

The Tractability Border of Reachability in Simple Vector Addition Systems with States

Henry Sinclair-Banks

Based on work with Dmitry Chistikov, Wojciech Czerwiński, Filip Mazowiecki, Łukasz Orlikowski, and Karol Węgrzycki to appear in FOCS'24.

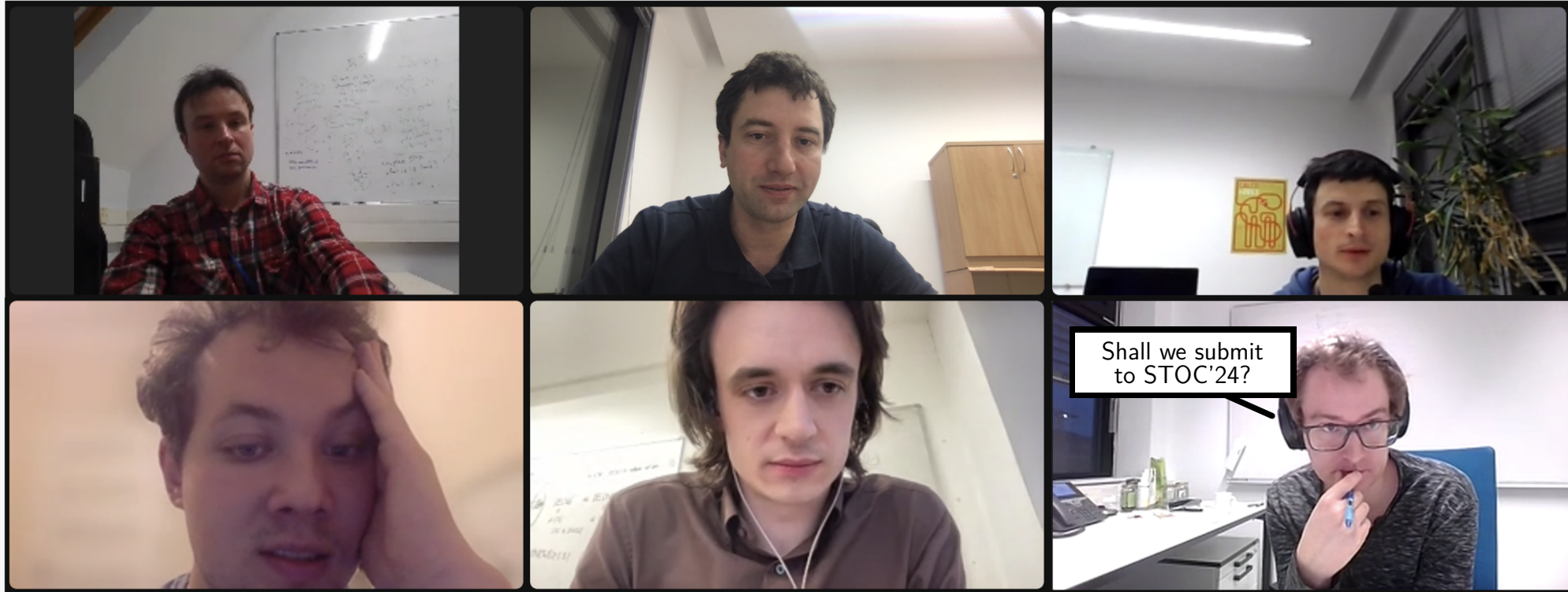


Highlights'24: Vector Addition Systems

19th September 2024

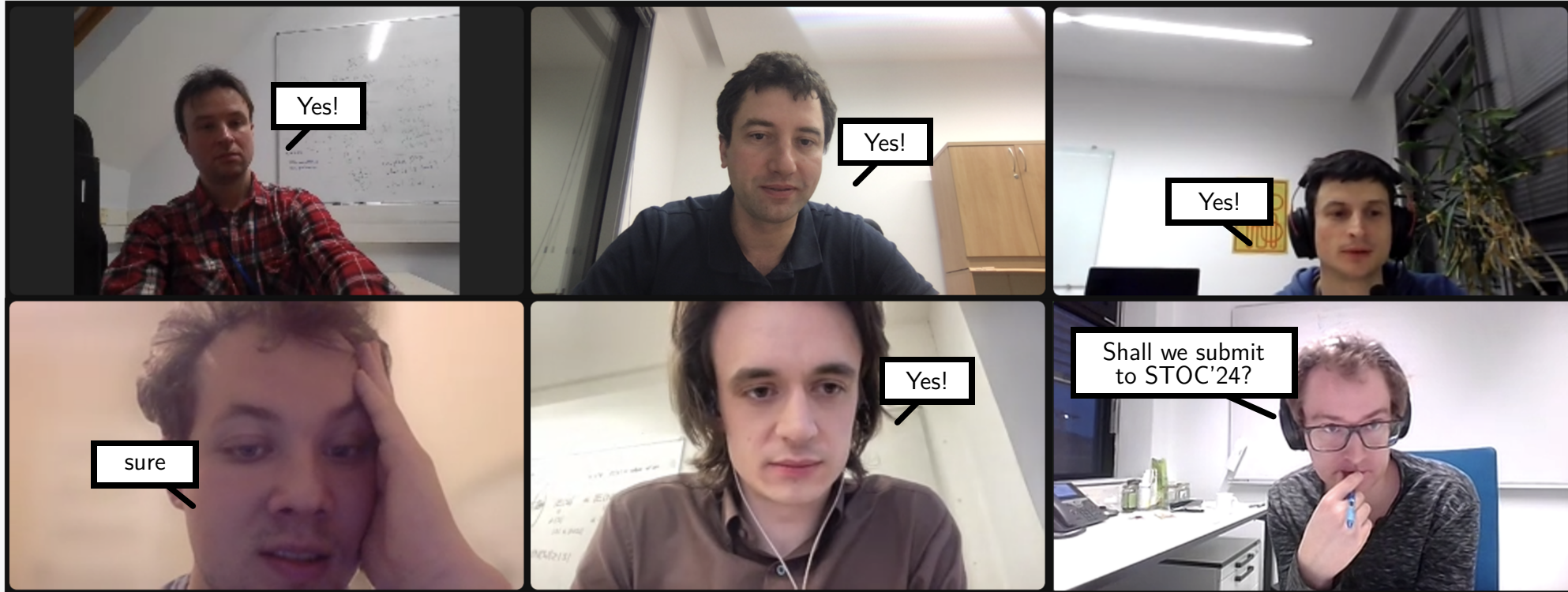
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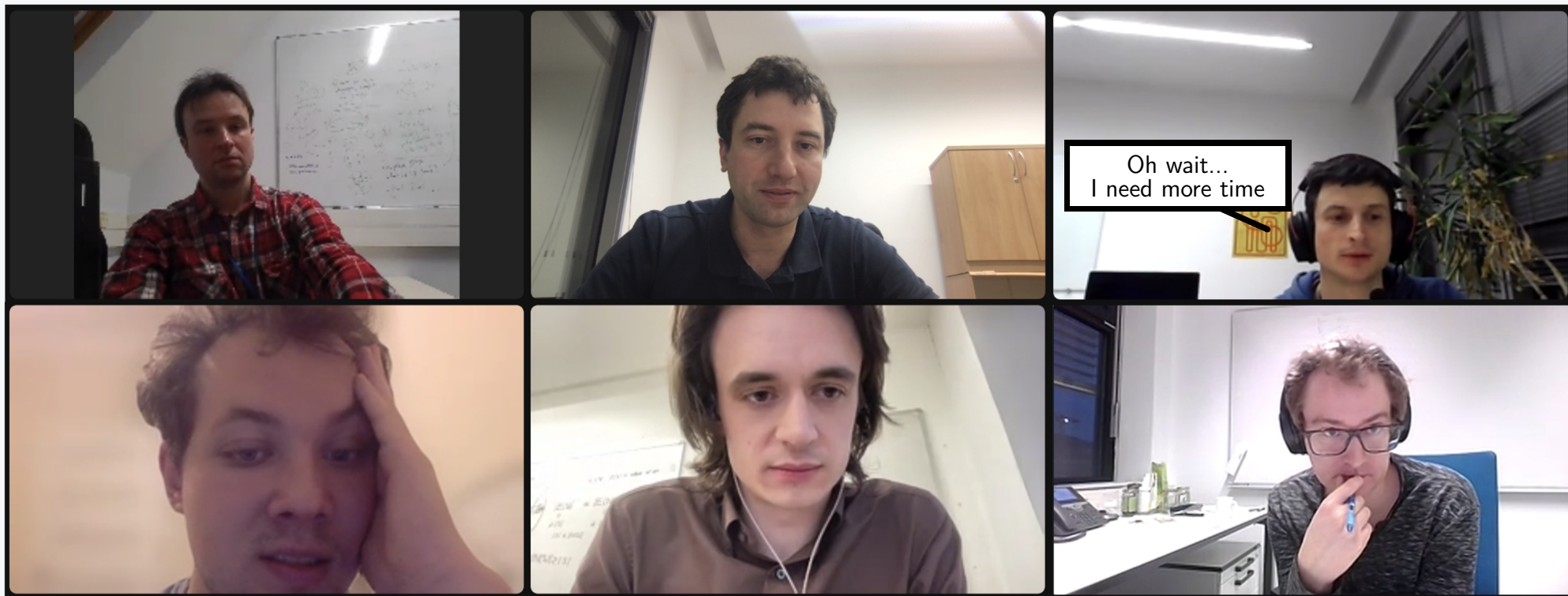
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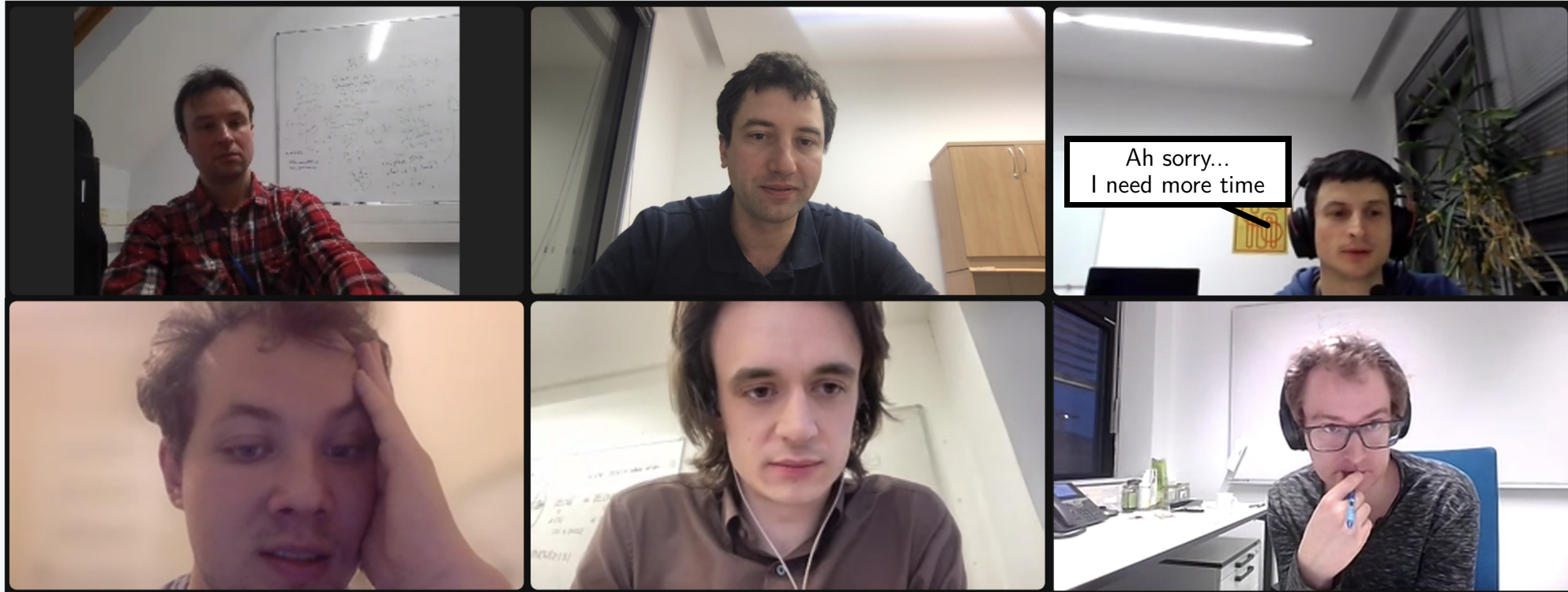
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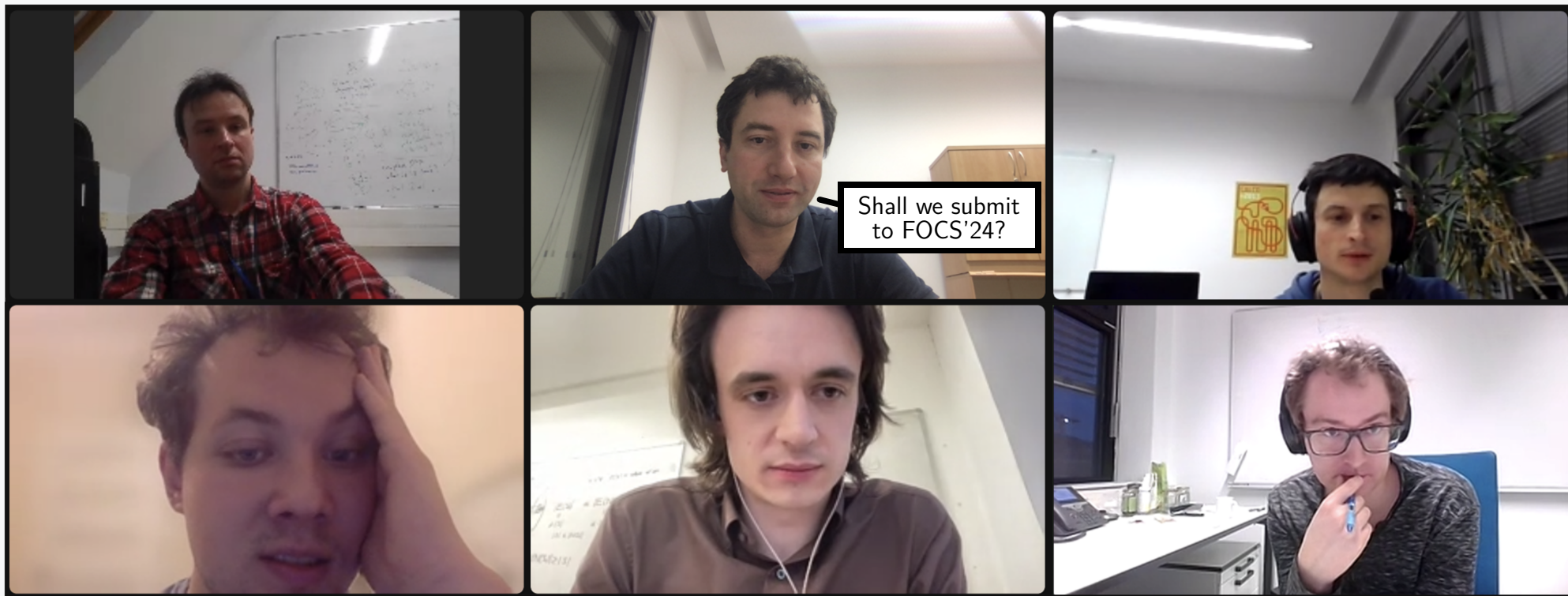
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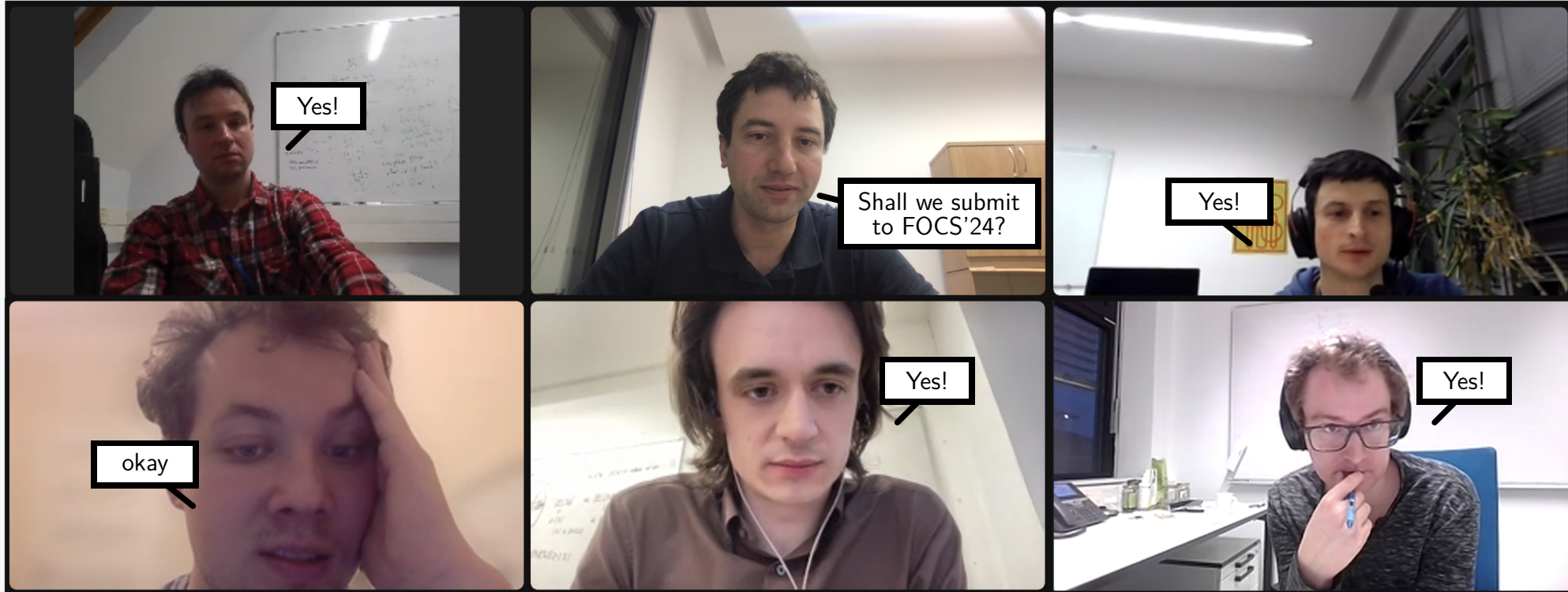
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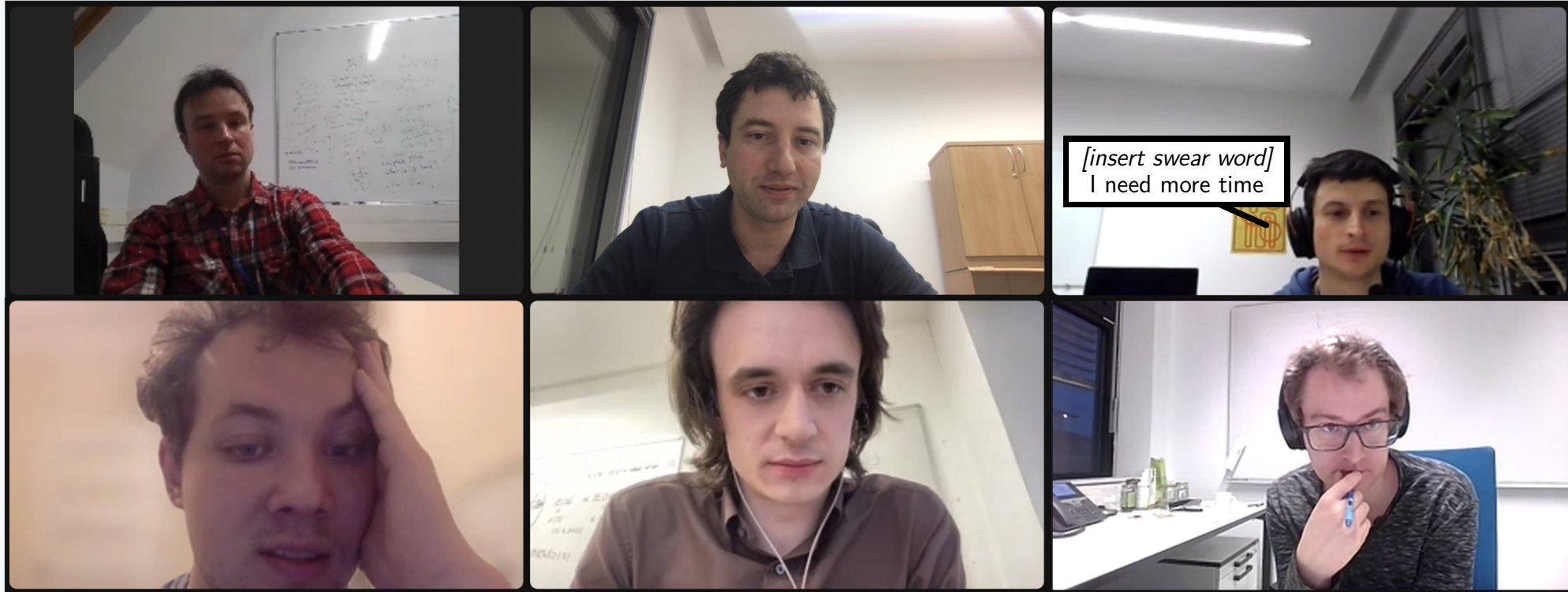
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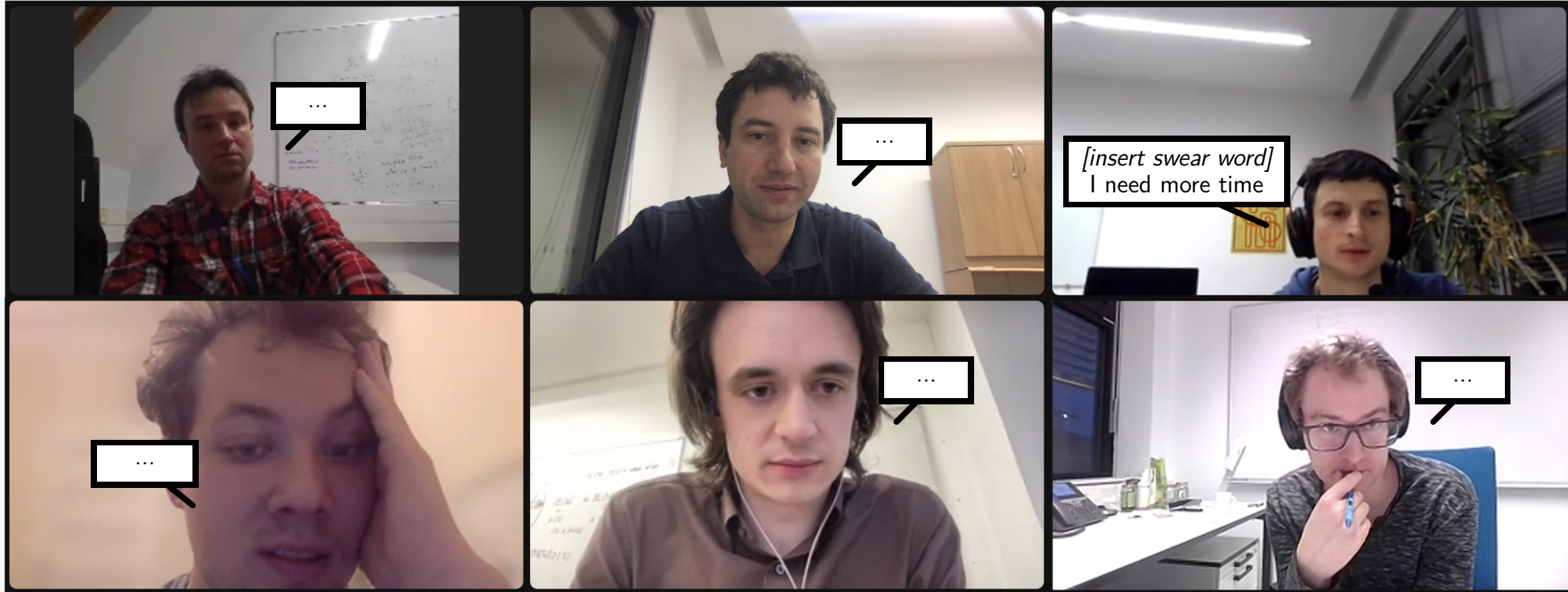
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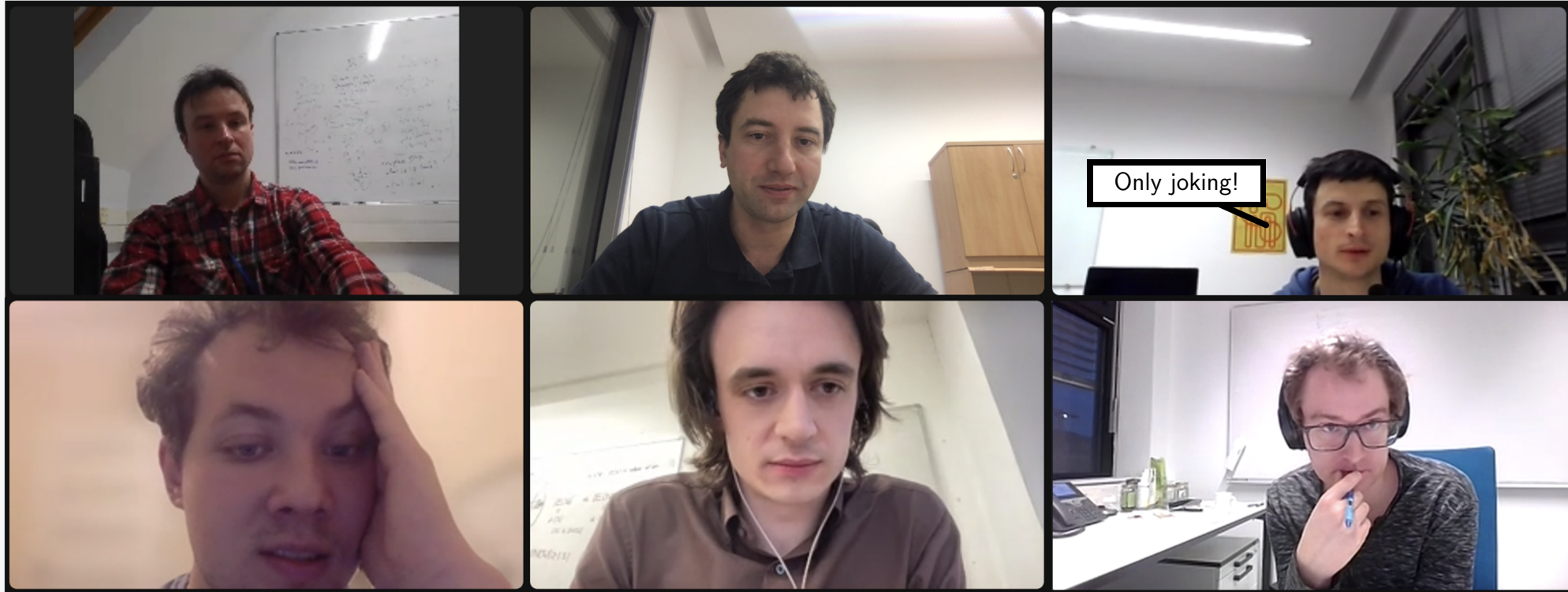
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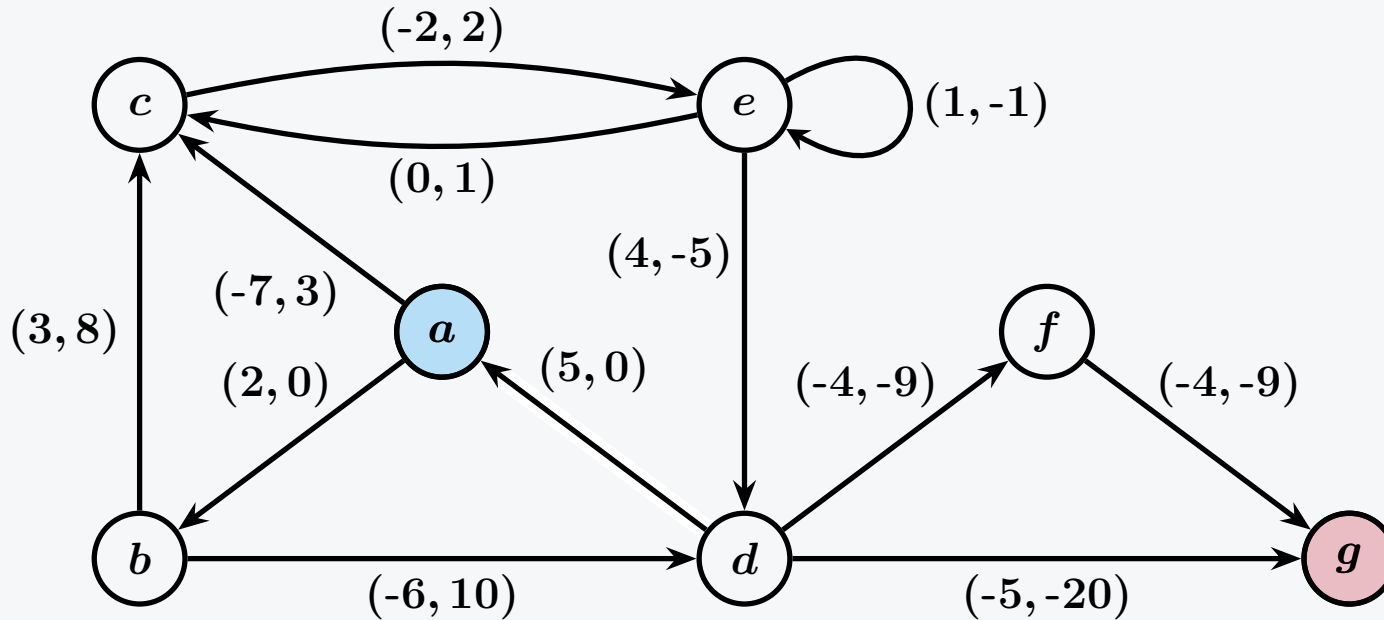
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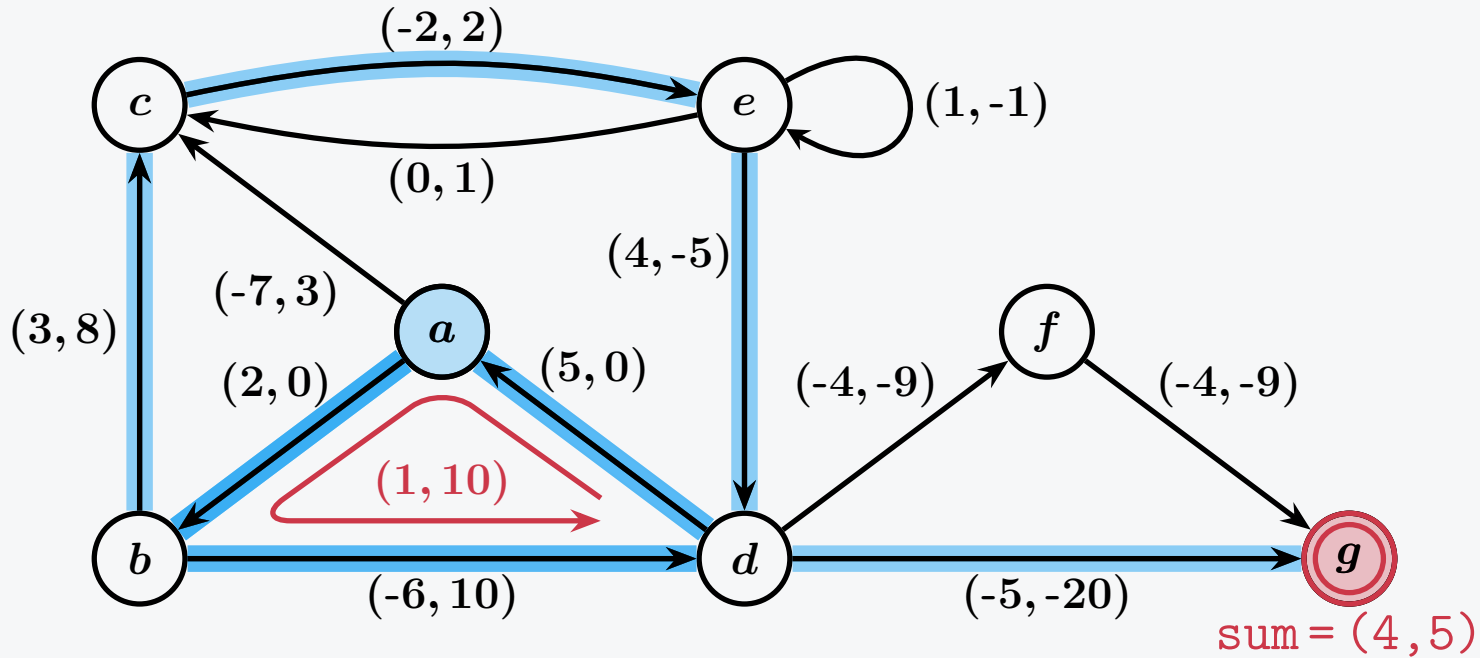
Reachability in 2-Dimensional VASS



Does there exist a run from a with counter values $(0, 0)$ to g with counter values $(4, 5)$?

(the counters must remain nonnegative at all times)

Reachability in 2-Dimensional VASS

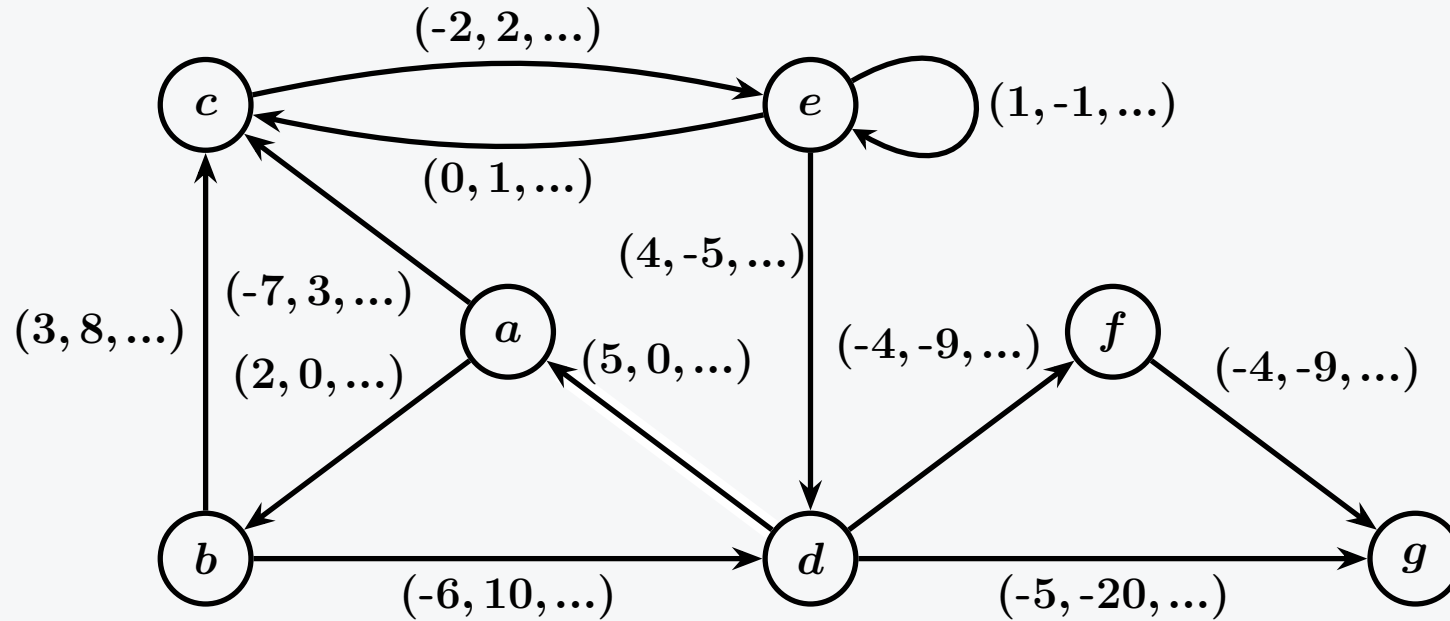


Does there exist a run from a with counter values $(0, 0)$ to g with counter values $(4, 5)$?

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YES!

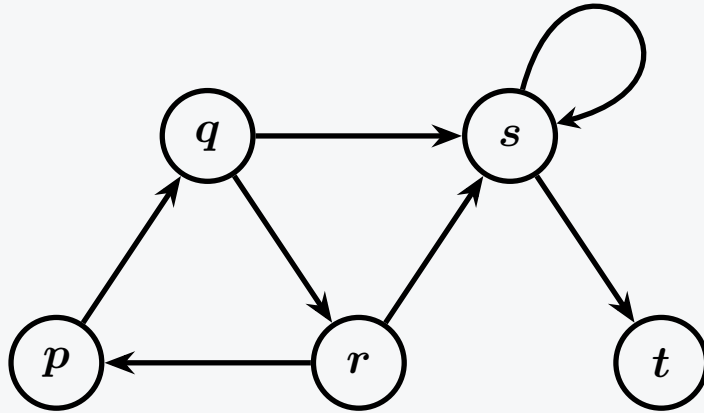
Reachability in VASS



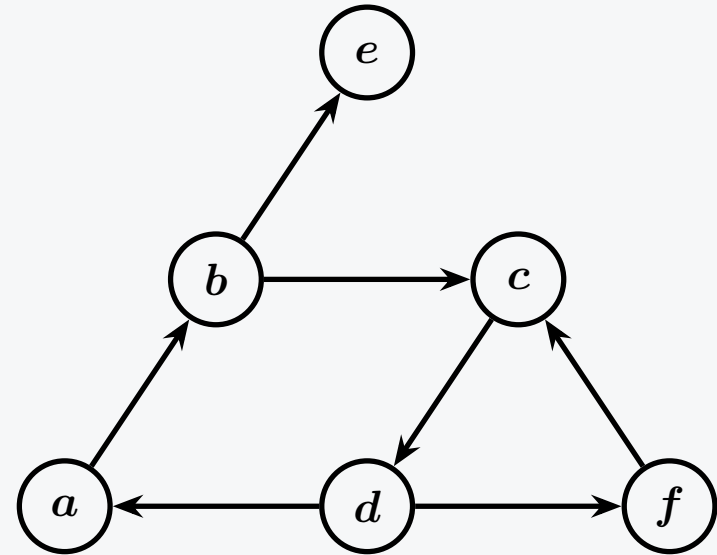
Reachability problem: does there exist a run from $p(u)$ to $q(v)$?

What is a Flat VASS?

Definition (Flat). For every state $q \in Q$, there is at most one simple cycle that contains q .



Flat :)



Not flat :(

Reachability in Flat VASS

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Theorem. Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Theorem. Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

Theorem. Reachability in unary (flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73]
[Englert, Lazić, and Totzke '16]

Theorem. Reachability in unary flat d -VASS is NP-hard for $d = 7$. [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]
... for $d = 5$. [Dubiak '20]
... for $d = 4$. [Czerwiński and Orlikowski '22]

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What is the complexity of reachability in unary flat 3-VASS?

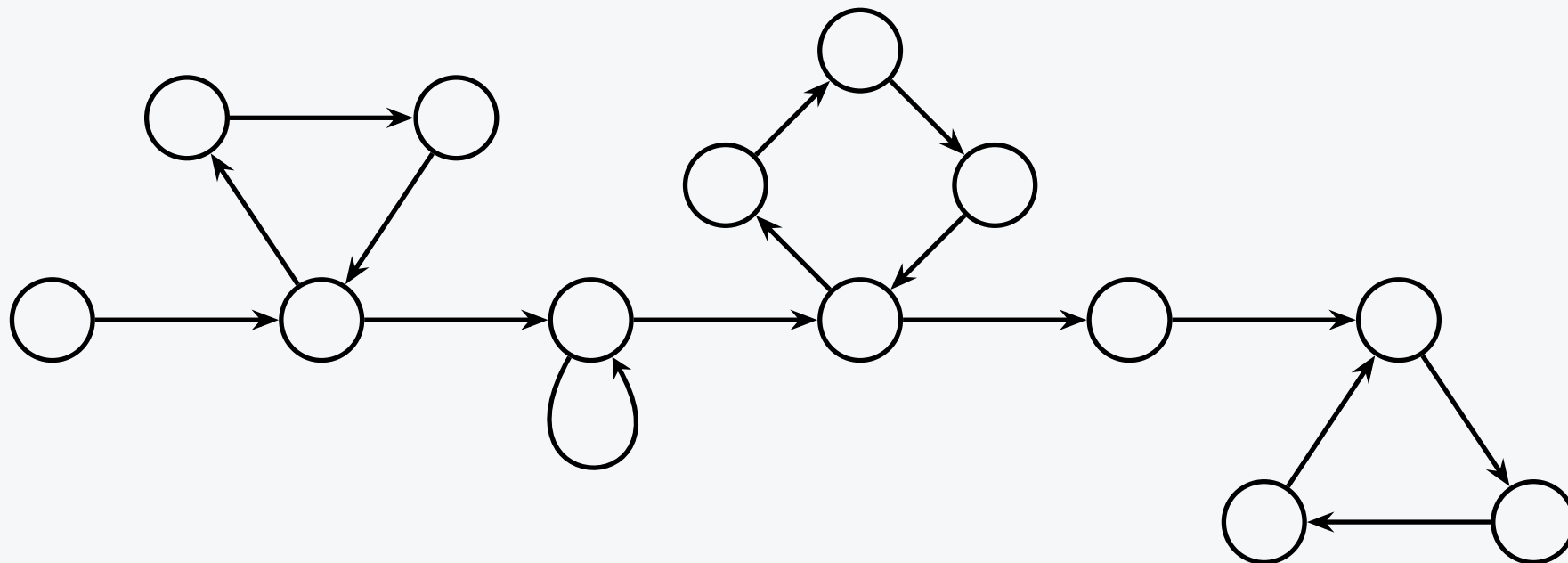
Theorem. Reachability in binary flat 3-VASS is NP-hard. [Kosier and Yen '85]
[Blondin, Finkel, Göller, Haase, and McKenzie '15]

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~~Flat VASS~~ Linear Path Schemes

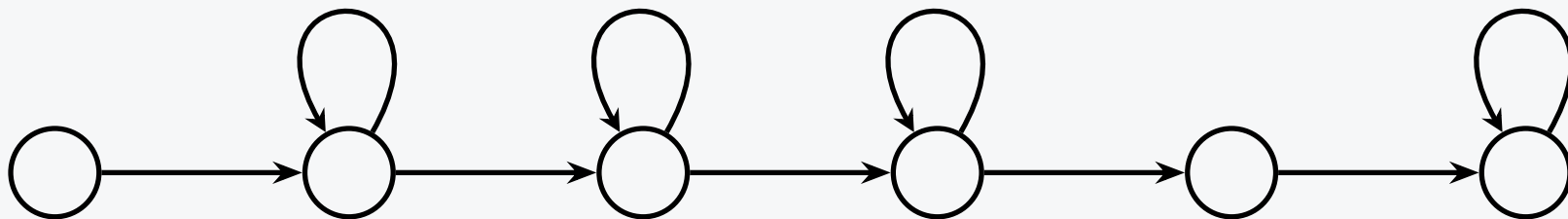
Definition (LPS). A VASS where the states and transitions form a simple path between disjoint cycles.



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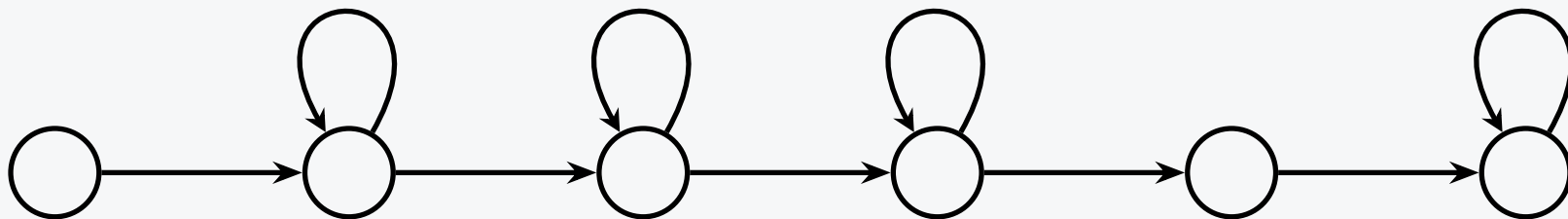
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Open problem. For $d \geq 3$, is reachability in unary d -dimensional linear path schemes in P?

[Englert, Lazić, and Totzke '16] [Leroux '21]

Theorem. Reachability in unary 3-dimensional *simple* linear path schemes is NP-hard. [This paper]

Reachability in Unary 3-SLPS is NP-hard

Proof idea:

1) Use “Chinese remainder encoding” of a SAT instance with k variables.

Let $n \in \mathbb{N}$ represent an assignment and let $p_1, \dots, p_k$ be the first k primes.

Encoding: $x_i = \text{true} \iff n \equiv 1 \pmod{p_i}$ and $x_i = \text{false} \iff n \equiv 0 \pmod{p_i}$.

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2) Turn the SAT instance into a conjunction of non-divisibility checks.

Example: $x_1 \vee \neg x_2 \vee x_3$; need to check $n \equiv 1 \pmod{2}$ or $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{5}$.

Only falsified when $n \equiv 10 \pmod{2 \cdot 3 \cdot 5}$, so check $2 \cdot 3 \cdot 5 \nmid n - 10$.

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Come and ask me for the construction – I promise its very neat!

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4) Simulate the zero tests with a dedicated third counter. We use the “controlling counter technique”.

[Czerwiński and Orlikowski '21]

The Tractability Border of Reachability in Simple Vector Addition Systems with States

Theorem. Reachability in unary 3-SLPS is NP-complete.

Theorem. Reachability in unary *ultraflat* 4-VASS is NP-complete.

Theorem. Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

Theorem. Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

Thank You!



Presented by Henry Sinclair-Banks

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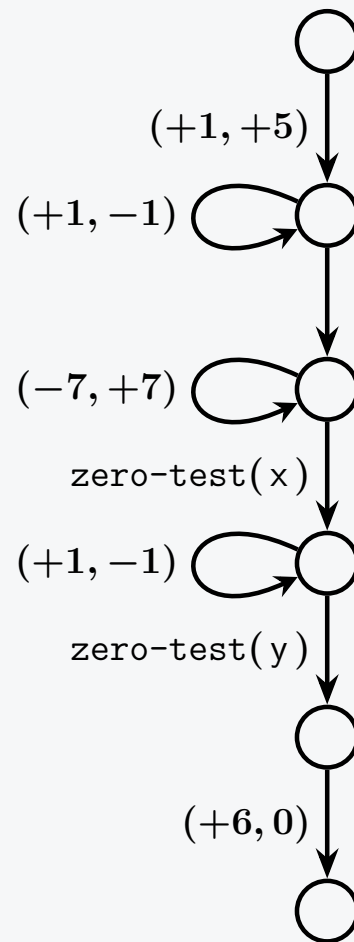


Non-Divisibility Testing Simple Linear Path Schemes

Suppose we want to perform a non-divisibility test $v \not\equiv 0 \pmod{7}$.

Let's construct a 2-SLPS with zero tests that:

- starts with $x = v$, $y = 0$,
- can only be passed if $v \not\equiv 0 \pmod{7}$, and
- ends with $x = v$, $y = 0$.



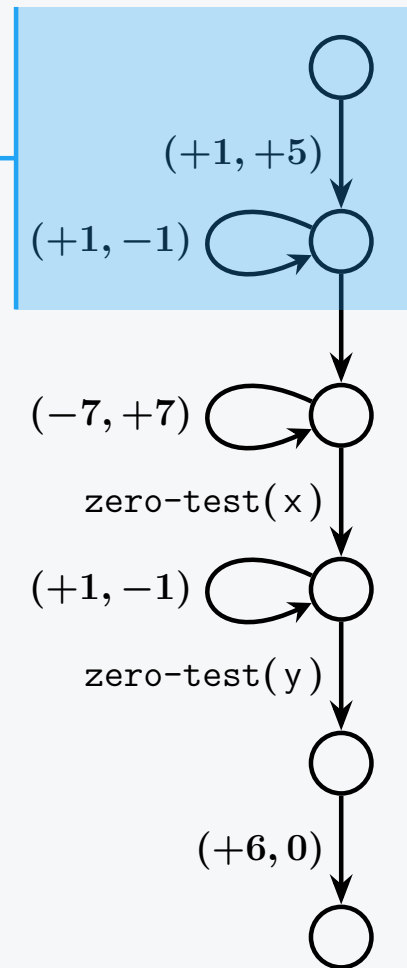
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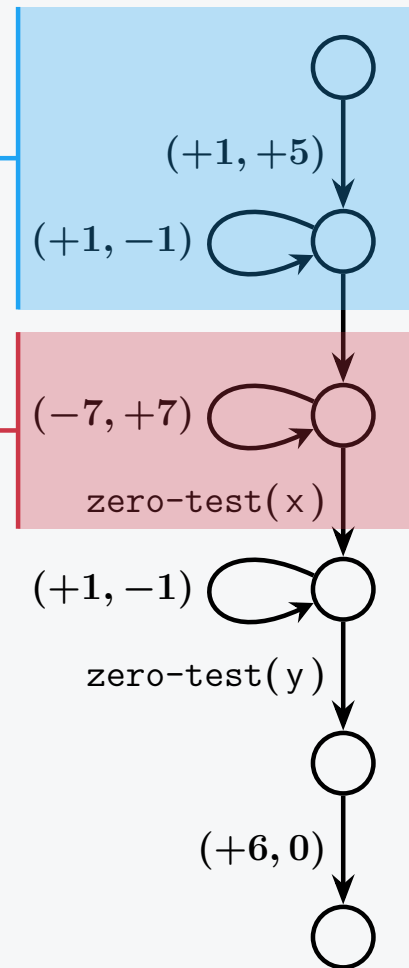
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(ii) ... such that $7 \mid v + r$.



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(ii) ... such that $7 \mid v + r$.

(iii) Restore $x = v, y = 0$.

