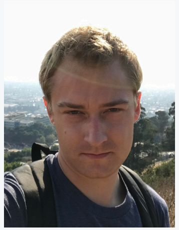
The Tractability Border of Reachability in Simple Vector Addition Systems with States

Henry Sinclair-Banks

Based on work with Dmitry Chistikov, Wojciech Czerwiński, Filip Mazowiecki, Łukasz Orlikowski, and Karol Węgrzycki to appear in FOCS'24.





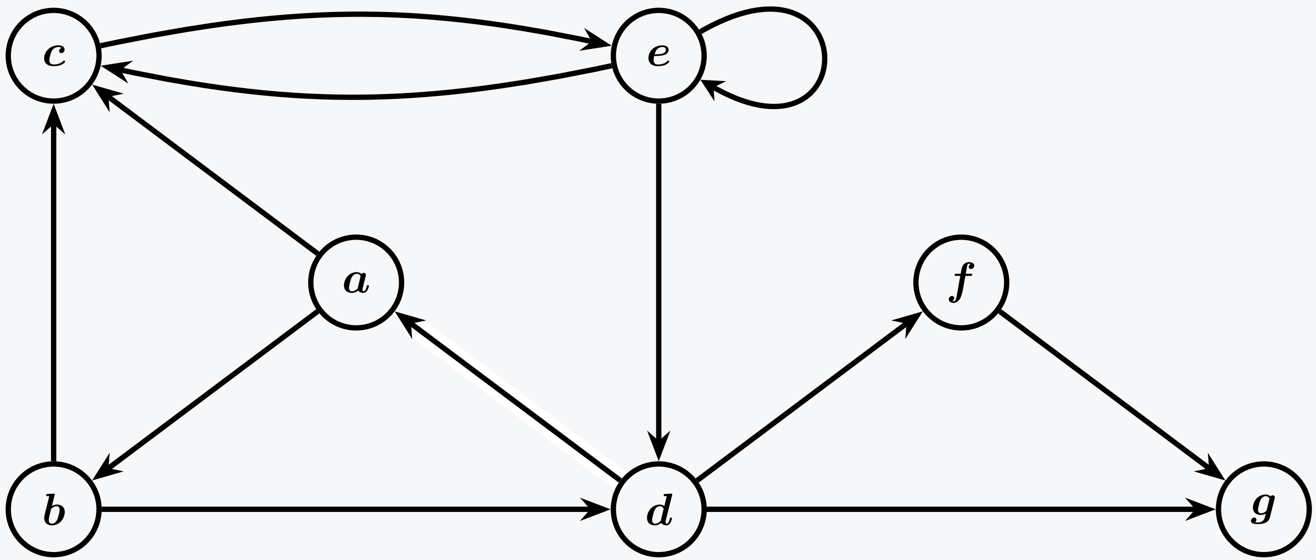



Algorithms & Complexity Seminar

27th August 2024

KIT, Karlsruhe, Germany

2-Dimensional Vector Addition System with States



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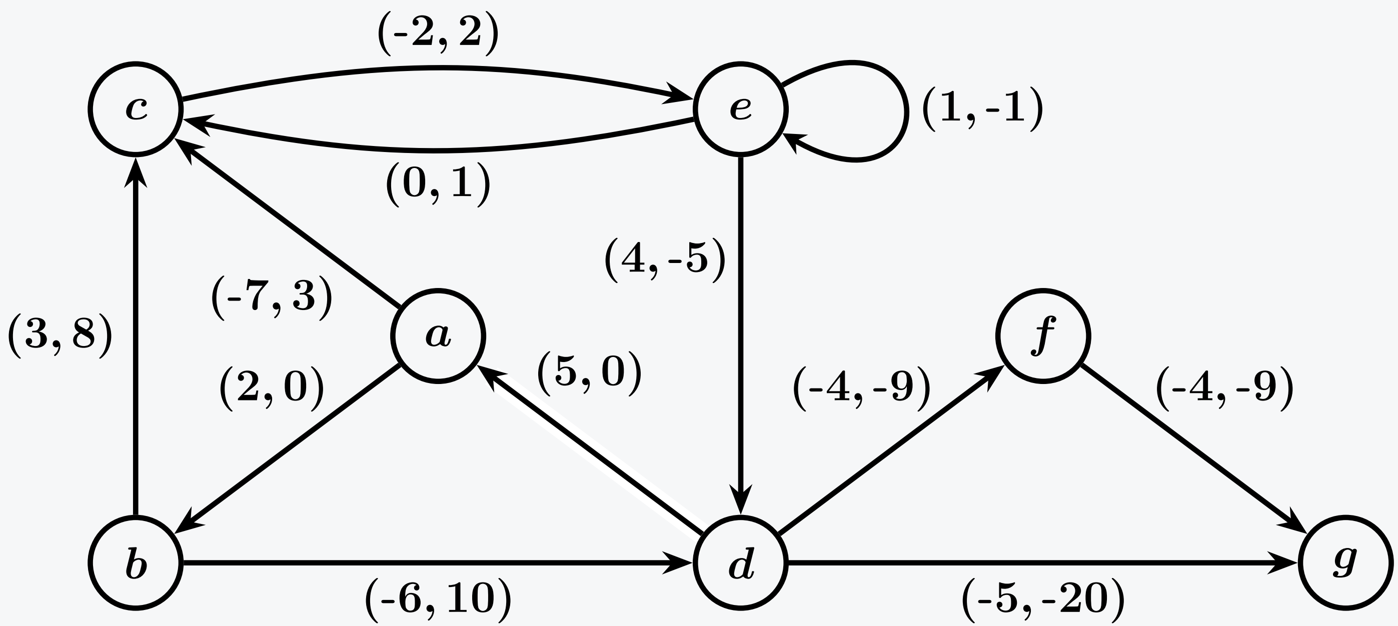
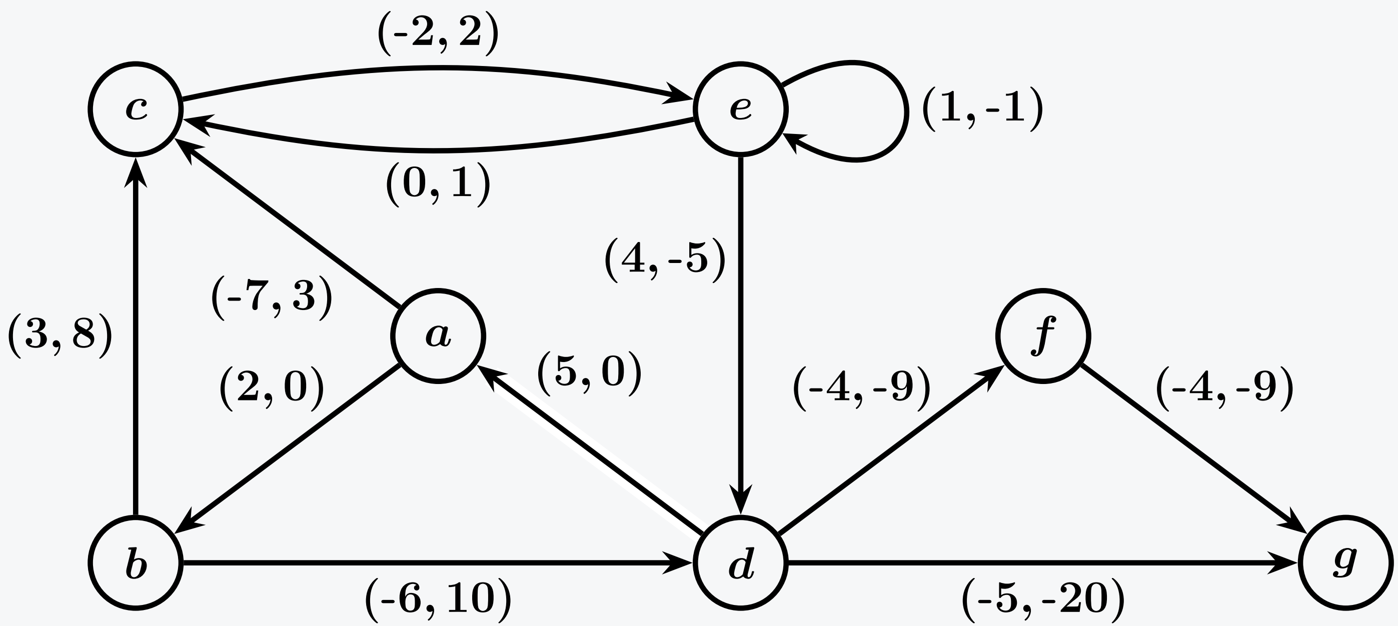
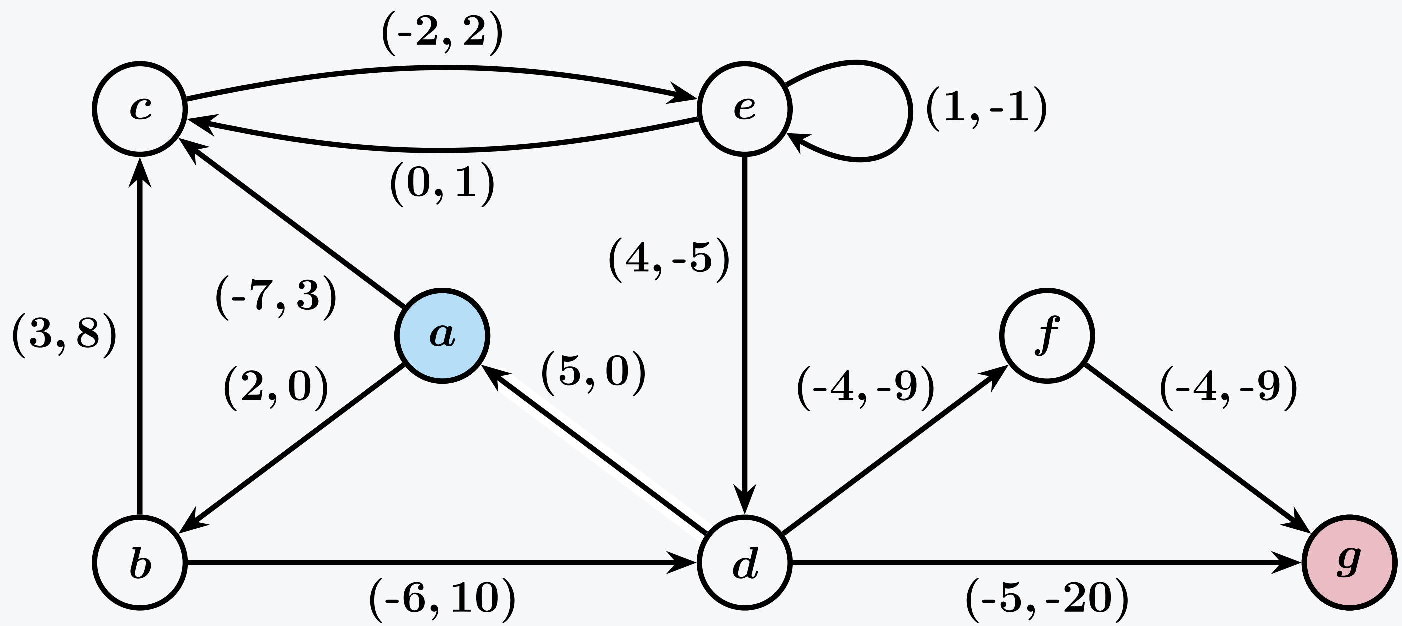


2-Dimensional VASS



Reachability in 2-Dimensional VASS



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(the counters must remain nonnegative at all times)

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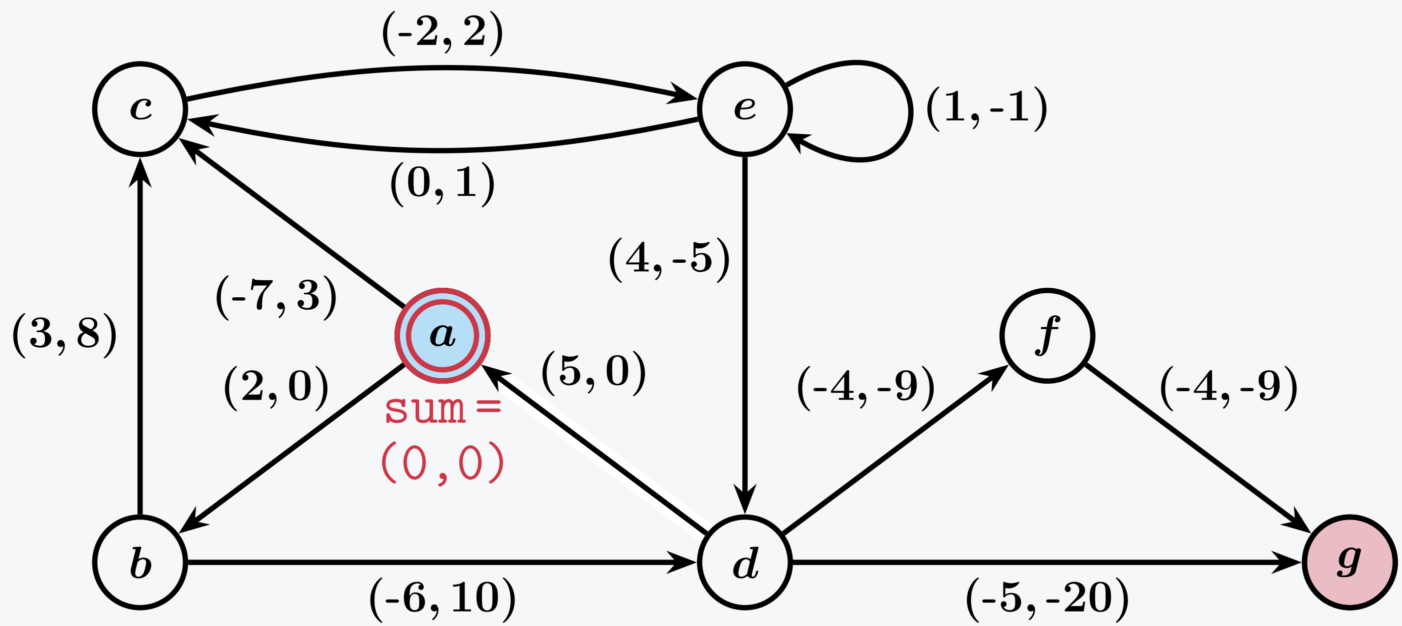

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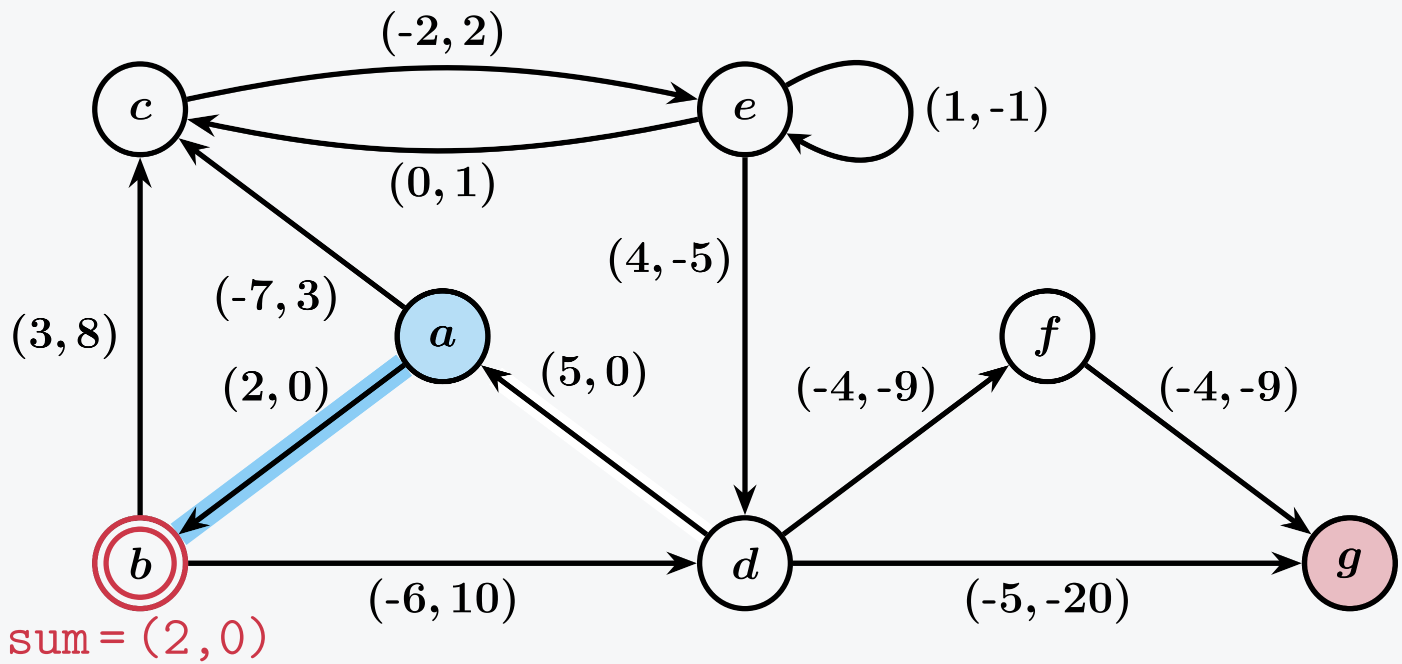

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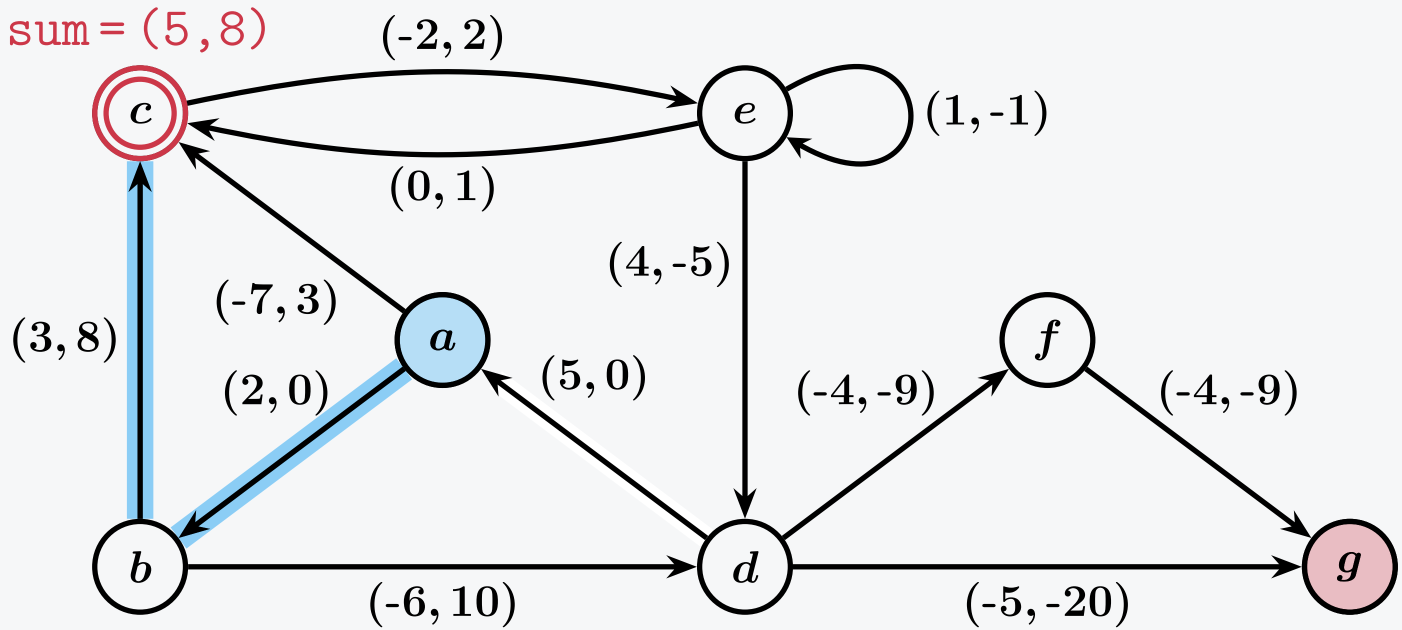

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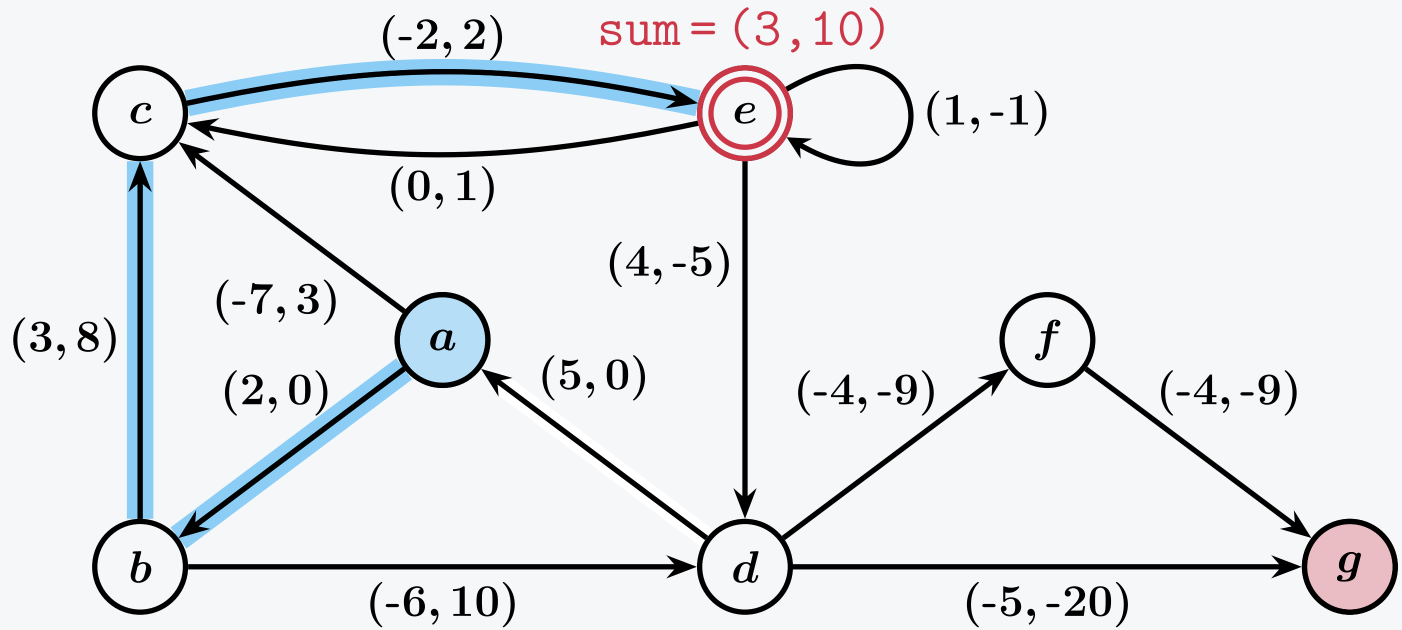

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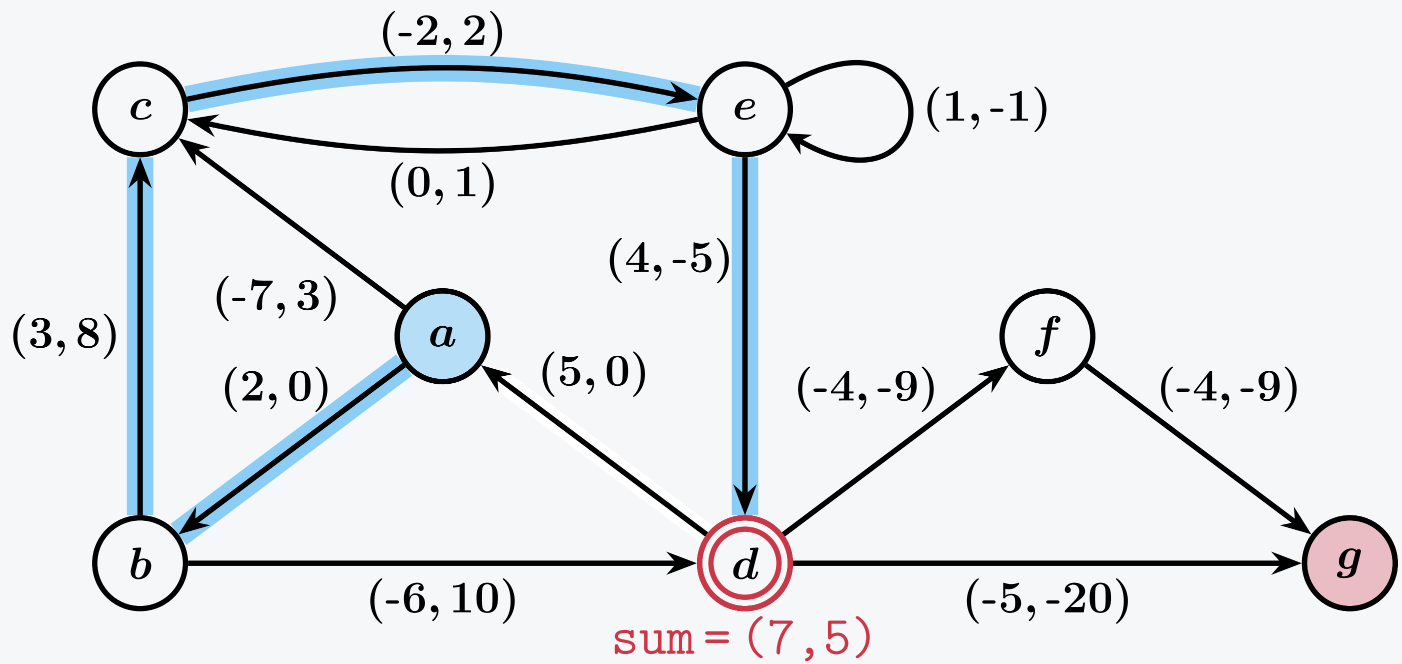

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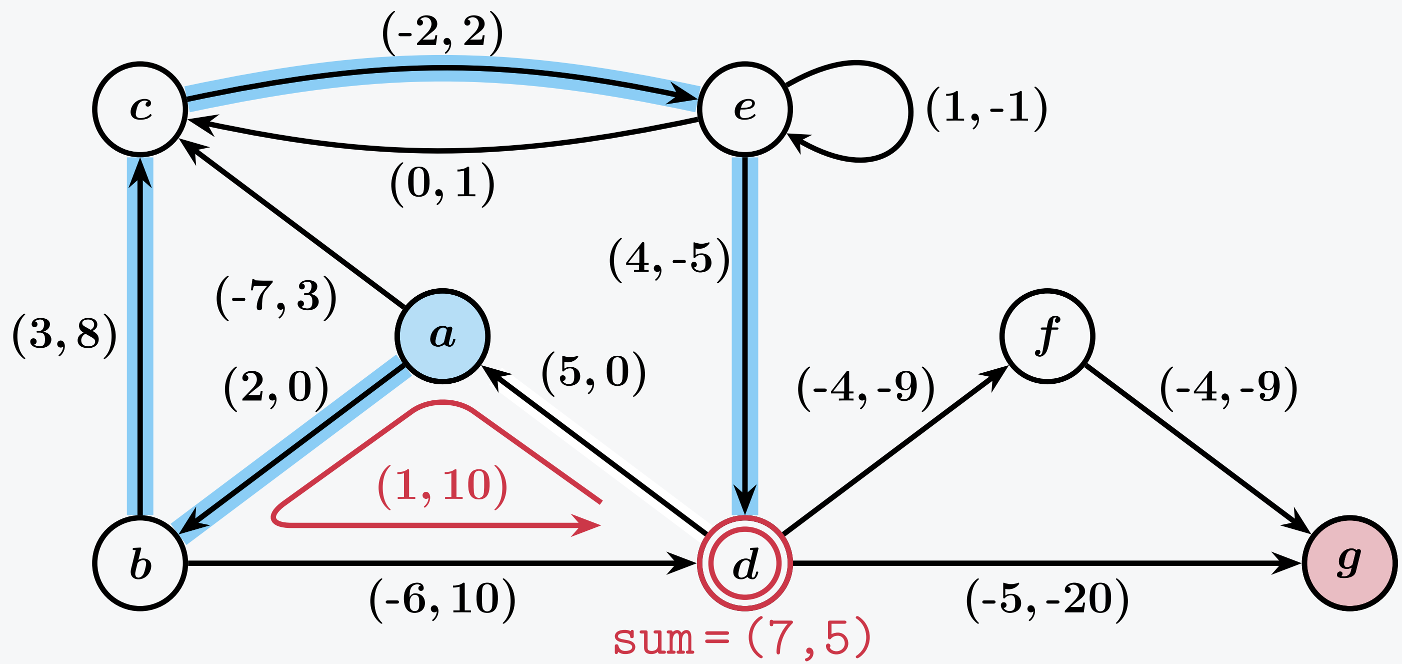

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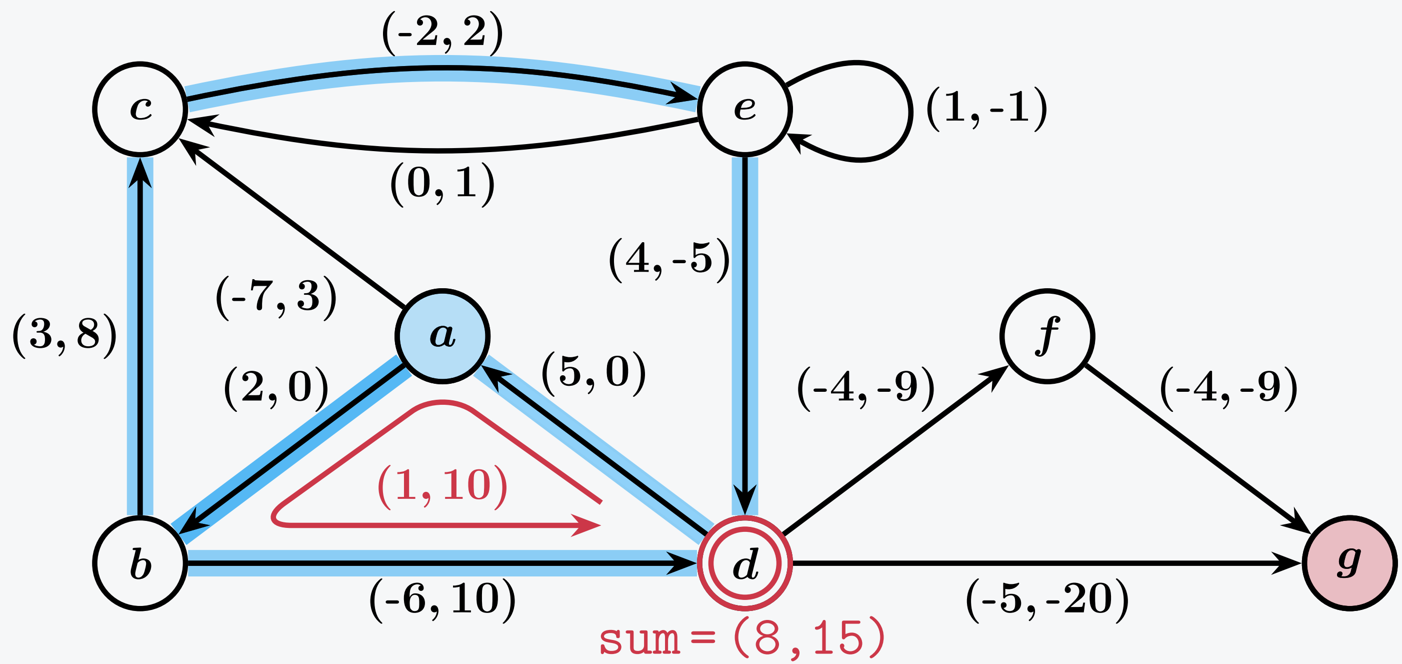

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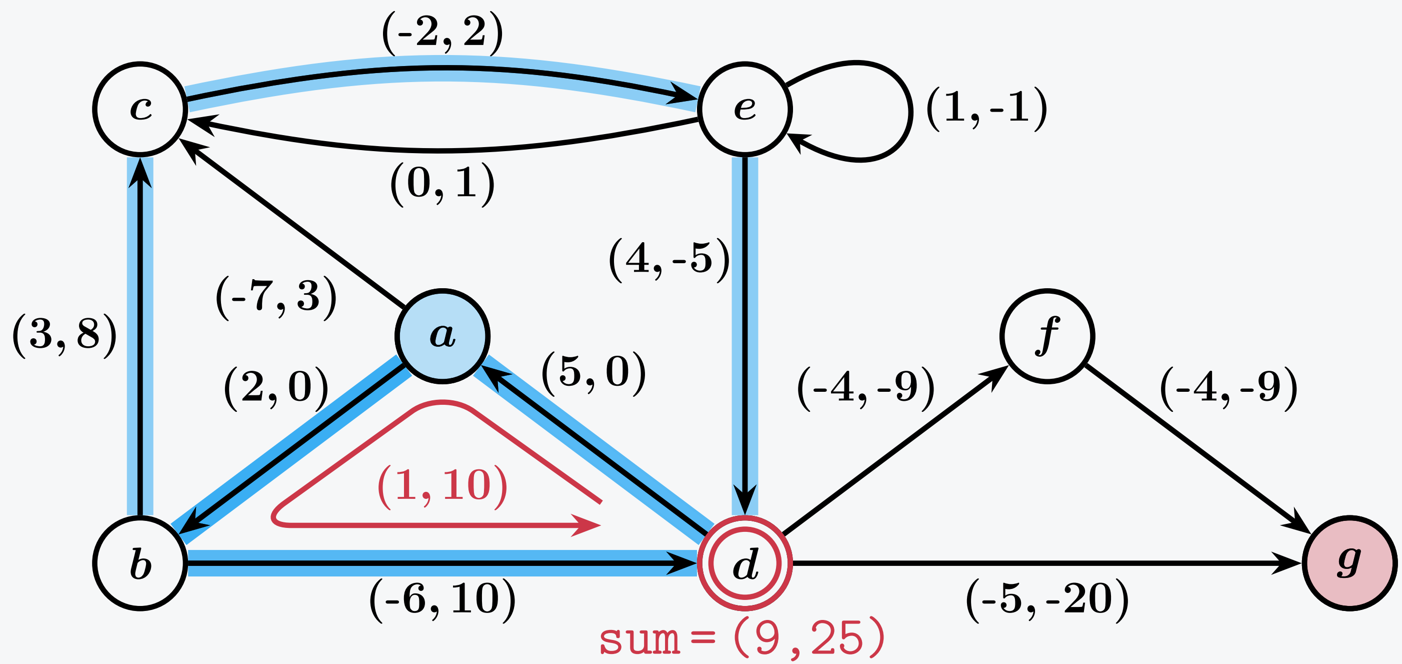

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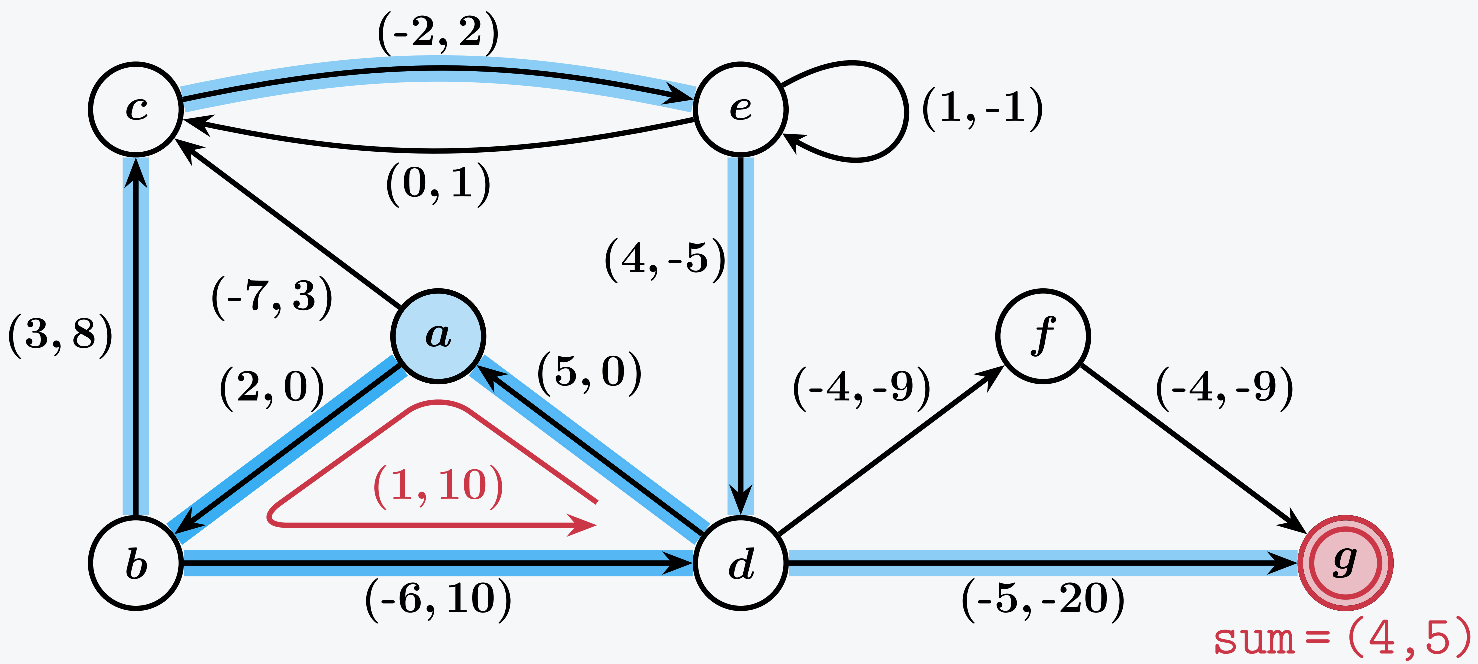

Does there exist a run from a with counter values (0,0) to g with counter values (4,5) ?

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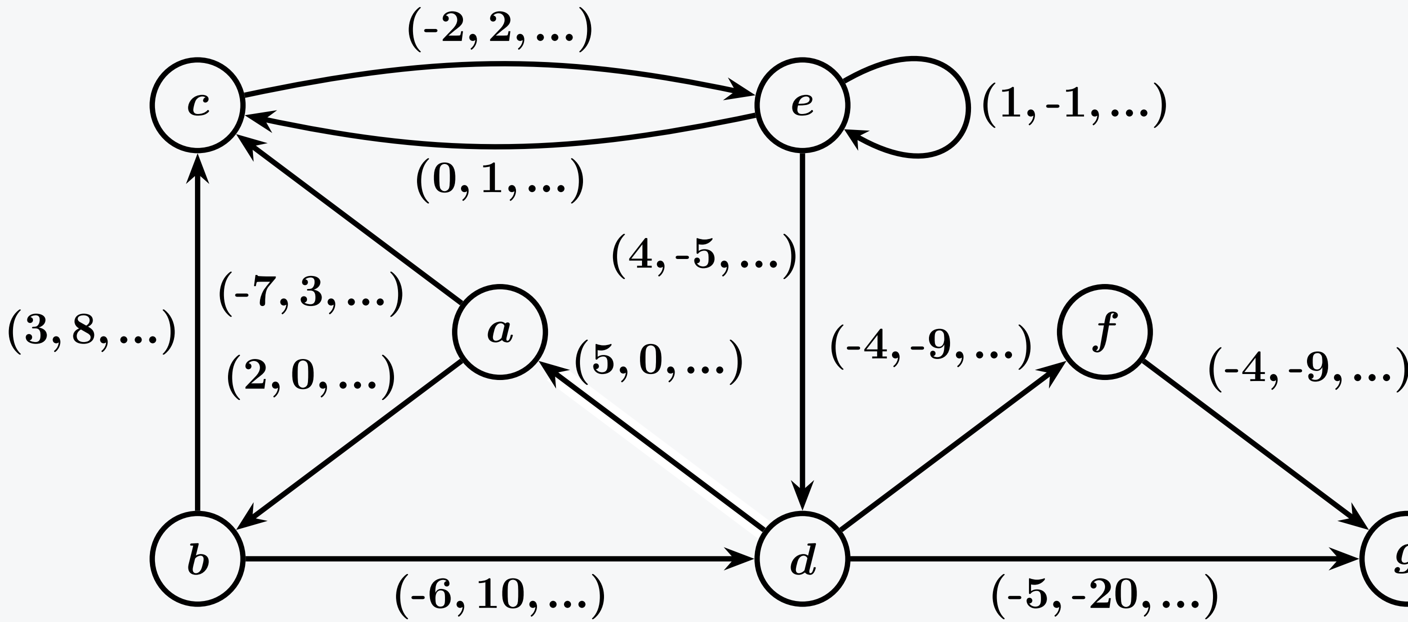
Reachability in VASS



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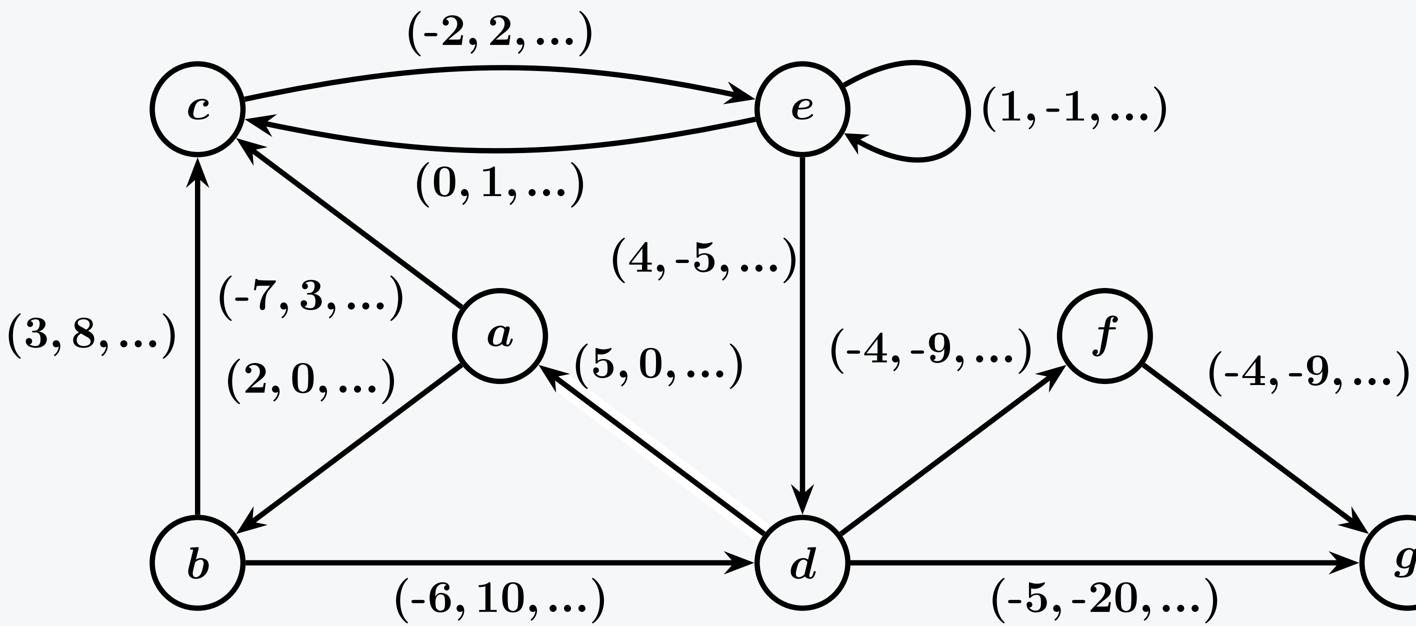

Reachability problem: does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v})$?

Flat VASS

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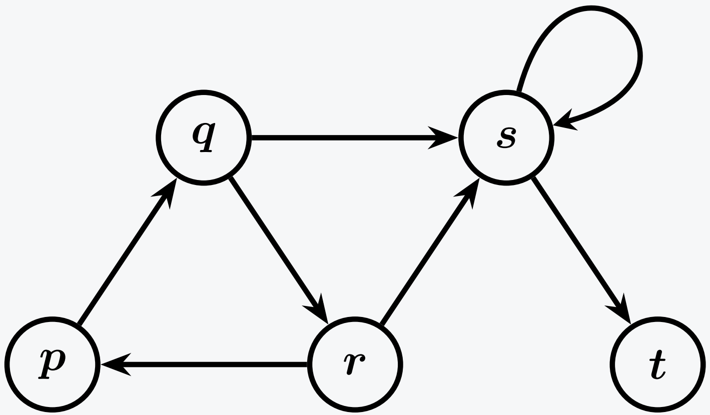

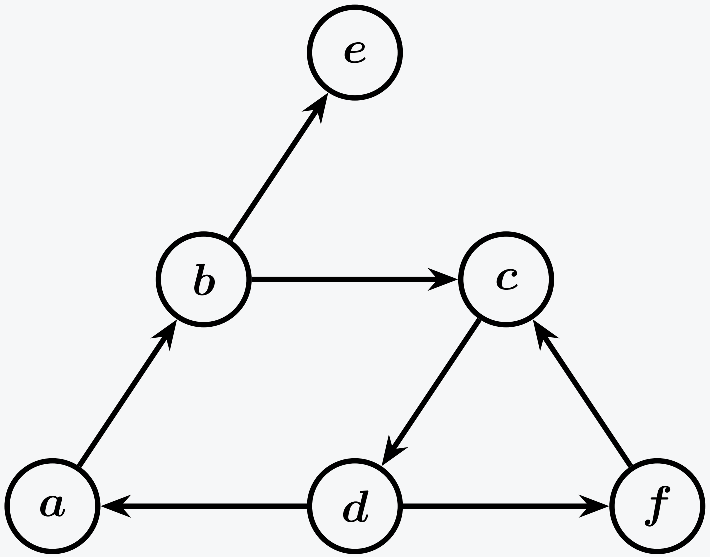

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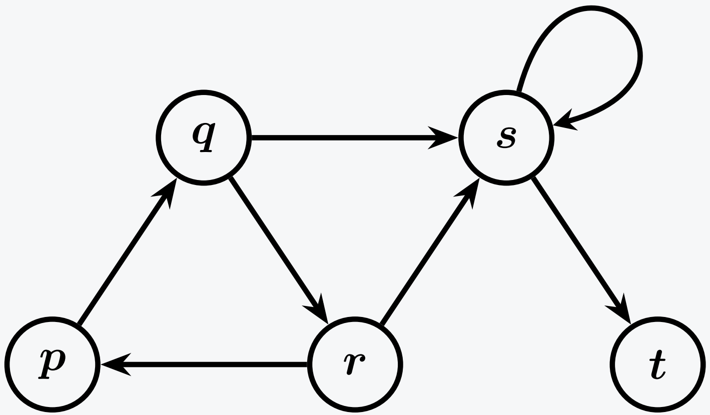

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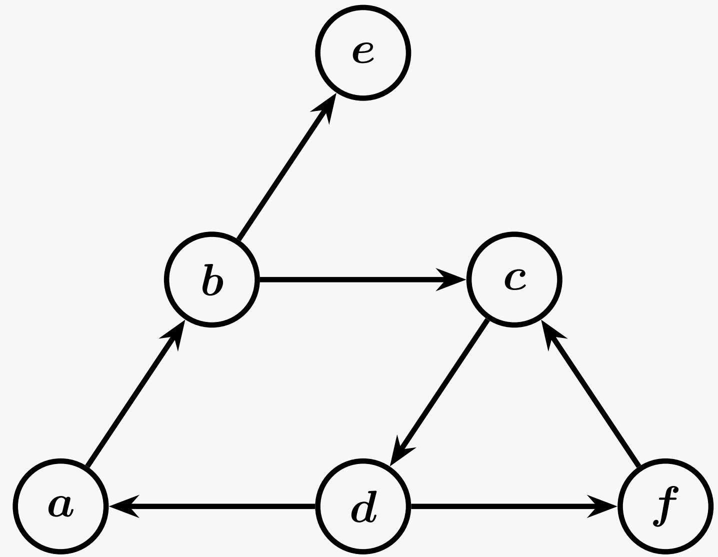

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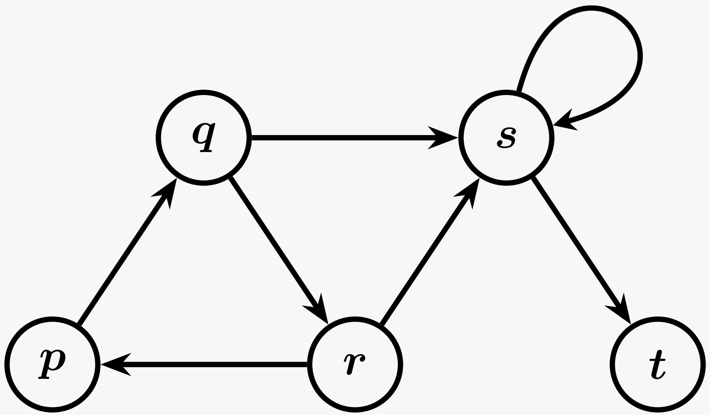

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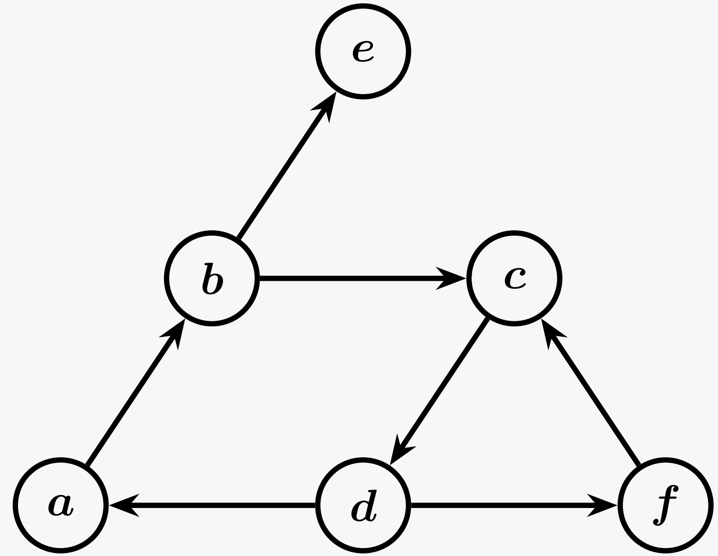

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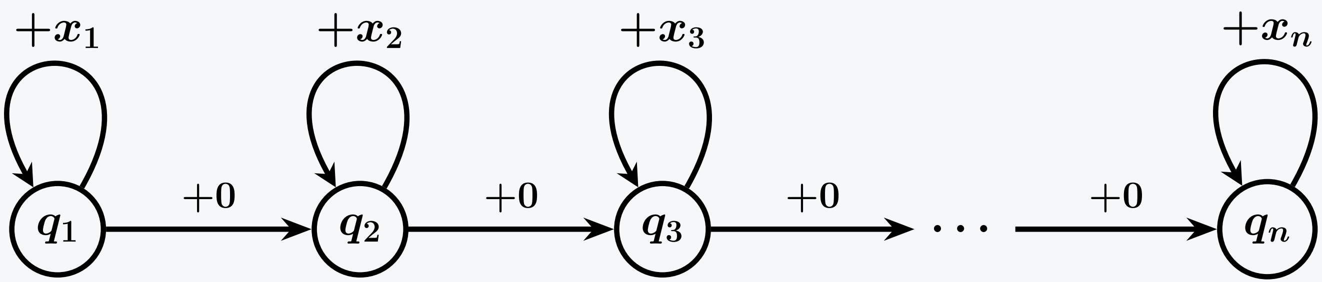

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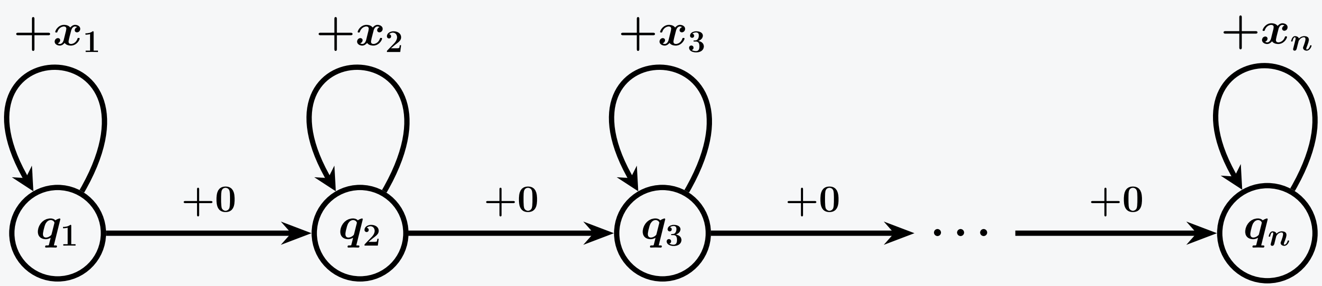

There exist $k_1, \ldots, k_n$ such that $t = \Sigma k_i \cdot x_i$ if and only if there is a run from $q_1(0)$ to $q_n(t)$.

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[Blondin, Finkel, Göller, Haase, and McKenzie '15]

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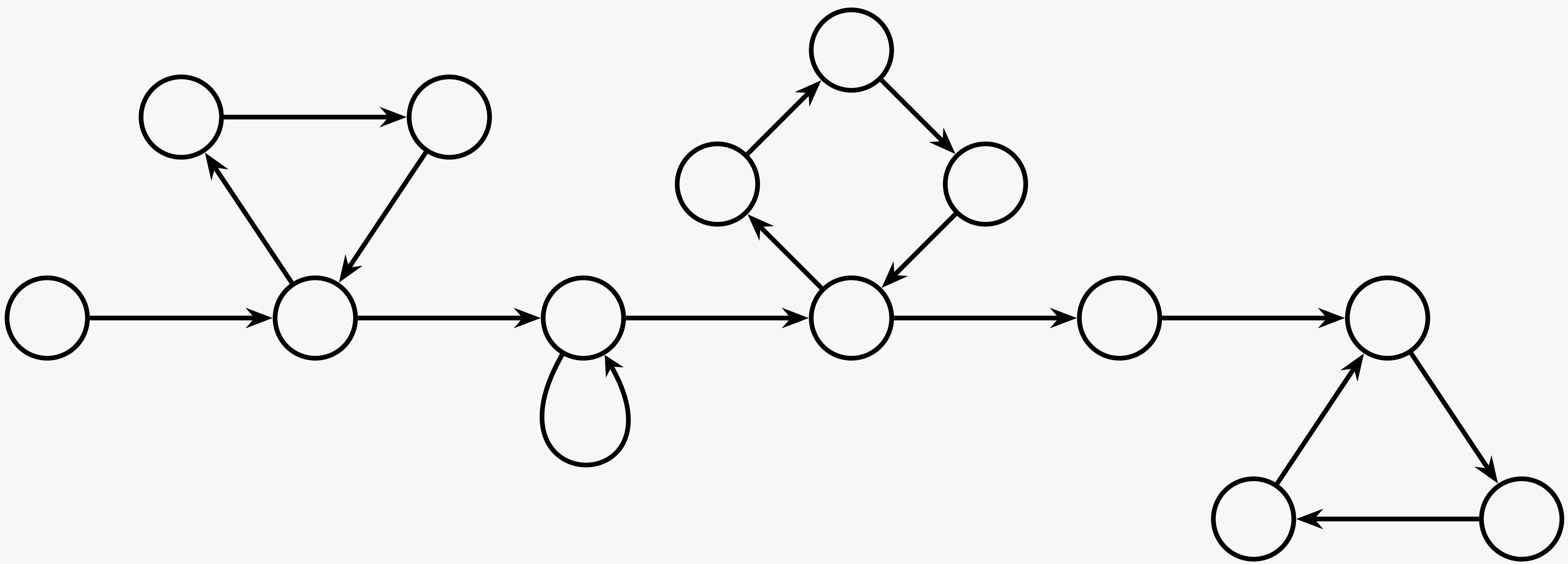

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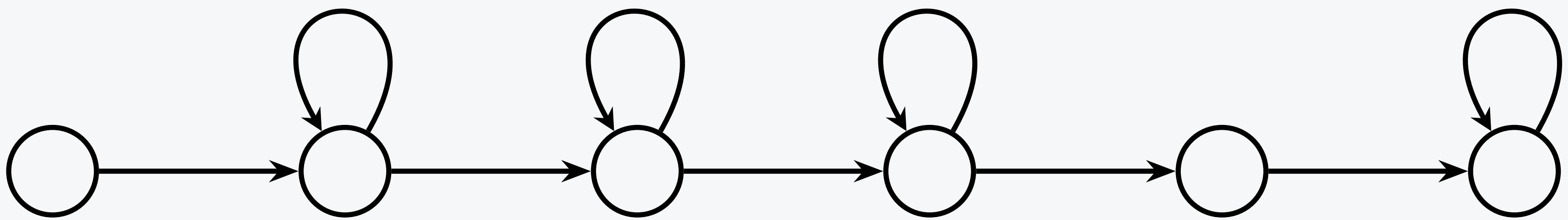

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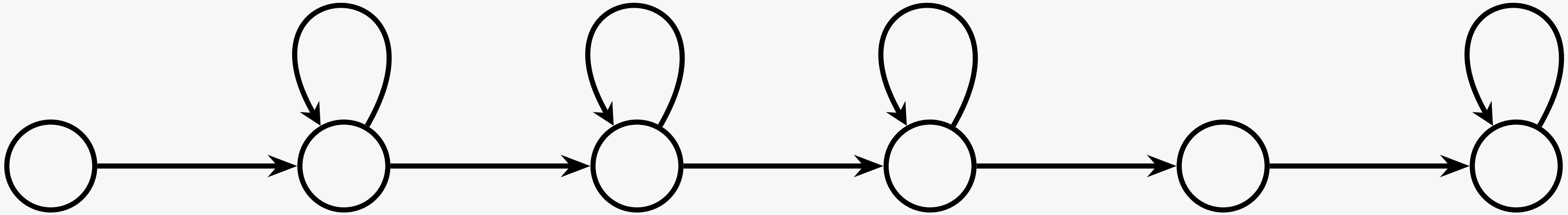

For $d \geq 3$, is reachability in unary d-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke '16]

[Leroux '21]

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   - A clause $x_1 \vee \neg x_2 \vee x_3$ is satisfied if $n \equiv 1 \bmod 2$ OR $n \equiv 0 \bmod 3$ OR $n \equiv 1 \bmod 5$.

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3) Again, use a conjunction of non-divisibility tests to verify that n represents a satisfying assignment.

   - A clause $x_1 \vee \neg x_2 \vee x_3$ is satisfied if $n \equiv 1 \bmod 2$ OR $n \equiv 0 \bmod 3$ OR $n \equiv 1 \bmod 5$.
   - Instead, check $2 \cdot 3 \cdot 5 \nmid n - 10$.

Main Contribution

Theorem. Reachability in unary 3-SLPS is NP-complete.

Proof approach. Reduce from 3-SAT. Suppose φ has k variables $(x_1, \ldots, x_k)$ and m clauses.

1) Use "Chinese remainder encoding" for SAT.

- Let $p_1, \ldots, p_k$ be the first k primes.
- Let $n \in \mathbb{N}$ such that $n \equiv 0 \bmod p_i \iff x_i$ is false and $n \equiv 1 \bmod p_i \iff x_i$ is true.

2) Use a conjunction of non-divisibility tests to verify that n represents a valid assignment.

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Let's construct a 2-SLPS with zero tests that:

- starts with $\mathsf{x} = \boldsymbol{v}$, $\mathsf{y} = \mathbf{0}$,
- can only be passed if $\boldsymbol{v} \nmid \mathbf{7}$, and
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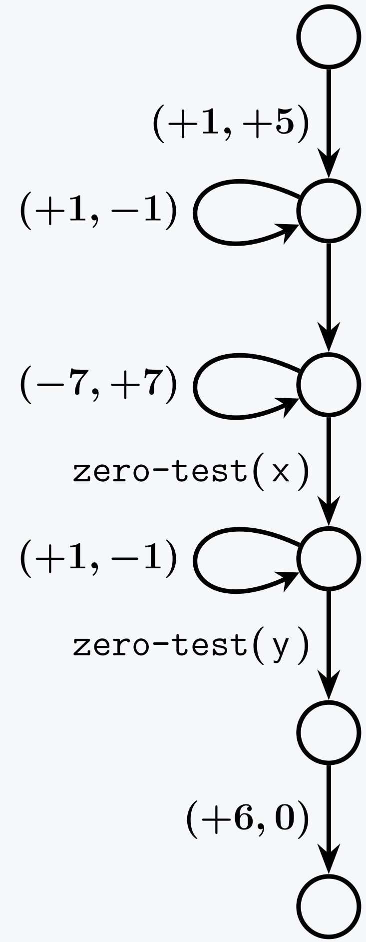

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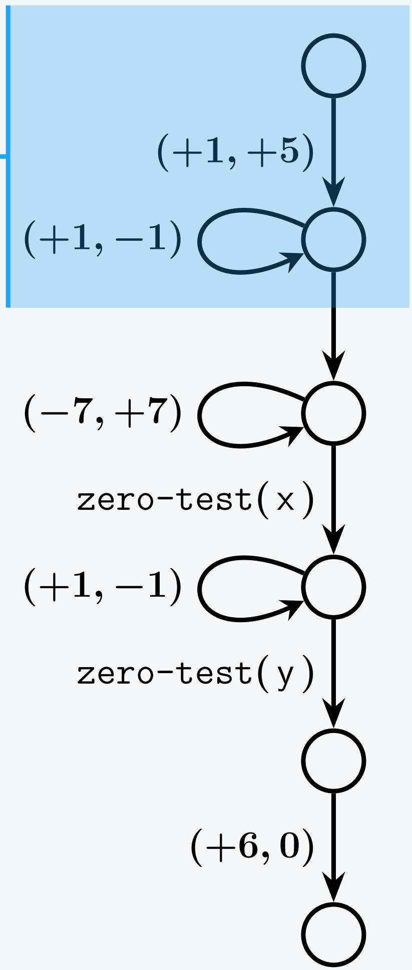
(i) Choose $r \in \{1, 2, 3, 4, 5, 6\}$ ...



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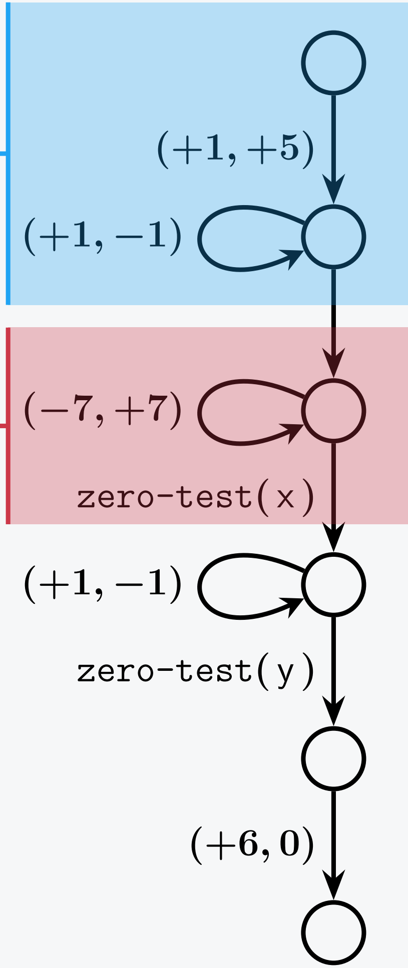

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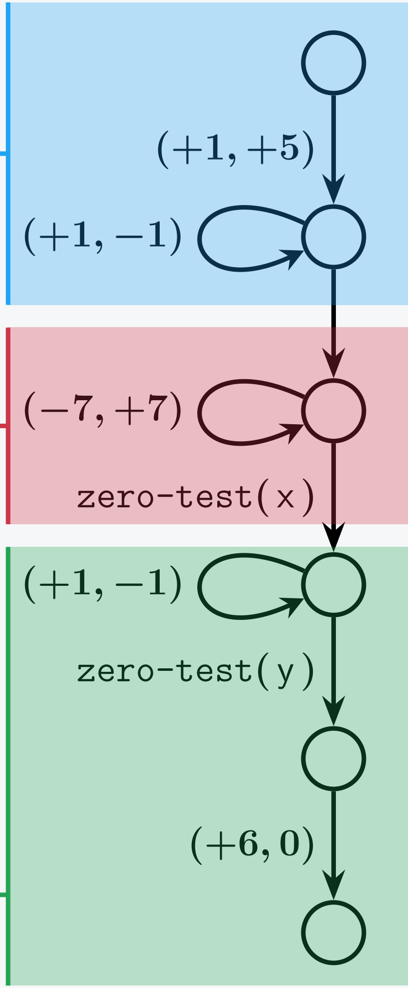
(iii) Restore x $= v$, y $= 0$.



Simulating Zero Tests

Simulating Zero Tests

Lemma 2.2 (Controlling Counter Technique). *Let $\mathcal{Z}$ be a d-VASS with zero tests and let $s(\mathbf{x}), t(\mathbf{y})$ be two configurations. Suppose $\mathcal{Z}$ has the property that on any accepting run from $s(\mathbf{x})$ to $t(\mathbf{y})$, at most m zero tests are performed on each counter. Then there exists a $(d+1)$-VASS $\mathcal{V}$ and two configurations $s'(\mathbf{0}), t'(\mathbf{y}')$ such that:*

(1) $s(\mathbf{x}) \xrightarrow{*}_{\mathcal{Z}} t(\mathbf{y})$ *if and only if* $s'(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}} t'(\mathbf{y}')$,

(2) $\mathcal{V}$ *can be constructed in* $\mathcal{O}((size(\mathcal{Z}) + \|x\|) \cdot (m+1)^d)$ *time, and*

(3) $\|\mathbf{y}'\| \leq \|\mathbf{y}\|$.

Moreover, if $\mathcal{Z}$ is a flat VASS or a (simple) linear path scheme in which no zero-testing transition lies on a cycle, then $\mathcal{V}$ can be assumed to be a flat VASS or a (simple) linear path scheme, respectively.

[Czerwiński and Orlikowski '21] [Chistikov, Czerwiński, Mazowiecki, Orlikowski, S., and Węgrzycki '24]

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Takeway message: A "small" number of zero tests can be simulated by an additional counter.

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- Use the controlling counter technique to obtain unary 3-SLPS for the SAT instance.

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Thank You!

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