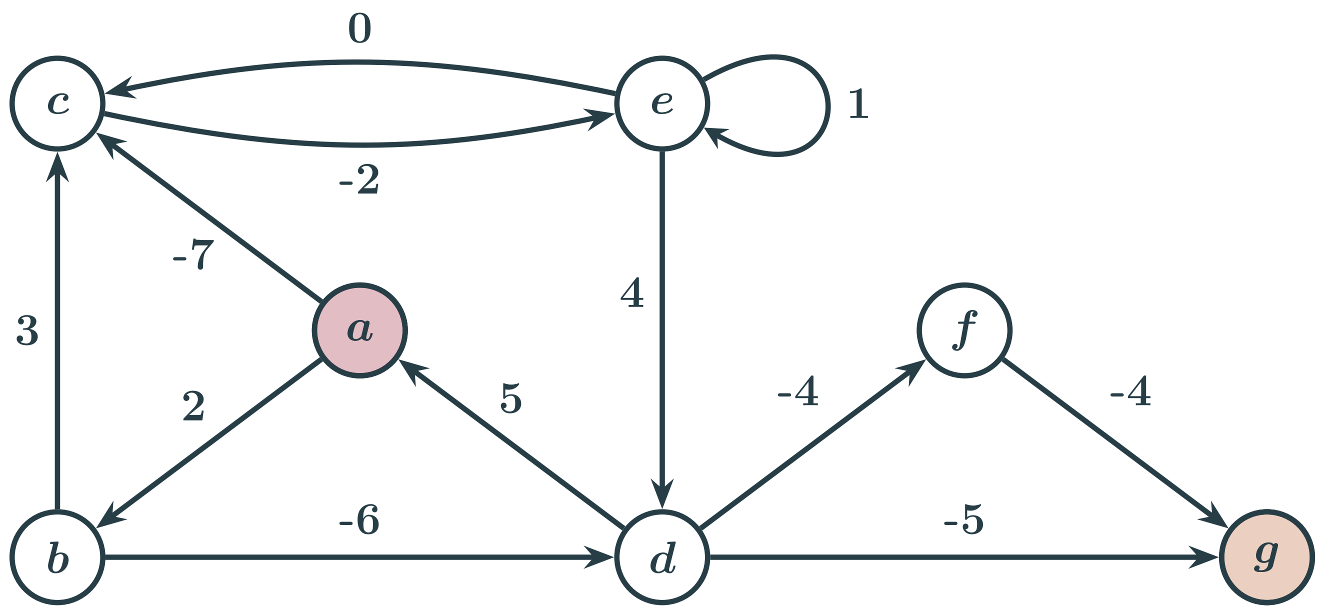


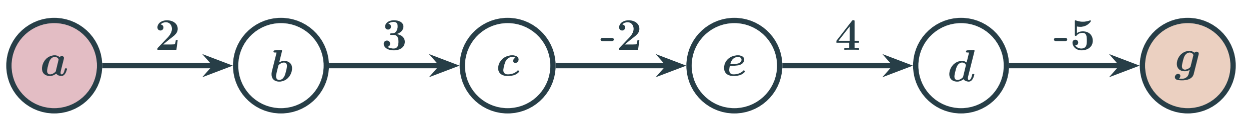

Warwick FoCS Theory Day

8th June 2023

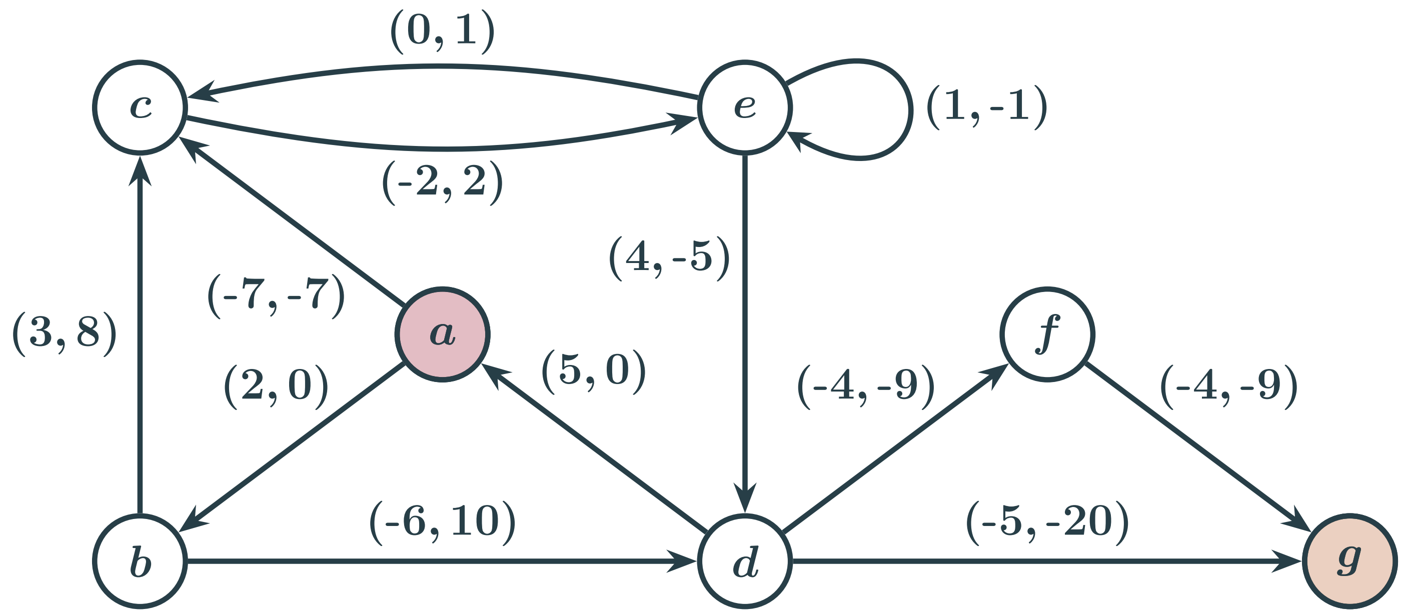
Never Negative Paths in Weighted Graphs



Question: from a can you reach g via a path that is *never negative* ? **YES!**

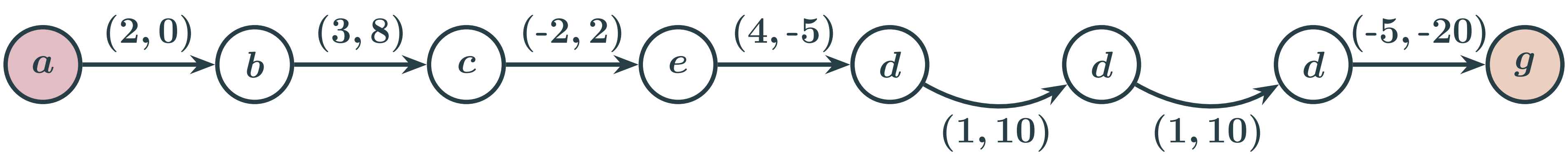


Never Negative Paths in Multi-Weighted Graphs

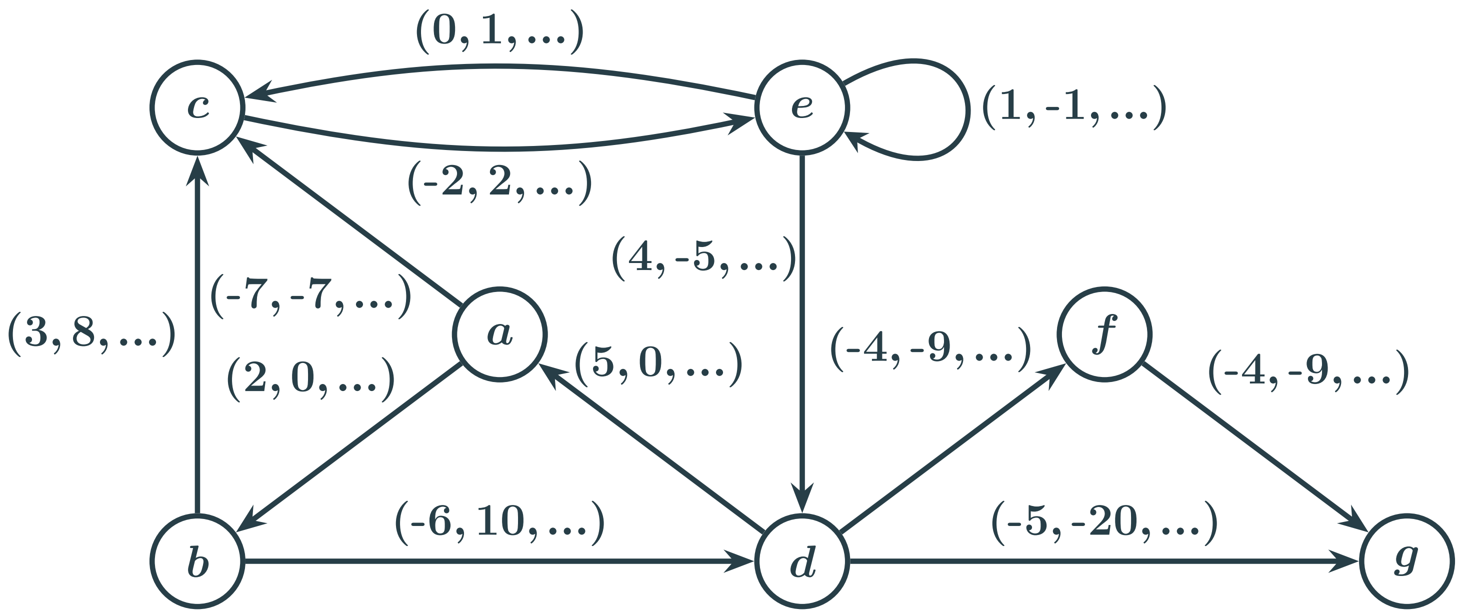


YES!

Question: from a can you reach g via a path that is *never negative on any component* ?



Coverability in Vector Addition Systems with States



Coverability problem: from p can you reach q via a path that is never negative on any component ?

VASS $\implies$ dimension is not fixed

d-VASS $\implies$ dimension d is fixed

Size of a *transition* is the absolute value of its maximum weight.

Size of a VASS n is the number of *states* plus sizes of all transitions.

History of Coverability

Theorem: Coverability in VASS is EXPSPACE-hard. [Lipton '76]

"Lipton's construction": there are instances only admitting $n^{2^{\Omega(d)}}$ length runs.

$\Longrightarrow$ Coverability in VASS requires $2^{\Omega(d)} \cdot \log(n)$-space.

> *Theorem 1:* The reachability problem for vector addition systems requires at least 2^{cn} space infinitely often for some constant $c > 0$.

Theorem: Coverability in VASS is in EXPSPACE. [Rackoff '78]

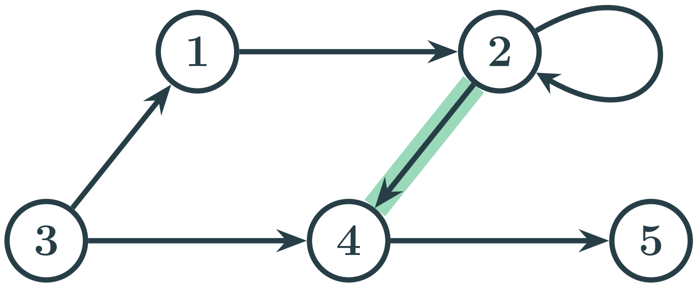
"Rackoff's bounding technique": argue (inductively) that there are always $n^{2^{\mathcal{O}(d \log d)}}$ length runs.

$\Longrightarrow$ Coverability in VASS can be decided in $2^{\mathcal{O}(d \log d)} \cdot \log(n)$-space.

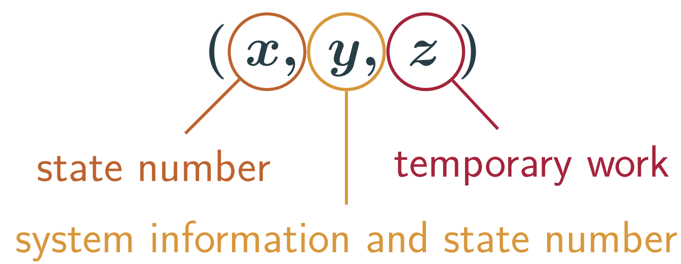
> **Theorem 3.5.** *The covering problem can be decided in space $2^{cn \log ..}$ for some constant c.*

Vector Addition Systems (without states)

Theorem: States and transitions can be *simulated* by 3 non-negative counters. [Hopcroft and Pansiot '79]



Fix $k = 6$, one more than the number of states.



Precondition —three steps→ Postcondition

$x \leftarrow 2,\quad y \leftarrow k(k-2),\quad z \leftarrow 0$ $\quad\longrightarrow\quad$ $x \leftarrow 4,\quad y \leftarrow k(k-4),\quad z \leftarrow 0$

Lemma 2.1. *An n-dim* VASS *can be simulated by an* $(n+3)$-dim VAS.

Motivation to Revisit Coverability

Coverability in VAS can be decided in $2^{\mathcal{O}(d \log d)} \cdot \log(n)$-space.

Coverability in *bidirected* VAS requires $2^{\Omega(d)} \cdot \log(n)$-space.

[Rackoff '78]

3. In [35] it has been shown that the *VRS covering problem*, to decide given $(\alpha, \beta, \mathcal{T})$ whether $\beta\gamma \in \mathcal{T}[\alpha]$ for some word γ, is decidable in space $c^{n^2 \log n}$. Our reduction of *ESC* to *CSG* implies a lower bound of space d^n for some $d > 1$. (This lower bound was originally obtained by Lipton [24].) Improve these bounds.

[Lipton '76]

[Mayr and Meyer '82]

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[Lipton '76]

[Mayr and Meyer '82]

**later refined by multiparameter analysis.

Using a similar approach, as was used in Section 2, an upper bound of $O((l+\log n)*2^{c*k*\log k})$ can be shown for the covering problem. (Finding better upper/lower bounds for this problem was mentioned as an open problem in [18].)

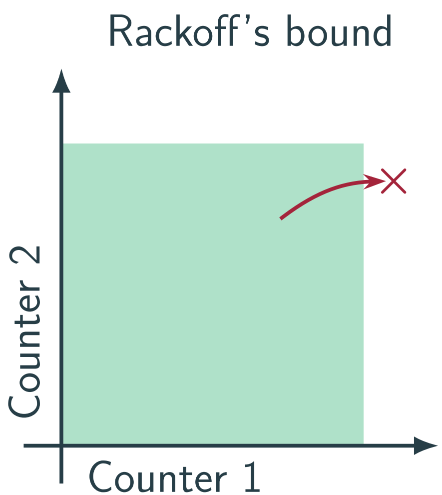
[Rosier and Yen '85]

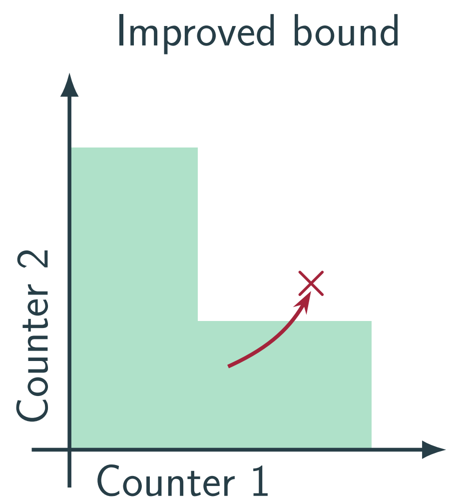
Improving Rackoff's Space Upper Bound

Theorem: Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.

[Künnemann, Mazowiecki, Schütze, Sinclair-Banks, and Węgrzycki '23]

Main idea is to carefully use "Rackoff's bounding technique" with sharper counter value bounds.





$\Longrightarrow$ Coverability in VASS can be decided in $2^{\mathcal{O}(d)} \cdot \log(n)$-space. **OPTIMAL!**

$\Longrightarrow$ Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$-time.

Conditionally Optimal Time Bound

$\implies$ Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$-time.

Theorem: Assuming the Exponential Time Hypothesis, there are no $n^{2^{o(d)}}$-time algorithms for coverability in VASS. [Künnemann, Mazowiecki, Schütze, Sinclair-Banks, and Węgrzycki '23]

Exponential Time Hypothesis $\implies$ there are no $n^{o(k)}$-time algorithms for finding a k-clique in a graph.

Main idea is to reduce the problem of finding a $k = 2^d$-clique in a graph to coverability in $\mathcal{O}(d)$-VASS.

$\implies$ Coverability in VASS conditionally requires $n^{2^{\Omega(d)}}$-time.

CONDITIONALLY OPTIMAL!

Conclusion

1976: Coverability in VASS requires $2^{\Omega(d)} \cdot \log(n)$-space**.

1978: Coverability in VASS can be decided in $2^{\mathcal{O}(d \log d)} \cdot \log(n)$-space**.

1985: **Refined by multiparameter analysis of coverability in VASS.

2023: Coverability in VASS can be decided in $2^{\mathcal{O}(d)} \cdot \log(n)$-space and can be decided in $n^{2^{\mathcal{O}(d)}}$-time.

2023: Coverability in VASS requires $n^{2^{\Omega(d)}}$-time, under the Exponential Time Hypothesis.

Thank You!

Presented by Henry Sinclair-Banks
Warwick FoCS Theory Day 2023

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